MÄLARDALENS HÖGSKOLA

MASTER THESIS IN MATHEMATICS /APPLIED MATHEMATICS

Valuation of exotic guaranteed equity bond by Monte Carlo simulation

By

Kwok-wai Choy

Magisterarbete i Matematik/tillämpad Matematik

DEPARTMENT OF MATHEMATICS AND PHYSICS
MÄLARDALEN UNIVERSITY
SE-721 23 VÄSTERÅS, SWEDEN

MÄLARDALENS HÖGSKOLA

DEPARTMENT OF MATHEMATICS AND PHYSICS

Master thesis in mathematics / applied mathematics

Date:	2006-07-10
Project name:	Valuation of exotic guaranteed equity bond by Monte Carlo simulation
Author:	Kwok-wai Choy
Supervisor:	Jan Röman
Comprising:	20 points

Acknowledgements

I would like to show my deep appreciation to Jan Röman who is the supervisor in my studies. During the last three months of thesis preparation, under his patient guidance, I learned a lot in quantitative finance and the programming techniques. I indeed, feel delighted for his trustfulness on my working attitude so that I can work independently under positive and encouraging working atmosphere. This valuable experience will definitely play an important part in my life. Besides, my family and my friends are considerate and concerned on my progress of studies. I really appreciate for all their contributions. I also like to thank Mr. Tomas Flodén for suggesting this topic for my thesis.

Abstract

This thesis valuates guaranteed equity bonds by Monte Carlo simulation. It is a combination of equity and fixed-income financial instrument. For example, a company sells a product with a guaranteed return plus some extra return, which is linked, to the performance of equity index when it comes to maturity. Such products have a certain value depending on the guaranteed return, the performance of equity index market and the interest rate risk during the time to maturity.

Instead of the general contracts embedded with typical options, this thesis is going to valuate the contracts with exotic options under stochastic interest rate and volatility modelling. We will calibrate the model parameter by the observed market data and then compare the value obtained in the model with those that are sold by the insurance companies.

Table of Contents

Chapter 1
Introduction

In this chapter, we will introduce the background of quantitative finance and the outlines of the thesis.

1.1 Background

Quantitative finance is one of the fastest growing research fields in applied mathematics in the past several decades. International banking and financial firms around the globe are looking for the advanced analytical and numerical techniques to price financial derivatives and manage portfolio risks. The trend has been accelerating in recent years on numerous fronts, driven both by substantial theoretical advances as well as by a practical need in the industry to develop effective methods to price and hedge increasingly complex financial instruments.

This subject naturally has a close relationship with the discipline of financial economics, but quantitative finance is narrower in scope and more abstract. A central difference is that financial economics study the structural reasons why a company may have a certain share price, but quantitative finance will take the share price as given, and attempt to use stochastic calculus to obtain the fair value of derivatives of the stock. The aim of this thesis is to valuate a derivative of the stock, options. When it comes to options pricing, we should mention the famous Black-Scholes formula.

The option pricing model developed by Fisher Black and Myron Scholes first appeared in the academic literature in 1973. Not only did this specify the first successful options pricing formula, but it also described a general framework for pricing other derivative instruments. That paper launched the field of financial engineering. It means that finance can be treated quantitatively in mathematical language. It now forms the basis for calculating the benchmark premia for regular and many forms of exotic options. The original model that Black and Scholes introduced to the world was based on some quite restrictive assumptions, subsequently researchers have been able to develop their own less constrained option pricing models. But the idea is still from the Black and Scholes. Thus it is worth getting an intuitive appreciation of the Black-Scholes option pricing model, the inputs it requires, and the assumptions which underpin it.

The Black–Scholes model is a model of the varying price over time of financial instruments, and in particular stocks. The Black–Scholes formula is a mathematical formula for the theoretical value of European put and call stock options that may be derived from the assumptions of the model. The equation was derived by Fischer Black and Myron Scholes. The fundamental insight of Black and Scholes is that the call option is implicitly priced if the stock is traded. The use of the Black–Scholes model and formula is pervasive in financial markets. We will come back to Black-Scholes model in Chapter 2.

1.2 Guaranteed equity bonds (GEB)

Risk and return always come together. For every investment, it will provide investors some return, but on the same time, he has to bear the risk of the investment due to the uncertainty of the future. Investors always want to maximize their return, and minimize the risk for every investment. One can have a large return by investing the whole capital on the equity market, but on the same time, he has to bear a risk that what he will get will be less than the initial capital. Another one can have a certain return by investing the whole capital on the bond market, but on the same time, what he will get will for sure be less than investing directly to the equity market. Thus the balance between the return and risk make the birth of a guaranteed equity bond, the hybrid financial instrument of equity and fixed income market. It provide another investment choice to the investor who prefer more return than bonds but afraid the risk of investing directly in the equity market.

Guaranteed equity bonds (GEBs) have been introduced to the market in early 1995 [1] and since then they have generated a lot of interest in the option market. A GEB is a financial instrument in which the issuer, usually an insurance company, guarantees a stated interest rate and some protection from loss of initial capital, and provides an opportunity to earn additional interest based on the performance of an equity market index. One of the most commonly used indices is Standard & Poor's 500 Composite Stock Price Index (the S&P 500). The value of any index varies from day to day and is not predictable. When one buys a GEB, one owns a contract and he is not buying shares of any stock of index.

GEBs credit interest using a formula based on changes in the index to which it is linked. The formula decides how the additional interest, if any, is calculated and credited. How much additional interest one get and when to get it depends on the features of the contract. GEBs also promise to pay a minimum interest rate. The rate that will be applied will not be less than this minimum guaranteed rate even if the index-linked interest rate is lower. The value of the contract also will not drop below a guaranteed minimum. For example, many contracts guarantee the minimum value will be 100 percent of the capital paid, plus at least 3% in annual interest. The guaranteed value is, the minimum amount available at maturity, is the capital plus the total interest earned during the term.

In other words, it enables investors to achieve potential capital appreciation by participating in the positive performance of the index but also provide investors with a guaranteed minimum return of their investment at maturity.

Two features that have the greatest effect on the amount of additional interest that may be credited to a GEB are the indexing method and the participation rate. It is important to understand these features and how they work together. In Chapter 3, we will describe some basic features of the GEBs in details and the indexing method.

GEBs are complex financial instruments. Many GEBs permit investors to participate in only a stated percentage of an increase in an index. Many of these investments also impose a cap rate that represents the maximum annual account value percentage increase allowed to investors. GEBs typically do not provide for investor participation in the dividends accumulated on the securities represented by the index. In addition, investors may assume mistakenly that GEBs provide the same returns as an index mutual fund.

Moreover, because of the product's complexity, some investors might have difficulty understanding all of the features of the product and determining the extent to which those features meet the needs to them. For example, the payoff of the contracts under varying combination of caps and participation rates are one of the main points that we are going to explore in this thesis. The result we obtained can give an idea to both the investors and the company to invest and synthesize GEBs wisely.

1.3 Outline of the thesis

In Chapter 2, a discussion on the mathematics of financial derivatives and Monte Carlo simulation will be presented. In Chapter 3, a detailed description on the guaranteed equity bonds, including the features, indexing method, types of various contracts will be given. Basic features of the valuation program on guaranteed equity bonds will be presented in Chapter 4. Some interesting results regarding the exotic contracts and the plain vanilla one will be discussed in Chapter 5. Chapter 6 will discuss the profitability to the insurance company and the investor by investing some real traded contracts on the market. Finally, a conclusion will be given in Chapter 7.

Chapter 2
Mathematics of financial derivatives and Monte Carlo Simulation

In this chapter, we will describe the mathematics framework that we used to valuate various types of GEBs.

2.1 Black-Scholes options pricing model

The main aim of this thesis is to valuate exotic options, so we would like to introduce the famous Black-Scholes options pricing model first.

The Black-Scholes options valuation model consists of a rather complex mathematical formula developed by two famous economists, which permits one to calculate what the theoretical price of an equity option should be. It prices European put or call options on a stock. The model assumes that the price of heavily traded assets follow a geometric Brownian motion with constant drift and volatility. When applied to a stock option, the model incorporates the current price of the stock, the option's strike price and the time to the option's expiry. The following describes the major determinants of the value of a stock option, which the Black-Scholes model takes into account.

Strike price

At any point in time, an option's intrinsic value is computed the difference between the strike price and the current underlying share price. In the case of a call option, if the current share price is below the strike price, then the option is said to be 'out of the money' and has no intrinsic value. Conversely, if the current share price is higher than the strike price, then the option is 'in-the-money' and has intrinsic value. In the case of put options, the same applies in reverse.

Time to maturity

All other things equal, the price of stock options decreases at an accelerating rate as the expiration date approaches. This is referred to as 'time decay' and is the primary reason that long-term options are more expensive than short-term options in the same underlying stock.

Volatility

In simple terms, volatility is a measure of stock moves around– whether up or down. There are two types of volatility. Historical volatility can be determined mathematically based on the size of price moves that a particular stock has actually made in the past. Implied volatility refers to the degree of price volatility that a particular stock is anticipated to make in the future. Different investors may have significantly different expectations about the future volatility of a stock. Hence, there will often be different perceptions of what the fair price of a particular stock option should be. Historical volatility is, however, often used as an estimate of the implied volatility of a stock to value a stock option.

Risk-free interest rate

The risk-free interest rate also affects option prices. When an investor buys an option, he has to pay a premium for the option contract. If the investor did not buy that option, he could have placed the amount of the option premium on time deposit and earned risk-free interest on it. The option price or premium will therefore reflect this lost interest. In particular, the higher the interest rate, the more lost interest and hence the lower the option premiums have to be to compensate for that foregone interest. In practice, when interest rates are very low, this effect is negligible and is often ignored when valuing an option, particularly a short-term option.

Dividends

Dividends that are due to be paid out on the underlying shares during the period of the option will reduce the price of the stock by that amount when actually paid out. This will have the effect of reducing the price of call option premiums and increasing put premiums. Accordingly, the anticipated payment of dividends must be taken into account when valuing options.

Next, the key assumptions of the Black–Scholes model are:

1) The price of the underlying instrument follows a geometric Brownian motion S_t , in particular with constant drift μ and volatility σ:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.1}$$

2) The underlying share can be shorted without penalty and short-sellers receive the cash benefits from the short sale in full.
3) The option is only exercisable at expiration.
4) The market operates continuously.
5) There are no transaction costs, zero taxes and no bid-offer spread.
6) The risk-free rate of interest is constant over the lifetime of the option.
7) The shares pay no dividend over the lifetime of the option.

Based on these assumptions and some calculation, it leads to the following formula for the price of a call on a stock currently trading at price S, where the option has an exercise price of *K*, at *T* years in the future. The constant interest rate is r and the constant stock volatility is σ.

$$C(S,K,r,\sigma,T) = SN(d_1) - Ke^{-rT}N(d_2) \tag{2.2}$$

where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{2.3}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Here *N* is the standard normal cumulative distribution function.

The price of a put option may be computed from this by put-call parity and simplifies to

$$P(S,K,r,\sigma,T) = Ke^{-rT}N(-d_2) - SN(-d_1) \qquad (2.4)$$

The Black-Scholes model is one of the most important concepts in modern financial theory. It was developed in 1973 and is still widely used today, and regarded as one of the best ways of determining fair prices of options. There are a number of variants of the original Black-Scholes model

For options on stocks may pay a dividend, it is reasonable to simplify and make the assumption that the dividends are paid continuously. The dividend payment paid over the time period $[t, t+dt]$ is then modelled as $qS_t dt$ for some constant q.

Under this formulation the arbitrage-free price implied by the Black–Scholes model could be shown to be

$$C(S,K,r,q,\sigma,T) = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2) \qquad (2.5)$$

where now

$$F = Se^{(r-q)T} \qquad (2.6)$$

is the modified forward price that occurs in the terms d_1 and d_2:

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)T}{\sigma\sqrt{T}} \qquad (2.7)$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

It is exactly the same formula that is used to price options on foreign exchange rates, except q plays the role of the foreign risk-free interest rate and S is the spot exchange rate.

Although Black-Scholes formula can give analytical solution to the theoretical price of the European option, it has some contradictions in practice to the real financial market. In Black–Scholes model, the implied volatility of an option on a particular stock would be constant, even as the strike and maturity varied, and roughly equal to the historic volatility. However, in practice, the volatility surface (implied volatility against strike and maturity) is not flat. In fact, in a typical market, the graph of strike against implied volatility for a fixed maturity is typically smile-shaped. That is, the at-the-money options have the lowest implied volatility; the out-of-the-money or in-the-money options have different implied volatility. In fact, the volatility surface of a given underlying instrument depends on, among other things, its historical distribution and is constantly changing as investors, market-makers, and arbitrageurs re-evaluate the probability of the underlying instrument reaching a given strike and the risk-reward associated to it.

In order to solve this problem, we will treat the volatility of the underlying index as a random variable and model it as a stochastic process. We will introduce the stochastic volatility modelling later in this chapter.

2.2 Risk-neutral pricing

In quantitative finance, a risk-neutral measure is a probability measure in which today's arbitrage-free price of a derivative is equal to the expected present value of the future payoff of the derivative.

The measure is named because, under this measure, all financial assets have the same expected rate of return, regardless of the risk of the asset. This is in contrast to the physical measure, the actual probability distribution of prices where more risky assets, assets with a higher volatility, have a greater expected rate of return than less risky assets.

From equation (2.5), it shows that the value of the option does not depend on the expected rate of return of the stock, but depend on the rate of risk free return. This observation led the financial economists Cox and Ross to develop an important tool known as the risk-neutral valuation method of security. Cox and Ross think that if a stock follows the geometric brownian motion of price movements, then the Black-Scholes differential equation says that to avoid arbitrage opportunities, option values must equal the values predicted by the Black-Scholes formula. But these formulas should be valid regardless of the average investor's view toward risk. Thus, as long as a given "investment world" satisfies the basic assumption of the Black-Scholes formula, the value given by the formula will hold. This world in which investors are completely neutral toward risk is called a risk-neutral world. It is characterized that investors require no risk premium for their investments.

Risk-neutral measures make it easy to express in a formula the value of a derivative. Suppose at some time T in the future a derivative, a call option on a stock, pays off Z_T units, where Z_T is a random variable on the probability space describing the market. Further suppose that the discount factor from now, time zero, until time T is $P(0, T)$, then today's fair value of the derivative is

$$Z_0 = P(0,T)E_Q(Z_T) \qquad (2.8)$$

where Q denotes the risk-neutral measure. This can be re-stated in terms of the physical measure P as

$$Z_0 = E_P(\frac{dQ}{dP} Z_T) \qquad (2.9)$$

where $\frac{dQ}{dP}$ is the Radon-Nikodym derivative of Q with respect to P.

A particular financial market may have one or more risk-neutral measures. If there is just one, then there is a unique arbitrage-free price for each asset in the market. This is the fundamental theorem of arbitrage-free pricing. If there is more than one such measure, then there is an interval of prices in which no arbitrage is possible. Suppose that the market consists of one stock, one risk free bond and that our model describing the evolution of the world is the Black-Scholes model. From equation (2.1), the stock has dynamics

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a standard Weiner process with respect to the physical measure. If we define

$$\tilde{W}_t = W_t + \frac{\mu - r}{\sigma} t \tag{2.10}$$

then Girsanov's theorem states that there exists a measure Q under which $\tilde{W}_t$ is a Weiner process. Substituting in we have

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t \tag{2.11}$$

Q is the unique risk neutral measure for the model. The discounted payoff process of a derivative on the stock $Z_t = E_Q(Z_T | F_t)$ is a martingale under Q. Since S and Z are Q-martingales we can find a replicating strategy, a holding of stocks and bonds that pays off Z_t at all times $t \le T$.

We will valuate various kinds of contracts under risk-neutral measure in our simulation program.

2.3 Pricing processes

Here we will introduce the mathematical models that are used to describe the behaviour of equity index, volatility and interest rate in our simulation model.

2.3.1 Black-Scholes model

First, we assume that the price of the equity index follows a geometric Brownian motion under risk-neutral measure. From equation (2.11),

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t^{(1)}$$

The equation gives the dynamics of the equity index: S_t denotes the index price at time *t*, r_t is the risk neutral drift, σ_t is the volatility and $W_t^{(1)}$ is Weiner process.

2.3.2 GARCH(1,1) volatility model

According to the classical Black-Scholes options pricing model, all options based on the same underlying sharing a constant implied volatility under the assumption of a geometric Brownian motion process. But if this model is used to back-test the market-traded option, we can observe that different contracts produce significantly different implied volatilities. Option's implied volatilities actually vary with the different time to maturity. This is the term structure of implied volatility. For a given time to maturity, implied volatilities for different strikes are not the same either. This is the implied volatility skew and often referred as the volatility smile. All these market evidences imply that the option market expects the future volatility of the underlying asset will not be a constant.

Thus a stochastic volatility model is more reasonable for option pricing. It can explain the basic shapes of the smile patterns and allow for more realistic theories of the term structure of implied volatility. A particular case is that volatility can be described with a GARCH model. GARCH is the acronym for 'Generalized Autoregressive Conditional Heteroskedastic'. In GARCH models, the variance is written as a function of past returns, but with exponentially smoothing and a certain time-decay factor. One more important feature of GARCH is that the constant term in the recursive equation allows GARCH to capture the notion that the volatility is mean reverting, and allow the model to be used for forecasting volatility. Other alternative models such as the GJR-GARCH and NGARCH allow negative returns to have more effect on the volatility than positive returns.

In our simulation, we will use the simplest one, GARCH(1,1), to describe the volatility of the equity index. GARCH(1,1) means that there is only one past return and one past σ are used for forecasting future volatility.

The process of the volatility of equity index is as follow

$$\sigma_{t+1}{}^2 = \kappa\vartheta + (1-\kappa)((1-\lambda)y_t^2 + \lambda\sigma_t{}^2)$$
$$y_t = \frac{\log\left(S(t)/S(t-1)\right) - (r_t - \frac{1}{2}\sigma_t{}^2)\Delta t}{\sqrt{\Delta t}} \qquad (2.12)$$

The equation gives the evolution of the variance as the weighted average with weights κ and $1-\kappa$, of two parts, one being the constant ϑ and the other being a weighted average of y_t^2 and σ_t^2. Whatever the variance might be at time *i*, the variance of y_j at any date *j* far into the future, computed without knowing the intervening y_{i+1}, y_{i+2},…, will be approximately the constant ϑ. The constant ϑ is called the unconditional variance, whereas σ_t^2 is the conditional variance of y_t.

To understand the unconditional variance, it is useful to consider the variance forecasting equation. Specifically, we can calculate $E_i[\sigma_{i+n}^2]$, which is the estimate made at date *i* of the variance of y_{i+n}; we estimate the variance without having observed $y_{i+1},\dots,y_{i+n-1}$. Note that by definition $E_i[y_{i+1}^2] = \sigma_{i+1}^2$, so (2.12) implies

$$E_i[\sigma_{i+2}^2] = \kappa\vartheta + (1-\kappa)((1-\lambda)E_i[y_{i+1}^2] + \lambda\sigma_{i+1}{}^2) = \kappa\vartheta + (1-\kappa)\sigma_{i+1}{}^2 \qquad (2.13)$$

Likewise,

$$E_i[\sigma_{i+3}^2] = \kappa\vartheta + (1-\kappa)E_i[\sigma_{i+2}^2] = \kappa\vartheta(1+(1-\kappa)) + (1-\kappa)^2\sigma_{i+1}{}^2 \qquad (2.14)$$

This generalizes to

$$E_i[\sigma_{i+n}^2] = \kappa\vartheta(1+(1-\kappa)+...+(1-\kappa)^{n-2}) + (1-\kappa)^{n-1}\sigma_{i+1}{}^2 \qquad (2.15)$$

Thus, there is decay at rate κ in the importance of the current volatility $\sigma_{i+1}{}^2$ for forecasting the future volatility. Furthermore, as $n \to \infty$, the geometric series $1+(1-\kappa)+...+(1-\kappa)^{n-2}$ converges to $1/\kappa$, so, as $n \to \infty$ we obtain

$$E_i[\sigma_{i+n}^2] \to \vartheta \qquad (2.16)$$

This means that our best estimate of the conditional variance, at some date far in the future, is approximately the unconditional variance ϑ.

The most interesting feature of the volatility equation is that large returns in absolute value lead to an increase in the variance and hence are likely to be followed by more large returns. This is called 'volatility clustering', which is quite observable in actual markets. This feature also implies that the distribution of the returns will be 'fat-tailed'. This means that the probability of the extreme returns is higher than under a normal distribution with the same standard deviation. It is agreed that daily and weekly returns in most markets have this 'fat-tailed' property.

2.3.3 CIR interest rate model

From the classical Black-Scholes options pricing model, the interest rate will be constant throughout the timeframe, it is acceptable for short life option traded in the market. However, the timeframe that we are considering is 5 years, it is for sure that the interest rate must change throughout the timeframe [2]. In order to better describe the reality, we consider the interest rate as a random variable, which follows a Cox-Ingersoll-Ross (CIR) process.

Cox, Ingersoll and Ross developed one of the first general equilibrium theories of the term structure of interest rates in 1985. That theory came with a model for the pricing of zero coupon bonds and derivatives. The CIR model is based on the following stochastic process for the short rate:

$$dr_t = \mu(\gamma - r_t)dt + \sigma_r\sqrt{r_t}\,dW_t^{(2)} \qquad (2.17)$$

where $W_t^{(2)}$ is Weiner process and μ, γ, σ_r are positive constants. The interest rate follows the square-root process: γ is the long-run mean variance, μ represents the speed of mean reversion, and σ_r is a parameter, which determines the volatility of the variance process. Also, we assume that $W_t^{(1)}$ and $W_t^{(2)}$ are two independent Weiner processes. This model has some realistic properties. First, negative interest rates are precluded. Second, the absolute variance of the interest rate increases when the interest rate itself increases. Third, the interest rates are elastically pulled to the long-term value γ, where μ determines the speed of adjustment.

2.4 Monte Carlo simulation

Simulation is widely used to solve problem that are intractable from an analytical point of view. This may be due to the inherent complexity of a problem to the presence of uncertainty. Basically, Monte Carlo simulation is based on statistical sampling, and it may be visualized as a black box in which a stream of psuedorandom numbers enter, an estimate of a quantity of interest is obtained by analyzing the output. Typically, we want to estimate an expected value with respect to an underlying probability distribution; for instance, an option price may be evaluated by computing the expected value of the payoff with respect to a risk-neutral probability measure. In other cases we want to evaluate a portfolio policy by simulating a suitably large number of scenarios. We may be interested not only in average values, but also in what happens on the tail of probability distributions, as in the case of value-at-risk; however, even this case boils down to computing expected values, possibly conditioned ones. Generally, a large number of simulations generally are required to achieve a high degree of pricing accuracy. However, its efficiency can be improved using control variates and quasi-random numbers.

In all these examples, the underlying issue is actually computing an integral in a possibly high-dimensional space.

Recall from the fundamental theorem of arbitrage-free pricing that the value of a derivative is equal to the discounted expected value of the derivative payoff where the expectation is taken under the risk-neutral measure. An expectation is simply an integral with respect to the measure. Monte Carlo methods are ideally suited to evaluate difficult integral. Thus suppose that our risk-neutral probability space is Q and that we have a derivative Z that depends on a set of underlying instruments $S_1,\ldots, S_n$. Then given a sample ω from the probability space the value of the derivative is $Z(S_1(\omega), S_2(\omega),\ldots,S_n(\omega)) = Z(\omega)$. Today's value of the derivative is found by taking the expectation over all possible samples and discounting at the risk-free rate. Thus the derivative has value:

$$Z_0 = P(0,T)\int_{\omega} Z(\omega)dQ(\omega) \tag{2.18}$$

where $P(0, T)$ is the discount factor corresponding to the risk-free rate at the final maturity date T years into the future.

Now, suppose the integral is hard to compute. We can approximate the integral by generating sample paths and then taking an average. Suppose we generate N samples then

$$Z_0 \approx \frac{1}{N}\sum_{\omega\varepsilon\ SampleSet} Z(\omega) \tag{2.19}$$

which is much easier to compute.

In our simulation program, we simulate thousand time of the future possible scenario of the equity index under stochastic interest rate and volatility model. By the strong law of large number, we will get the fair price of the derivative.

To facilitate an understanding of the technique, we shall look at how Monte Carlo simulation has been used to price standard European options. Of course, we know that the Black-Scholes model is the correct method of pricing these options so Monte Carlo simulation is not really needed. It is useful, however, to conduct this experiment because it demonstrates the accuracy of the technique for a simple option of which the exact price is easily obtained from a known formula.

The assumptions of the Black-Scholes model imply that for a given stock price at time t, simulated changes in the stock price at a future time $t + dt$ can be generated by the following formula:

$$\Delta S = Sr\Delta t + S\sigma\varepsilon\sqrt{\Delta t} \qquad (2.20)$$

where S is the current stock price, ΔS is the change in the stock price, and Δt is the length of the time interval over which the stock price change occurs. The variable ε, is a random number generated from a standard normal probability distribution. Recall that the standard normal random variable has a mean of zero, a standard deviation of one and occurs with a frequency corresponding to that associated with the famous bell shaped curve. Generating future stock prices according to the above formula is actually quite easy. A standard normal random variable can be approximately generated by the computer. After generating one standard normal random variable, we simply insert it into the right hand side of the above formula for ΔS. This gives the price change over the life of the option, which is then added to the current price to obtain the price of the asset at expiration. We then compute the price of the option at expiration according to the standard formulas, $\max(0, S_T - K)$ for a call or $\max(0, K - S_T)$ for a put, where S_T is the asset price at expiration. This produces one possible option value at expiration. We then repeat this procedure many thousands of times, take the average value of the call at expiration and discount that value at the risk-free rate.

Naturally every simulation is different because each set of random numbers is different. A Monte Carlo procedure written in Excel's Visual Basic produced the following values for this call option, whose actual Black-Scholes price is 5.79, where the number of random drawings is the sample size, n.

n	Call Price
1,000	5.58
10,000	5.51
50,000	5.83
100,000	5.75

It would appear that a sample of at least 50,000 is required for a standard European option. The option price obtained from a Monte Carlo simulation is a sample average. Thus, its standard deviation is the standard deviation of the sample divided by the square root of the sample size. Consequently, the error reduces at the rate of 1 over the square root of the sample size. Notice what this means for increasing the sample accuracy by increasing the sample size. Suppose σ is the standard deviation of the sample. We first conduct a Monte Carlo simulation using n_1 random drawings. Since the option value is a sample mean, the standard deviation of our estimate of the option value is $\sigma / \sqrt{n_1}$. Now, suppose we wanted to reduce that standard deviation in half. Let this new sample size be n_2. Then its standard deviation of the estimate of the option price is $\sigma / \sqrt{n_2}$. Now note that,

$$0.5\, \sigma / \sqrt{n_1} = \sigma / \sqrt{n_2} \text{ if and only if } n_2 = 4n_1$$

Thus, to achieve a 50% reduction in error, i.e., a 50% increase in accuracy, we must quadruple the number of random drawings. That is, the standard error reduces only at the rate of the square root of the sample size, not at the rate of the sample size itself.

After the introduction of the mathematical background, we are ready to study the financial instruments, Guaranteed equity bonds, in depth.

Chapter 3
Guaranteed equity bonds

In this chapter, we will describe the features of GEBs that are sold in the real financial market and the mathematical formulation of various kinds of contracts.

3.1 Basic terminology

In short, Guaranteed equity bonds (GEBs) are a combination of equity and fixed-income financial instrument. It run for a fixed term, usually three to six years, and the returns are linked to the performance of a market index. Investors get a share of the gain if the market rises, or your capital back if the market falls. The simplest GEBs pay a fixed return if the index has not fallen by the end of the term. So a bond linked to the index might offer a fixed return of, says 19%, after three years. For example, if the index is at 5,700 when one invest and have climbed to 6,000 by the end of the term, he will get back the original investment, plus 19%. However, if the index is below 5,700, he will get back only the original capital. But a 19% return over three years works out at 5.97% a year. If we put the money in an ordinary savings account today, we could earn interest of 5% a year, or more. Moreover, a lot can happen in three years. A bond that looks great in the summer of 2006 might look pretty silly in the summer of 2009. However, not all bonds pay a fixed return. Instead, the bonds that we are going to study have a return that linked to the proportion of gains in the index at the end of the term. Due to its complex payoff structure, we should be familiar with the features of the contracts before going further.

Term

The index term is the period over which index-linked interest is calculated. In most product designs, interest is credited to the investor at the end of a term. Terms are generally from three to seven years, with five years being most common. Some contracts offer single terms while others offer multiple, consecutive terms. If the contract has multiple terms, there will usually be a window at the end of each term, typically 30 days, during which money can be withdrew without penalty.

Participation Rate

The participation rate decides how much of the increase in the index will be used to calculate index-linked interest. For example, if the calculated change in the index is 9% and the participation rate is 70%, the index-linked interest rate for the contract will be 6.3% (9% x 70% = 6.3%). A company may set a different participation rate for newly issued contracts as often as each day. Therefore, the initial participation rate in the contract will depend on when it is issued by the company. The company usually guarantees the participation rate for a specific period, from one year to the entire term. When that period is over, the company sets a new participation rate for the next period. Some contracts guarantee that the participation rate will never be set lower than a specified minimum or higher than a specified maximum.

The participation rate may vary greatly from one contract to another and from time to time within a particular contract. Therefore, it is important to know how the contract's participation rate works with the indexing method. A high participation rate may be offset by other features, such as lower guarantee interest, cap rate, or indexing method. For example, an insurance company may offset a lower participation rate by also offering a feature such as an annual reset indexing method.

Cap Rate

Some contracts may put an upper limit, or cap, on the index-linked interest rate. This is the maximum rate of interest the contract will earn. In the example given above, if the contract has a 6% cap rate, 6%, and not 6.3%, would be credited. Not all contracts have a cap rate. While a cap limits the amount of interest one might earn each year, contracts that have a cap may have a higher participation rate.

Floor

The floor is the minimum index-linked interest rate that will be paid. The most common floor is 0%. A 0% floor assures that even if the index decreases in value, the index-linked interest that can earn will be zero and not negative. As in the case of a cap, not all contracts have a stated floor on index-linked interest rates. But in all cases, the fixed contract will have a minimum guaranteed value.

Guaranteed Interest Compounding

Some contracts pay simple interest during an index term. That means index-linked interest is added to the original premium amount but does not compound during the term. Others pay compound interest during a term, which means that index-linked interest that has already been credited also earns interest in the future. It is important to know whether the contract pays compound or simple interest during a term. While it may earn less from a contract that pays simple interest, it may have other features, such as a higher participation rate.

Dividends

Depending on the index used, stock dividends usually are not included in the index value. For example, the S&P 500 is a stock price index and only considers the prices of stocks. It does not recognize any dividends paid on those stocks.

Early Withdrawal

In most cases, investors cannot take all or part of the money out of contract at any time during the term. There will be a cost and the index-linked interest on the amount withdrawn will not be paid.

3.2 Indexing method

The indexing method means the approach used to measure the amount of change, if any, in the index. Some of the most common indexing methods, which are explained more fully below, include point-to-point, annual reset (ratchet), high water mark (lookback). We also synthesize other indexing method like point-to-point with barrier**,** annual reset with barrier, average [3]**.**

3.2.1 Point-to-Point

The index-linked interest, if any, is based on the difference between the index value at the end of the term and the index value at the start of the term. Interest is added to the contract at the end of the term. It works like a European call option.

Since interest cannot be calculated before the end of the term, this design may permit a higher participation rate than contracts using other designs. However, as the payoff is totally dependent on the index value at the maturity, investors have to bear a risk if the index fluctuates highly at that time.

3.2.2 Annual Reset (Ratchet)

The index-linked interest, if any, is determined each year by comparing the index value at the end of the year with the index value at the start of the year during the term of the contract. Interest is added to the contract each year during the term.

Since the interest earned is ‘locked in’ annually and the index value is ‘reset’ at the end of each year, future decreases in the index will not affect the interest that have already earned. Therefore, the contract using the annual reset method may credit more interest than contracts using other methods when the index fluctuates up and down often during the term. However, the participation rate may change each year and generally will be lower than that of other indexing methods because of the high cost to synthesize. Also an annual reset design may use a cap or averaging to limit the total amount of interest that might earn each year.

3.2.3 High Water Mark (Lookback)

The index-linked interest, if any, is decided by looking at the index value at various points during the term, usually the annual anniversaries of the date that the contract was bought. The interest is based on the difference between the highest index value and the index value at the start of the term. Interest is added to the contract at the end of the term.

Since interest is calculated using the highest value of the index on a contract anniversary during the term, this design may credit higher interest than some other designs if the index reaches a high point early or in the middle of the term, but drops off at the end of the term. However, interest is not credited until the end of the term. If the investor surrenders the contract before the end of the term, he may not get index-linked interest for that term. Also, contracts with this design may have a lower participation rate than contracts using other designs or may use a cap to limit the total amount of interest that might earn.

3.2.4 Point-to-Point with Barrier

A point-to-point GEB measures the index increment from the start to the end of a contract term. The increment divided by the start index is the index return of the GEBs. The payoff of the GEBs is the greater of either the index return times a participation rate or a minimum guaranteed return. However, the payoff will be the same regardless of the path taken by the index to attain its final value. If investors believe that the underlying index will hit a barrier in a certain period, then they may not want to pay a high premium for the option embedded in a point-to-point GEB and will be reluctant to buy the GEBs. Thus, we propose to synthesize a point-to-point with barrier contracts.

The index-linked interest, if any, is based on the difference between the index value at the end of the term and the index value at the start of the term provided that the index value during the term is higher than a specified level. Interest is added to the contract at the end of the term. It works like a European barrier option.

3.2.5 Annual Reset with Barrier

The returns of annual reset GEBs are locked in periodically, such as annually, throughout the contract term. In each period, the annual reset GEBs provide investors with the greater of either the annual index return times the participation rate or a minimum guaranteed rate. Thus the annual reset GEBs credit better interest to the policy in a very volatile market. A drawback of annual reset GEBs is that the participation rate is relatively low because these GEBs have one embedded option in each policy year. Thus we propose to apply an up-and-in barrier option to this contract to increase the participation rate.

The index-linked interest, if any, is determined each year by comparing the index value at the end of the year with the index value at the start of the year during the term of the contract provided that the index value during the term is higher than a specified level. Interest is added to the contract each year during the term.

3.2.6 Average

The design of point-to-point contract was always being criticized that the rate of return process depends solely on the index level at the end of the term. In the event of a sudden adverse movement in the equity market, this feature can be devastating to investors. To mitigate the risk of depending on the index level on one particular day, an averaging scheme has been proposed. This design is commonly referred as the Asian design since it depends on the daily, weekly, or monthly index levels in the year when the contract mature. Since the averaging design is less volatile than the design that depends only on a single observation, this implies that the value of a GEB with Asian design is cheaper than the corresponding GEB with point-to-point.
Averaging at the beginning of a term protects investors from buying the contract at a high point, which would reduce the amount of interest that might earn. Averaging at the end of the term protects the investors against severe declines in the index and losing index-linked interest as a result. On the other hand, averaging may reduce the amount of index-linked interest to earn when the index rises either near the start or at the end of the term.

The index-linked interest, if any, is decided by looking at the average of the index value at various points during the term, usually the annual anniversaries of the date that the contract was bought.

The interest is based on the difference between the average of index value during the term and the initial index value. Interest is added to the contract at the end of the term.

3.3 Mathematical formulation

3.3.1 Point-to-Point

Let us consider one of the simplest classes of GEBs. The crediting strategy for this family is known as the point-to-point design and, in general, its contingent claim *C*(*t*) [4] in year *t* for one unit of GEB can be represented as followed

$$C(t) = \max(\min(1+\alpha R_t, (1+\zeta)^t), (1+g)^t) \qquad (3.1)$$

The above payoff structure is ideal for many investors. While subject to the maximum cap rate ζ that can be earned under this design, the first random term allows the investors to have a participation rate α in any potential upside gain in the equity market. More importantly, in the event of an adverse market environment, the downside risk is constrained to the minimum guarantee floor component, that is, $(1+g)^t$. The presence of the cap rate, although placing an upper bound on the rate of return of the contract, could reduce the cost of such design substantially. Various variations of point-to-point GEBs exist, depending on how we define the equity return random variable R_t in the equation. As R_t denotes the gain for the reference index over the time interval (0, *t*], without loss of generality, we can define R_t as $R_t = \frac{S(t)}{S(0)} - 1$ where *S*(*t*) is a random variable yet to be determined.

3.3.2 Annual Reset (Ratchet)

Then we focus on one of the most popular type of GEBs known as the annual reset or annual ratchet GEBs. Generically, the payoff [4] in year t for one unit of GEB is given by

$$C(t) = \max(\prod_{s=1}^{t} \max(\min(1+\alpha R_s, 1+\zeta), 1), (1+g)^t) \qquad (3.2)$$

where the random variable R_s again measures the appreciation of the referenced index fund in year s. Variations of annual reset GEBs exist, depending on the precise definition of the random variable R_s. The most common type of design is based on the term-end point. In this case, the random variable R_s is solely determined by the index levels at the beginning and the end of year s through the following structure: $R_s = \frac{S(s)}{S(s-1)} - 1$

Comparing annual reset GEBs to point-to-point GEBs, the former can be more appealing to the investors for two reasons. First, the interest is credited each year for the annual reset GEBs. The credited interest cannot be lost even if the index subsequently goes down. Second, the index level used to determine the appreciation of the index is reset annually. This 'lock in' feature can be extremely valuable, particularly in a more volatile market. For instance, let us consider an annual reset design and also assume that the index drops by 20% in the first year. By the nature of the design, no interest is credited in the first year, but the index appreciation for the second year would be based on the lower level. This implies that, even if the index returns to its initial level in the following year, the crediting rate would become 20% for that year. For these reasons, annual reset GEBs would be more expensive than the corresponding point-to-point GEBs.

3.3.3 High Water Mark (Lookback)

The crediting strategy for the high water mark design and its contingent claim C(t) [5] in year t for one unit of GEB is the same as the point-to point design. It can be represented as that in equation

$$C(t) = \max(\min(1+\alpha R_t, (1+\zeta)^t), (1+g)^t) \qquad (3.3)$$

The only difference is the definition of the equity return random variable R_t in the equation. R_t will be defined as the highest value that the index had during the time interval (0, *t*], thus we can define R_t as $R_t = \frac{\max_{s\in(0,t]} S(s)}{S(0)} - 1$

3.3.4 Point-to-Point with Barrier

Consider barrier options, whose payoffs are the same as those of their underlying plain-vanilla options if the path of an underlying asset satisfies an activating condition, but will be zero otherwise. A barrier option is cheaper than its underlying plain-vanilla option because the payoff of the barrier option is less than or equal to that of the plain-vanilla option. To increase the participation rate, let us now apply an up-and-in barrier option to the point-to-point GEBs. An up-and-in barrier GEB provides purchasers with the greater of the index return times the participation rate and a minimum guaranteed return if the index rises above a barrier for the monitoring period and offers the minimum guaranteed return otherwise.

Let us take a close look at the payoff of the up-and-in barrier GEBs. Assume that the minimum guaranteed return is g for the contract term, the participation rate is α, and the barrier is B. Then the payoff [6] can be expressed as follows:

$$C(t) = \begin{cases} \max\left\{\min\left[1+\alpha R_t, (1+\zeta)^t\right], (1+g)^t\right\} & \text{if } S(s) > B \\ (1+g)^t & \text{else} \end{cases} \qquad (3.4)$$

where R_t defined as $R_t = \frac{S(t)}{S(0)} - 1$

3.3.5 Annual Reset with Barrier

Similarly, we can apply an up-and-in barrier option to the annual reset GEBs [6] to reduce the cost of synthesis of annual reset contract.

Consider an annual reset GEB with up-and-in barriers. In each period, the GEBs will provide customers with the greater of either the annual index return times the participation rate or a minimum guaranteed rate if the index value for the monitoring period rises above a barrier. Otherwise, the GEBs will credit to the policyholder the minimum guaranteed rate as annual return.

$$C(t) = \begin{cases} \max\left\{ \prod_{s=1}^{t} \max\left[\min(1+\alpha R_s, 1+\zeta), 1\right], (1+g)^t \right\} & \text{if } S(s) > B \\ (1+g)^t & \text{else} \end{cases} \tag{3.5}$$

3.3.6 Average

The crediting strategy for average design and its contingent claim C(t) [5] in year t for one unit of GEB is the same as the point-to point design. It can be represented as that in equation

$$C(t) = \max(\min(1+\alpha R_t, (1+\zeta)^t), (1+g)^t) \tag{3.6}$$

The only difference is the definition of the equity return random variable R_t in the equation. R_t will be defined as the average value that the index had during the time interval (0, *t*], thus we can define R_t as $R_t = \frac{\sum_{s=1}^{t} S(s)}{S(0)} - 1$

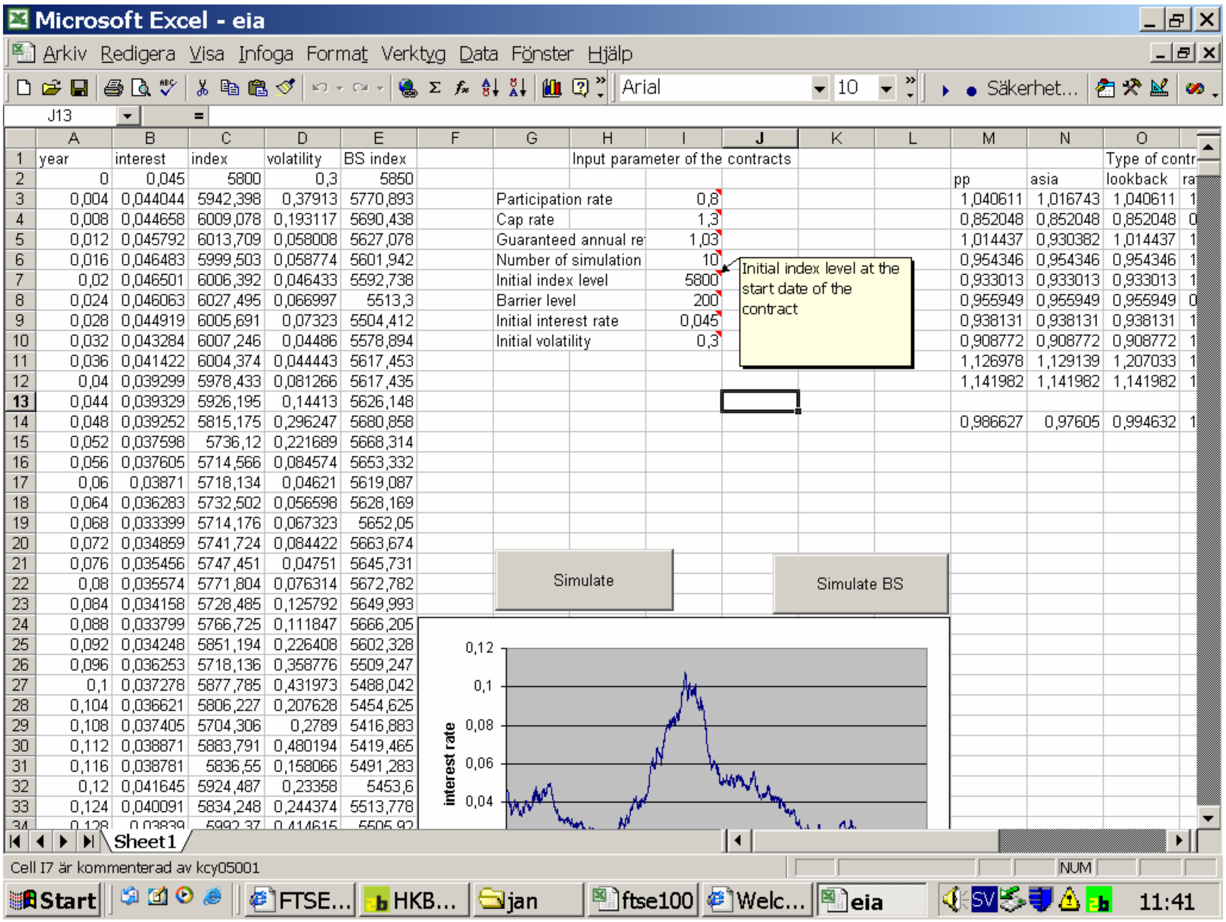
After the introduction of the contract features, we will move to the implementation of the financial models on GEBs in computer simulation.

Chapter 4
Program details

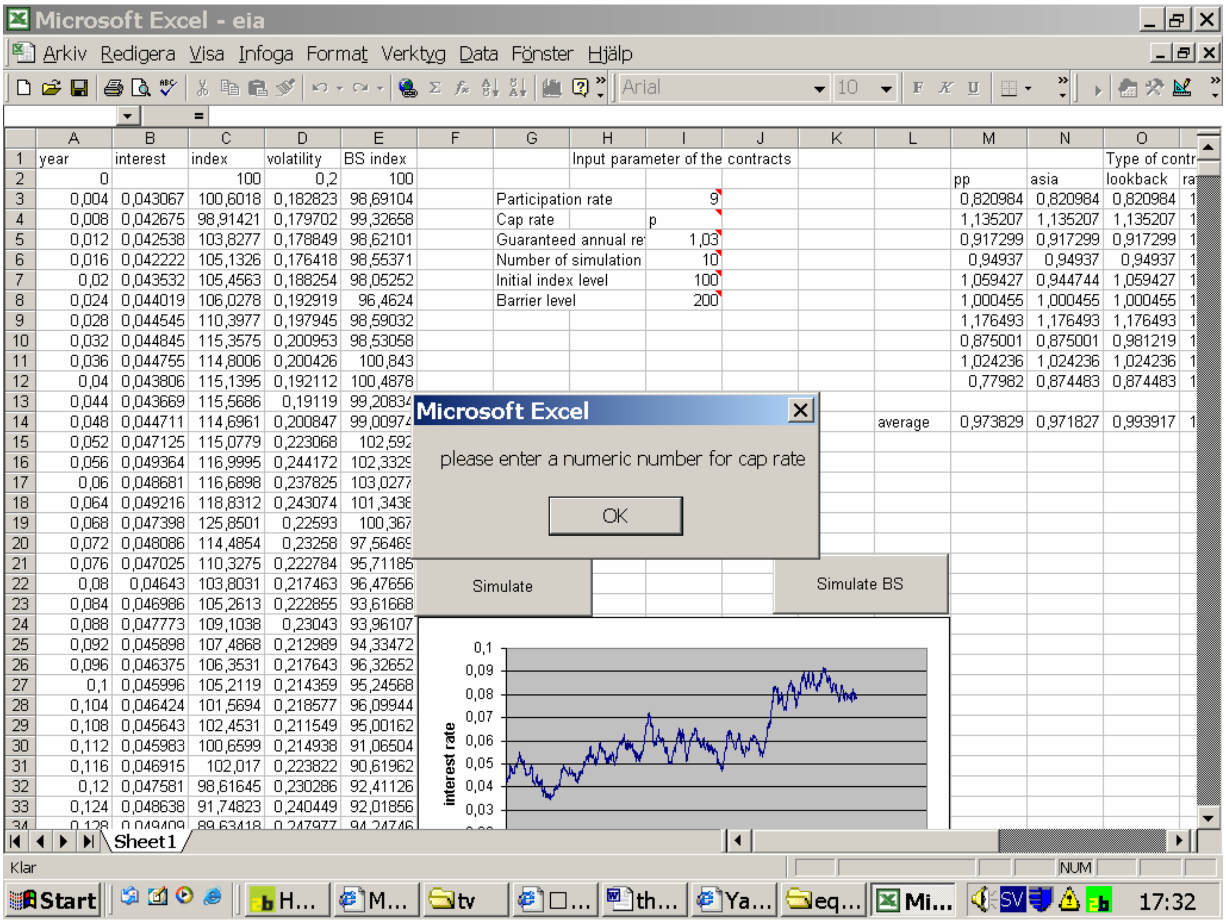
In this chapter, we will describe the program that we used to valuate various kinds of GEBs, and then we will calibrate the parameters in the GARCH(1,1) model to the historical equity index data and the CIR interest rate model to the current term structure.

4.1 Program layout

In order to valuate the fair price of different types of exotic GEBs, we have developed a computer program in Excel to simulate a 5 year long of the equity index, volatility and the interest rate. The program will valuate the present value of exotic contracts by Monte Carlo simulation. The layout of the program is as follow,

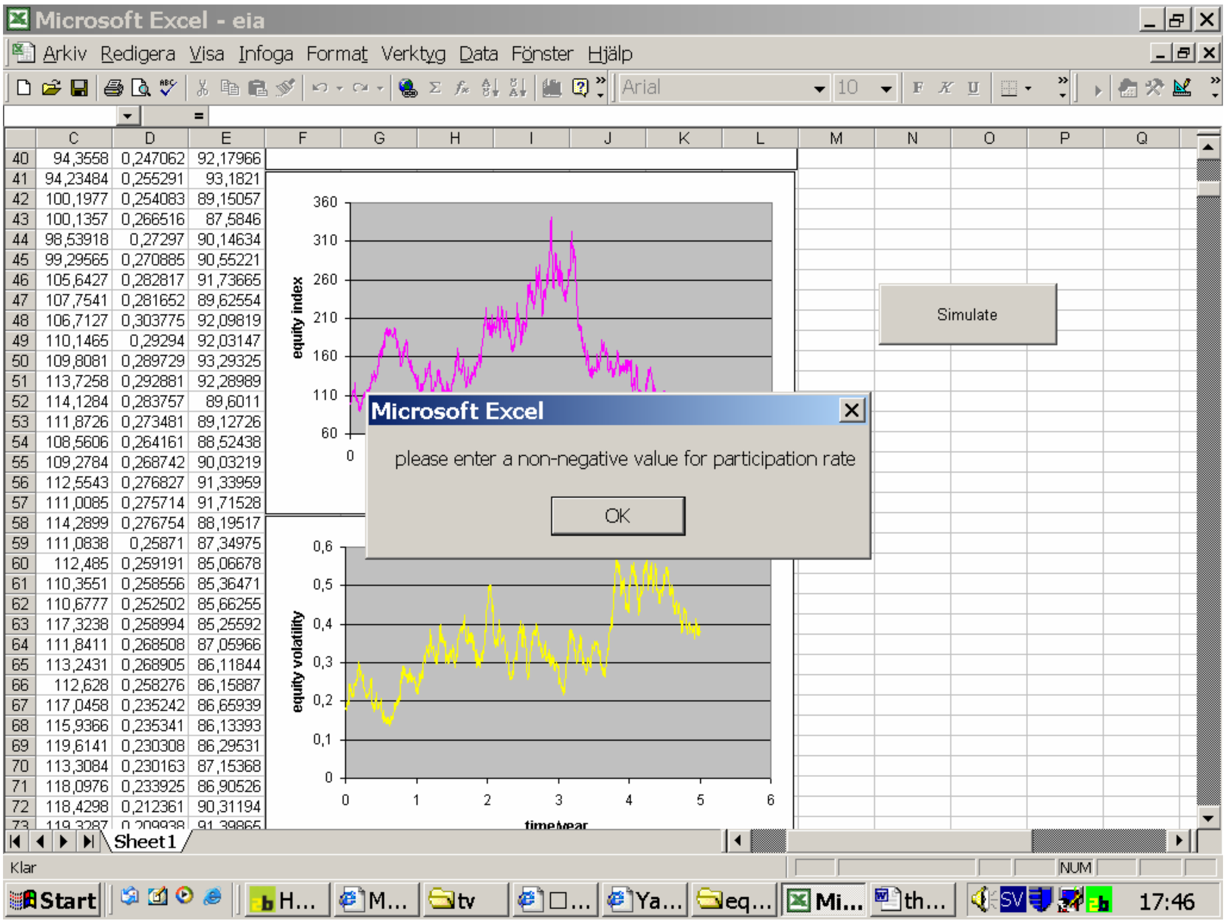


The program includes the input and output section. Users have to specify the details of the contracts before the valuation starts. For example, the participation rate of the contract, cap rate of the contract, guaranteed annual return. The initial index level and the initial volatility and interest rate, barrier level and the number of simulation are also required. Tool tips are present to help the users to input the value of the parameters. Then the program will start simulation after all input is valid, otherwise error message will show to ask for input the parameters of the contract again.



This program can check the validity of the input parameters, for example, users will be asked to input a numeric number of the cap rate if it was a character.

Also, negative value is not allowed for these parameters, so an error box will be present to ask the user to input valid parameters again before the simulation starts. The presence of the parameters checking is important, as it can prevent the ‘crush’ of the program before calculation starts.



The output of the program includes the graph of the 5 year long simulation of the equity index, volatility, interest rate and the fair price of the different types of contracts. They are then calculated once the 'Simulate' button is pressed.

We have also provided the simulation of the equity index under constant interest rate and volatility, and the fair price of the different types of contracts are then calculated once the 'Simulate BS' button is pressed.

4.2 Calibration of model parameters

In order to make our program truly describe the current financial market and valuate the contracts correctly, the value of the parameters used in our simulation model should be well calibrated to the real financial market. That means to calibrate the parameters of the pricing process used in our simulation to the observed financial data. However, we want to stress that model calibration consists of solving a multi-dimensional reverse engineering problem. As discussed by many literatures it is impossible, in general, to determine a set of parameters such that market prices are exactly reproduced by any model. Even from a pure financial point of view this is impossible to achieve. In fact, market imperfections and inefficiencies do not allow to identify option prices exactly (a bid/ask spread is always present). By 'model calibration', we mean that the difference between market and model option prices is within the bid/ask spread and the calibrated solution is statistically robust.

4.2.1 Calibration of CIR model parameters

First, we consider the calibration of the CIR process of the instantaneous interest rate. From equation (2.17),

$$dr_t = \mu(\gamma - r_t)dt + \sigma_r \sqrt{r_t} dW_t^{(3)}$$

Zero bond prices can be computed in the CIR model by solving the partial differential equation. Let $P(t, u)$ be the price at date t of a bond maturing at date u. Thus the bond price will depend on the time to maturity and the short rate at date t. Then there must be some deterministic function f such that $P(t, u) = f(r(t), u - t)$. The PDE is obtained by applying Ito's formula to f to compute df in term of the partial derivatives of f and then using the fact that the expected return of the zero bond must equal to the short rate under the risk-neutral measure.

$$-\frac{\partial f}{\partial \tau} + \frac{\partial f}{\partial r}\mu(\gamma - r_t) + \frac{1}{2}\frac{\partial^2 f}{\partial r^2}\sigma_r{}^2 r = rf \tag{4.1}$$

This equation should be solved for the function f subject to the boundary condition that the value of the bond is one at maturity, $f(r, 0) = 1$ for all r.

We assume that the bond price has the form

$$f(r,\tau) = \exp(-a(\tau) - b(\tau)r) \tag{4.2}$$

for deterministic functions a and b.

Thus we obtain

$$\begin{cases} b(\tau) = \dfrac{2(e^{q\tau} - 1)}{c(\tau)} \\ a(\tau) = -\dfrac{2\mu\gamma}{\sigma_r^2}\left\{\dfrac{(\mu + q)\tau}{2} + \log\dfrac{2q}{c(\tau)}\right\} \\ c(\tau) = (\mu^2 + q)(e^{q\tau} - 1) + 2q \end{cases} \tag{4.3}$$

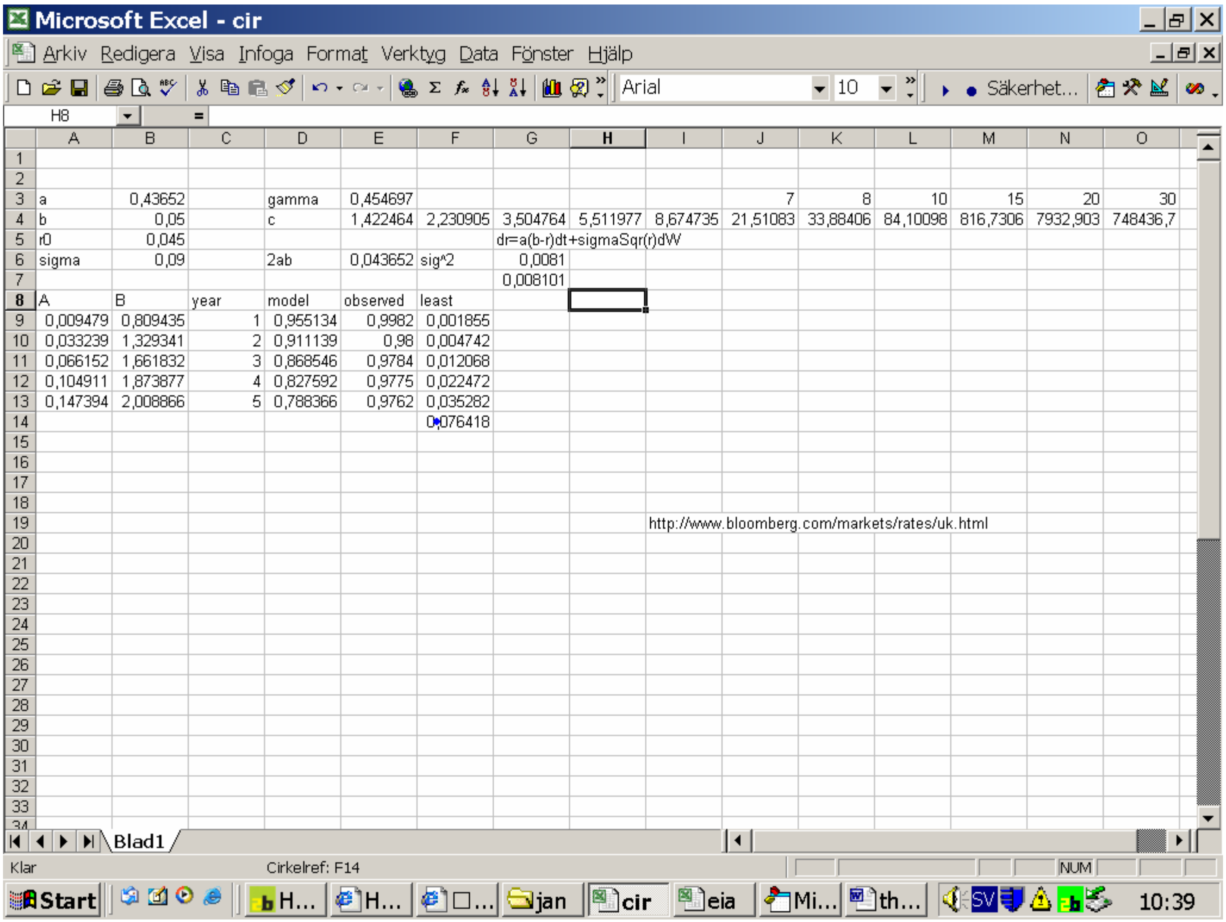
Where

$$q = \sqrt{\mu^2 + 2\sigma_r^2}$$

To calibrate the parameters μ, γ, σ_r, we used the UK government bond as the underlying bonds to our model. It is issued by the Bank of England and supported by the government of the United Kingdom. The bonds market can trace its origins to the creation of the Bank of England and the British national debt in 1694. Since 1998 they have been issued by the UK Debt Management Office (DMO) on behalf of HM Treasury. The UK government has a AAA credit rating from Standard & Poor's and Aaa from Moody's, and all other rating agencies also award UK government debt, whether in sterling or foreign currency, the highest possible rating. So we can assume that the government bond as risk free bonds.

The price is quoted from Bloomberg at 23 June 2006. We used the Excel built-in Solver to minimize the square error between the model price and the observed price for 1, 2, 3, 4, 5 year zero bonds.

year	model	observed	least
1	0,955134	0,9982	0,001855
2	0,911139	0,98	0,004742
3	0,868546	0,9784	0,012068
4	0,827592	0,9775	0,022472
5	0,788366	0,9762	0,035282
			0,076418



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
3	a	0,43652		gamma	0,454697					7	8	10	15	20	30
4	b	0,05		c	1,422464	2,230905	3,504764	5,511977	8,674735	21,51083	33,88406	84,10098	816,7306	7932,903	748436,7
5	r0	0,045					dr=a(b-r)dt+sigmaSqr(r)dW								
6	sigma	0,09		2ab	0,043652	sig^2	0,0081								
7							0,008101								
8	A	B	year	model	observed	least									
9	0,009479	0,809435	1	0,955134	0,9982	0,001855									
10	0,033239	1,329341	2	0,911139	0,98	0,004742									
11	0,066152	1,661832	3	0,868546	0,9784	0,012068									
12	0,104911	1,873877	4	0,827592	0,9775	0,022472									
13	0,147394	2,008866	5	0,788366	0,9762	0,035282									
14						0,076418									
19									http://www.bloomberg.com/markets/rates/uk.html						

Thus we obtained the parameters as follow,

$$\mu = 0.43652$$
$$\gamma = 0.05012$$
$$\sigma_r = 0.09221$$

4.2.2 Calibration of GARCH(1,1) model parameters

Then, we consider the calibration of the GARCH(1,1) process of the volatility of equity index to historical value. We choose FTSE 100 as the underlying index and calibrate our model to the historical value of FTSE 100. The FTSE 100 Index is a capitalization-weighted index of the 100 most highly capitalized companies traded on the London Stock Exchange. The equities use an investibility weighting in the index calculation. The index was developed with a base level of 1000 as of 3 January, 1984. The historical data of FTSE 100 index ranges from 1 January 1990 to 31 December 2005 will be used in our calibration. To estimate the GARCH(1,1) model parameters, we will perform the maximum likelihood estimation (MLE) for the model. Before going further, we would like to give a background of maximum likelihood estimation [7] for a mathematical model first.

In financial mathematics, we seek to uncover general laws and principles that govern the behaviour under investigation. As these laws and principles are not directly observable, they are formulated in terms of models. The goal of modelling is to deduce the form of the underlying process by testing the viability of such models. Once a model is specified with its parameters, and data have been collected, one is in a position to evaluate its goodness of fit, that is, how well it fits the observed data. Goodness of fit is assessed by finding parameter values of a model that best fits the data, a procedure called parameter estimation.

There are two general methods of parameter estimation. They are least-squares estimation (LSE) and maximum likelihood estimation (MLE). The former has been a popular choice of model fitting in many fields and is tied to many familiar statistical concepts such as linear regression, sum of squares error and root mean squared deviation. Also, we have used the LSE to calibrate our CIR interest rate model. Unlike MLE, LSE requires no or minimal distributional assumptions. It is useful for obtaining a descriptive measure for the purpose of summarizing observed data, but it has no basis for testing hypotheses or constructing confidence intervals. On the other hand, MLE is not as widely recognized among modellers in finance, but it is a standard approach to parameter estimation and inference in statistics. In contrast, most statisticians would not view LSE as a general method for parameter estimation, but rather as an approach that is primarily used with linear regression models. Further, many of the inference methods in statistics are developed based on MLE.

In short, MLE determine the parameters that maximize the probability, or likelihood of the sample data. LSE, on the other hand, seek the parameter values that provide the most accurate description of the data, measured in terms of how closely the model fits the data under the square-loss function.

Formally, in LSE, the sum of squares error between observations and predictions is minimized; in MLE, the sum of likelihood function generated by the model is maximized.
To estimate the model parameters, we first have to define the likelihood function of the model, and then maximized the function by optimising the parameters. As the same as previous, we used the Excel built-in Solver to maximize the likelihood function.

First, we take the daily closing index level of FTSE 100 from 1 January 1990 to 31 December 2005. Then, we calculate the daily return and the volatility for each day by equation (2.12). The log-likelihood of each day is calculated by the formula

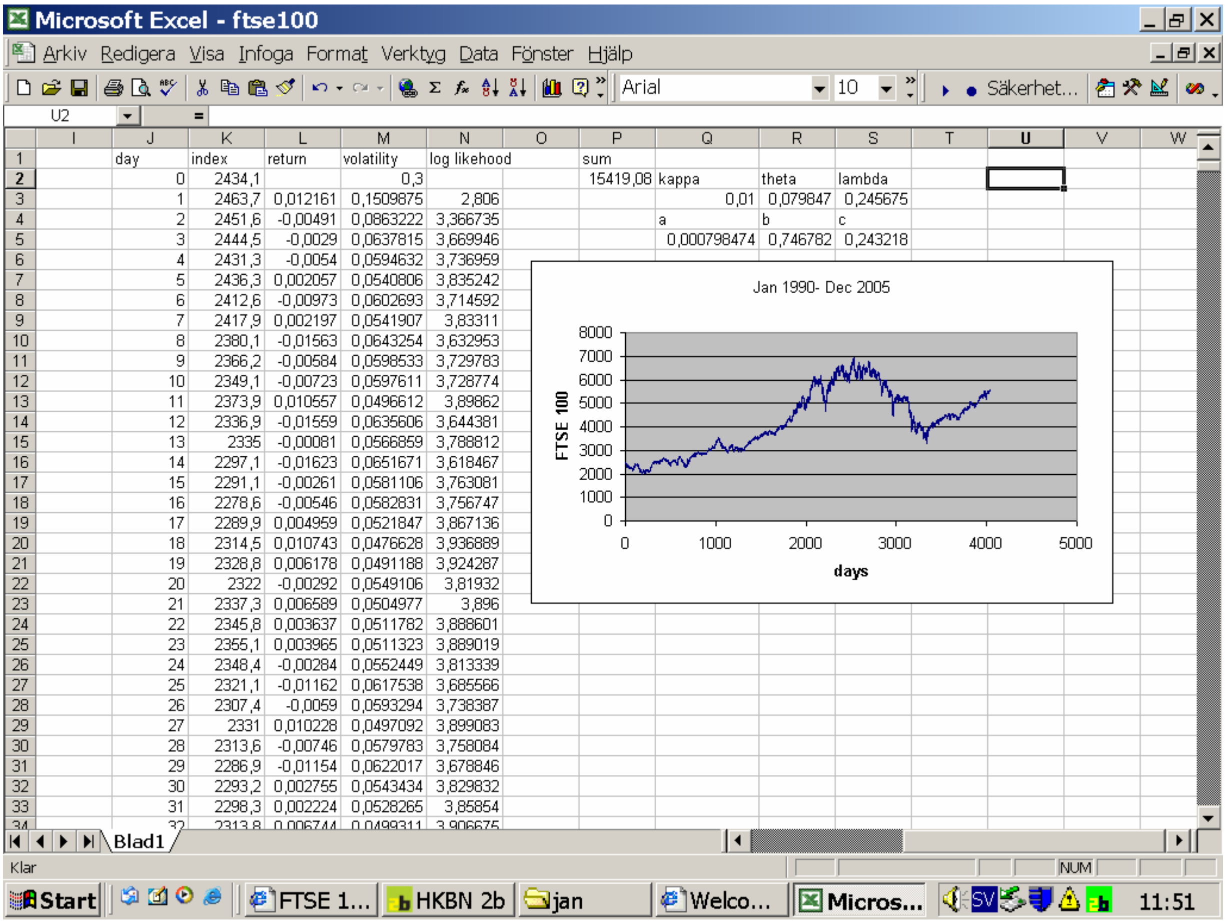
$$-\frac{1}{2}(\log(\sigma^2)+(\frac{\varepsilon}{\sigma})^2) \tag{4.4}$$

then the summation for whole set of log-likelihood is calculated.

Finally, we used the Excel Solver to maximize the likelihood value by varying the models parameters. (kappa, theta, lambda)

Thus we obtained the parameters as follow,

$$\kappa = 0.01041$$
$$\vartheta = 0.07985$$
$$\lambda = 0.24568$$



	J	K	L	M	N	O	P	Q	R	S
1	day	index	return	volatility	log likehood		sum			
2	0	2434,1		0,3			15419,08	kappa	theta	lambda
3	1	2463,7	0,012161	0,1509875	2,806			0,01	0,079847	0,245675
4	2	2451,6	-0,00491	0,0863222	3,366735			a	b	c
5	3	2444,5	-0,0029	0,0637815	3,669946			0,000798474	0,746782	0,243218
6	4	2431,3	-0,0054	0,0594632	3,736959					
7	5	2436,3	0,002057	0,0540806	3,835242					
8	6	2412,6	-0,00973	0,0602693	3,714592					
9	7	2417,9	0,002197	0,0541907	3,83311					
10	8	2380,1	-0,01563	0,0643254	3,632953					
11	9	2366,2	-0,00584	0,0598533	3,729783					
12	10	2349,1	-0,00723	0,0597611	3,728774					
13	11	2373,9	0,010557	0,0496612	3,89862					
14	12	2336,9	-0,01559	0,0635606	3,644381					
15	13	2335	-0,00081	0,0566859	3,788812					
16	14	2297,1	-0,01623	0,0651671	3,618467					
17	15	2291,1	-0,00261	0,0581106	3,763081					
18	16	2278,6	-0,00546	0,0582831	3,756747					
19	17	2289,9	0,004959	0,0521847	3,867136					
20	18	2314,5	0,010743	0,0476628	3,936889					
21	19	2328,8	0,006178	0,0491188	3,924287					
22	20	2322	-0,00292	0,0549106	3,81932					
23	21	2337,3	0,006589	0,0504977	3,896					
24	22	2345,8	0,003637	0,0511782	3,888601					
25	23	2355,1	0,003965	0,0511323	3,889019					
26	24	2348,4	-0,00284	0,0552449	3,813339					
27	25	2321,1	-0,01162	0,0617538	3,685566					
28	26	2307,4	-0,0059	0,0593294	3,738387					
29	27	2331	0,010228	0,0497092	3,899083					
30	28	2313,6	-0,00746	0,0579783	3,758084					
31	29	2286,9	-0,01154	0,0622017	3,678846					
32	30	2293,2	0,002755	0,0543434	3,829832					
33	31	2298,3	0,002224	0,0528265	3,85854					
34	32	2313,8	0,006744	0,0499311	3,906675					

4.3 Program algorithm

After the calibration of the model parameters is done, the program is ready to run. The program contains two parts. First, it will simulate a 5 years long equity index, interest rate and the volatility of the index according to the equations stated in Chapter 2.3. Then, various exotic contracts will be priced according to the payoff structure stated in Chapter 3.3. Here, we will describe the implementation of the equations to programming in details.

First, we convert the differential equation (2.11)

$$dS_t = r_t S_t dt + \sigma_t S_t dW_t^{(1)}$$

to difference equation

$$\begin{cases} \Delta s = rs\Delta t + \sigma s z_1 \sqrt{\Delta t} \\ s = s + \Delta s \end{cases} \tag{4.5}$$

Then convert the differential equation (2.12)

$$\sigma_{t+1}^{2} = \kappa\vartheta + (1-\kappa)((1-\lambda)y_t^2 + \lambda\sigma_t^{2})$$

$$y_t = \frac{\log\left(S(t)/S(t-1)\right) - (r_t - \frac{1}{2}\sigma_t^{2})\Delta t}{\sqrt{\Delta t}}$$

to difference equation

$$\sigma = \sqrt{\kappa\vartheta + (1-\kappa)(1-\lambda)\left(\frac{\left\{\log(s_i / s_{i-1}) - (r - \frac{1}{2}\sigma^2)\Delta t\right\}}{\sqrt{\Delta t}}\right)^2 + (1-\kappa)\lambda\sigma^2} \quad (4.6)$$

Similarly, we convert the differential equation (2.17)

$$dr_t = \mu(\gamma - r_t)dt + \sigma_r\sqrt{r_t}\,dW_t^{(2)}$$

to difference equation

$$\begin{cases} dr = 0.43(0.05 - r)\Delta t + 0.09\sqrt{r}z_2\sqrt{\Delta t} \\ r = r + \Delta r \end{cases} \quad (4.7)$$

We assumed that there is 250 trading days per year and Δt = 5/1250 = 0.004. After one 5 year long simulation, the payoff of various kinds of contracts is calculated from the value of the simulated index according to the equation (3.1) to (3.6), with a discounted factor to present value. We run the simulation thousands times, the value of different kinds of GEBs are obtained as the mean of the present value. We have also provided the simulation of the equity index under constant interest rate and volatility. We used the observed initial interest rate, 4.5%, and the current observed volatility, 15% in equation (2.11). Similarly, we run the simulation thousands times, the value of different kinds of GEBs are also obtained as the mean of the present value.

In our thesis, we will focus on six types of exotic GEB contracts. We will study in depth how the combination of different features in the contracts affects the payoff. The results will be discussed in chapter 5.

Chapter 5
Analysis of the contracts with different parameters

In this chapter, we will verify the stability of Monte Carlo simulation of our program and valuate different kinds of contracts with varying parameters under stochastic interest rate and volatility model.

5.1 Stability of the contracts' price by Monte Carlo simulation

Monte Carlo simulation is a robust but time-consuming technique to valuate derivatives, like the path-dependent options. It simulates a lot of scenarios to describe the real picture in the future and is well known that a large number of simulations are required to obtain the stable value. In our thesis, we will use the Monte Carlo simulation to valuate the exotic GEB contracts. However, we should first know the required number of simulation to obtain a stable value. Thus we performed 10, 100, 200, 500 simulation 10 times and calculate the standard error between them.

The following is the parameters used in our valuation of the contracts.

Participation rate	0.8
Cap rate	1.3
Guaranteed annual return	1.03
Initial index level	5800
Barrier level	9000
Initial interest rate	0.045
Initial volatility	0.15

For the point-to-point contracts,

No. of simulation	10	100	200	500
1	0.995799	0.934222	0.948654	0.950524
2	0.962222	0.957062	0.956616	0.950536
3	0.965238	0.956171	0.948651	0.950524
4	0.937908	0.942077	0.956617	0.950537
5	0.983359	0.963088	0.948661	0.950554
6	0.935145	0.934221	0.956612	0.950573
7	1.003786	0.957062	0.948681	0.950561
8	0.970149	0.956171	0.956611	0.950551
9	0.945828	0.942077	0.948651	0.950532
10	0.931449	0.963088	0.956611	0.950544
Standard Error	**0.025618**	**0.011251**	**0.004192**	**1.61E-05**

For the annual reset (ratchet) contracts,

No. of simulation	10	100	200	500
1	1.116632	1.110981	1.123596	1.123542
2	1.190108	1.138361	1.135706	1.123553
3	1.081955	1.133052	1.123211	1.123542
4	1.115715	1.099107	1.135911	1.123561
5	1.154436	1.136211	1.123212	1.123578
6	1.091689	1.110982	1.135611	1.123589
7	1.140691	1.138362	1.123514	1.123571
8	1.125437	1,133052	1.135719	1.123558
9	1.156166	1,099107	1.123111	1.123539
10	1.189285	1,136211	1.135718	1.123551
Standard Error	**0.036995**	**0.016502**	**0.006552**	**1.66E-05**

For the high water mark (lookback) contracts,

No. of simulation	10	100	200	500
1	1.002946	0.953101	0.965434	0.968227
2	0.965949	0.977361	0.975443	0.968246
3	0.977566	0.973526	0.965481	0.968234
4	0.947831	0.959381	0.975431	0.968245
5	1.001422	0.977767	0.965491	0.968262
6	0.950491	0.953101	0.975412	0.968281
7	1.042975	0.977361	0.965481	0.968269
8	0.994508	0.973526	0.975442	0.968252
9	0.959107	0.959381	0.965431	0.968233
10	0.934874	0.977767	0.975461	0.968245
Standard Error	**0.032819**	**0.010641**	**0.005257**	**1.7E-05**

For the point-to-point with barrier contracts,

No. of simulation	10	100	200	500
1	0.959806	0.909085	0.920037	0.921719
2	0.919714	0.926676	0.926352	0.921735
3	0.914246	0.926027	0.920112	0.921724
4	0.903633	0.915821	0.926331	0.921743
5	0.937909	0.930989	0.920081	0.921761
6	0.916066	0.909085	0.926912	0.921777
7	0.990849	0.926676	0.920071	0.921761
8	0.939609	0.926027	0.926370	0.921746
9	0.926331	0.915821	0.920031	0.921727
10	0.901727	0.930989	0.926354	0.921738
Standard Error	**0.027472**	**0.008478**	**0.003376**	**1.83E-05**

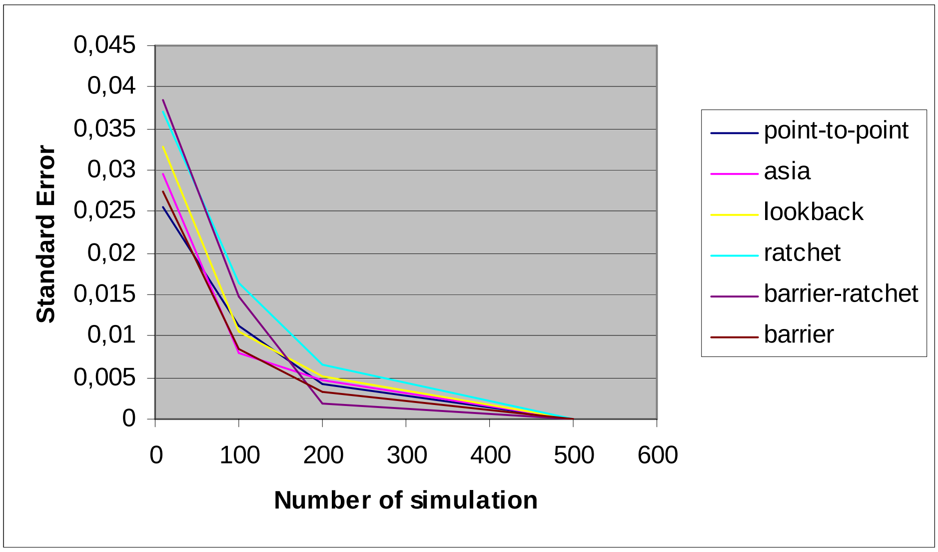
For the annual reset with barrier contracts,

No. of simulation	10	100	200	500
1	0,959806	0.957486	0.974253	0.971359
2	1.006631	0.977242	0.977504	0.971374
3	0.940512	0.977767	0.974291	0.971363
4	0.977953	0.953282	0.977531	0.971381
5	0.974902	0.991021	0.974258	0.971397
6	0.972035	0.957486	0.977502	0.971416
7	1.055938	0.977242	0.974257	0.971404
8	0.965056	0.977767	0.977502	0.971394
9	1.006652	0.953282	0.974271	0.971375
10	1.050735	0.991021	0.973411	0.971387
Standard Error	**0.038384**	**0.014768**	**0.001769**	**1.81E-05**

For the average contracts,

No. of simulation	10	100	200	500
1	0.967461	0.910365	0.919638	0.922641
2	0.905662	0.928269	0.928264	0.922655
3	0.925637	0.928259	0.919711	0.922644
4	0.887981	0.917398	0.928231	0.922663
5	0.938855	0.928912	0.919641	0.922681
6	0.917437	0.910365	0.928712	0.922699
7	0.980193	0.928269	0.919638	0.922687
8	0.949711	0.928259	0.928264	0.922677
9	0.912312	0.917398	0.919611	0.922658
10	0.903885	0.928912	0.928291	0.922671
Standard Error	**0.029642**	**0.007899**	**0.004589**	**1.87E-05**

Here we show the graph of standard error versus number of simulation, we can see that the standard error is strictly decreasing as the number of simulation increase. In other words, the price does converge to a stable value if there is enough number of simulations.



The standard error of 100 simulations for six types of contracts is about 0.01, which is surprisingly small. It is acceptable in our calculation. Thus, in order to save time, the number of simulation that we will use to study the features of the contracts in the following section will be 100.

5.2 Valuation of the contracts with different parameters

According to our simulation results, it is obviously that the fair value of some of the GEBs is less than 1. In other words, contract's holders pay an amount 1 dollar today to buy something, which is less than 1. This gives another source of profit to the company.

From the previous table, we can see that the price of the average type contracts is the cheapest, then the point-to-point one, and the high water mark type. The most expensive one is the ratchet contracts. This agrees with the results from other literatures. Since the averaging design is less volatile than the design that depends only on a single observation, this implies that the value of a GEB with average design is cheaper than the corresponding GEBs with point-to-point. For the high water mark design, the highest realized index level over the term of the contract is used to calculate the return, thus the high water mark design is more expensive than the corresponding GEBs with point-to-point design, which, in turn, is more costly than the GEBs with average design. For the ratchet design, the interest is credited each year. The credited interest cannot be lost even if the index subsequently goes down. Second, the index level used to determine the appreciation of the index is reset annually. This "lock in" feature can be extremely valuable, so the annual reset GEBs would be the most expensive one. When we include an up-and-in barrier option to the point-to-point and ratchet contracts, the cost to synthesize is cheaper than the original one. It agrees with the fact that barrier option is cheaper than the corresponding vanilla one. Investors can pay a lower cost to buy these contracts if they believe the future index market is bullish.

Cap Rate

Some contracts may put a cap, on the index-linked interest rate. This is the maximum rate of interest the contract will earn. Not all contracts have a cap rate. The presence of the cap reduces the cost to synthesize the contracts. We vary the cap rate of each contract to see the effect of cap to the contracts.

The following is the parameters used in our valuation of the contracts

Participation rate	0.8
Guaranteed annual return	1.03
Number of simulation	100
Initial index level	5800
Barrier level	9000
Initial interest rate	0.045
Initial volatility	0.15

Cap Rate (%)	Point-to-Point	Ratchet	Lookback	Point-to-Point with barrier	Ratchet with barrier	Average
No Cap	1.015731	1.145941	1.053112	0.976734	1.000487	0.935367
50	0.963433	1.116318	0.993219	0.929419	0.962648	0.914589
40	0.975355	1.140566	1.004696	0.938573	0.978622	0.934108
30	0.956171	1.133052	0.973526	0.926027	0.977767	0.928259
20	0.928098	1.123112	0.933465	0.917057	0.981053	0.919391
10	0.911383	1.075609	0.911383	0.911383	0.956211	0.911383

In general, we can see that the decreasing cap rate will lower the price of the contracts. The price will be the highest if no cap is on the contracts. We will have the same price for point-to-point, lookback, and average contracts if the cap rate is 10%. That means that the interest earned in the contracts is much more than 10% and capped to 10%. For the average design, the price does not change so much when the cap rate decrease. Thus we concluded that the interest earned in the average contracts is lower than 10%. For the ratchet design, the price also does not change so much when the cap rate decrease until 10%. Thus we concluded that the annual interest earned in the ratchet contracts is more than 10%.

Participation Rate

The participation rate decides how much of the increase in the index will be used to calculate index-linked interest. We vary the participation rate of each contract to see the effect of participation rate to the contracts.
The following is the parameters used in our valuation of the contracts

Cap rate	1.3
Guaranteed annual return	1.03
Number of simulation	100
Initial index level	5800
Barrier level	9000
Initial interest rate	0.045
Initial volatility	0.15

Participation Rate (%)	Point-to-Point	Ratchet	Lookback	Point-to-Point with barrier	Ratchet with barrier	Average
150	0.956431	1.298844	0.984182	0.909085	1.000207	0.934207
130	0.973574	1.288121	0.997821	0.926676	1.018307	0.947832
110	0.968545	1.217861	0.985055	0.926027	1.004369	0.943764
90	0.946537	1.126441	0.966122	0.915821	0.960818	0.921269
70	0.957623	1.107188	0.970936	0.930989	0.981477	0.924691
50	0.918295	1.026742	0.926781	0.909042	0.933262	0.898979

In general, we can see that the change of participation rate does not place great effect to the point-to-point, lookback, and average contracts until 50%. We will have the lowest price for all the contracts if the participation rate is 50%. That means that, in general, the change of participation rate in the contracts is not significant to the change of price if it is not too low, less than 50%.

Annual Guaranteed Interest

Some contracts pay guarantee interest during an index term if the index-linked return is less than the guarantee return or negative. We assumed that the contract pays a simple interest during a term.

The following is the parameters used in our valuation of the contracts

Participation rate	0.8
Cap rate	1.3
Number of simulation	100
Initial index level	5800
Barrier level	9000
Initial interest rate	0.045
Initial volatility	0.15

Annual Guarantee Return Rate	Point-to-Point	Ratchet	Lookback	Point-to-Point with barrier	Ratchet with barrier	Average

(%)						
6	1.050396	1.310486	1.050396	1.050396	1.126012	1.050396
5	1.006993	1.244254	1.009694	1.003458	1.073681	1.001639
4	0.966467	1.148995	0.975274	0.953988	0.998599	0.952589
3	0.963088	1.136211	0.977767	0.930989	0.991021	0.928912
2	0.914577	1.065549	0.944937	0.872169	0.914411	0.881798
1	0.927131	1.053979	0.965791	0.854578	0.893834	0.883096
No	0.916407	1.025659	0.951387	0.820625	0.859302	0.872415

In general, we can see that the increasing guarantee interest will increase the price of the contracts. The price will be the highest if the guarantee interest is 6% on all the contracts, while it will be the lowest if there is no guarantee interest on all the contracts. We have the same price for point-to-point, lookback, and average contracts if the guarantee interest is 6%. That means that the interest earned in the contracts is the guarantee interest 6% and the index-linked interest is lower than 6%. Thus we concluded that the higher the annual guarantee interest, the more favourable to the contract's holder.

Comparing contract's price under Black-Scholes model with constant and stochastic interest rate and volatility

In our program, we have also provided the simulation of the equity index under constant interest rate and volatility. We used the observed initial interest rate, 4.5%, and the current observed volatility, 15% in equation (2.11). As usual, we run the simulation hundreds times, the value of different kinds of GEBs are obtained as the mean of the present value.

The following is the parameters used in our valuation of the contracts

Participation rate	0.8
Cap rate	1.3
Guaranteed annual return	1.03
Number of simulation	100
Initial index level	5800
Barrier level	9000
Initial interest rate	0.045
Initial volatility	0.15

Valuation Method	Point-to-Point	Ratchet	Lookback	Point-to-Point with barrier	Ratchet with barrier	Average
Stochastic	0.934221	1.110981	0.953101	0.909085	0.957486	0.910365
Constant	0.978212	1.214037	0.986011	0.950421	1.052912	0.956329

We can see that the price of different types of contracts will be over-priced under constant interest rate and volatility model. That means that the volatility used, 15%, in the constant model is higher than that in the stochastic model, while the interest rate used, 4.5%, in the constant model is lower than that in the stochastic model. This agrees with the mean-reverting features of the GARCH (1,1) model of volatility and CIR interest rate model that the initial value will revert to the mean value in the future.

After we have the ideas of how the basic features of the GEBs affect the payoff of various kinds of contracts, we are ready to analyze the profitability of the current traded contracts in the market.

Chapter 6
Analysis of the profitability to the company and investor of the current traded contracts in the market

In this chapter, we will analyze the profitability of the current traded GEBs in the market with our valuation program. We will take three currently traded GEBs to illustrate our analysis.

6.1 Abbey's Capital Guaranteed UK Equity Bond

The Abbey Capital Guaranteed UK Equity Bond is offered by the Abbey Company. Its return linked to the performance of the FTSE 100 Index and guarantee initial capital. It is a five-year plan linked to the FTSE 100. Investors receive 130% of the growth in the FTSE, although this is capped at 50% of the initial investment. A minimum deposit of £3,000 is required for five years. At the end of this fixed term, investors are guaranteed to get the money back, plus a return in interest of up to 50% of the investment. If the overall performance of the Index is negative or zero, investors will not receive a return on the investment. Here are the extracts from Abbey Company.

What will I get back?

Provided you leave your money invested for the full term, you'll get back your original deposit plus interest. Interest will depend on how the FTSE 100 Index has performed over the five years, averaged over the final year and capped at a maximum of 50% interest, as explained below. The FTSE 100 Index can go down as well as up and there's no guarantee that it will behave the way it has done in the past. For illustrative purposes, the table to the right shows some examples of what you might get back at the end of the term in circumstances where the Index has gone up, stayed the same and gone down.

How is growth calculated?

We take the percentage difference between the Initial Index Level and the Final Index Level to work out the percentage change since the start date. We then multiply the answer by 130%. If the result is 50% or more, your return will be 50%, the maximum interest you can earn with this Bond. The Initial Index Level used is the closing level of the FTSE 100 Index on the start date (14th July 2006). The Final Index Level is the average of the daily closing levels of the FTSE 100 Index during the 12 months prior to the maturity date (18th July 2011). Averaging will have an effect on the return you get at maturity. If the Index rises in the final year, then averaging is likely to reduce the return. On the other hand, if the Index falls in the final year, the return is likely to be greater than if averaging didn't apply.

Why is there a limit on the maximum interest I can earn with the Bond?

By putting a limit on the maximum interest that we pay you, we can offer you a higher return if the growth in the Index, as calculated above, is only moderate. For example, if the growth is calculated as 25%, you'll get 32.5%. If it's 38.46%, you'll get the maximum interest of 50%.

Please note that there is no guarantee that the FTSE 100 Index will rise and past performance must not be seen as an indication of future performance.

Based on an original investment of £20,000	After 5 years	
Change from Initial Index Level to Final Index Level	Investment return	Total repayment at maturity, including capital repayment
+55%	+50%	£30,000
+40%	+50%	£30,000
+25%	+32.5%	£26,500
0%	0%	£20,000
-25%	0%	£20,000
-55%	0%	£20,000

The above is a guide and is for reference only. It is not intended to portray the exact return you may receive and does not take into account Early Bird Interest, deductions of income tax and the effects of inflation.

If the Index stays the same or falls, you're guaranteed to get your capital back, but remember that the effect of inflation will mean that it's worth less in real terms than it was when you originally put your money in.

Investing in the Bond can be done tax efficiently but, depending on how you've chosen to invest in it, there may be some tax to pay (for example if you have made a Direct Investment).

It's very important to remember that the above figures are just examples and they assume that you don't cash in your Bond early – if you do you may get back less than you originally invested, as we explained on pages 4 and 5.

Thus we input the following features to the program to valuate this contract:
Underlying index: FTSE 100
Bond term: 5 years
Participation rate: 130%
Annual guaranteed rate: 0%
Indexing method: averaged over the final year
Cap rate: 50%
Number of simulation: 200

The price of the contract is 0.992154

As the minimum deposit to the contract is £3,000. We calculate the profit to the company,

£3000(1-0.992154) ~ £23.5 ~ 300 SEK

So we can see that although GEBs carry no explicit charges – you basically pay £3,000 to enter the contract, the company still can make a big profit if they can sell thousands of contracts to the clients. As the present value of the future payoff is close to 1, we suggest that this contract is worth to consider when the bank interest is low.

6.2 Barclays' Five-year Protected FTSE Plan

The Barclays' Five-year Protected FTSE Plan is offered by the Barclay Company. Its return linked to the performance of the FTSE 100 Index and guarantee initial capital. It is a five-year plan linked to the FTSE 100. Investors receive 106% of the growth in the FTSE. A minimum deposit of £4,000 is required for five years. At the end of this fixed term, investors are guaranteed to get the money back, plus a return of the investment. If the overall performance of the Index is negative or zero, investors will not receive a return on the investment. Here are the extracts from Barclay Company.

How does the Plan work?

If you decide to invest in the Plan, your money, inclusive of any interest, would be held for a five-year term. Once the five-year term has ended, we will calculate your investment return.

How is the investment return calculated?

Your return will be linked to the performance of the FTSE 100 Index. If you hold your investment for the full five-year term it is designed to give you:

- Your money back plus
- 106% of the rise, if any, in the FTSE 100 Index with the final level of the index averaged over the last year

The Plan is designed to be held for the full five-year term. On 14 July 2011, the Plan will mature, at which time you will receive back your initial investment. In addition, we will work out the difference between the starting level the 'Initial Index Level') and the closing level (the 'Final Index Level') of the FTSE 100 Index. The Final Index Level is calculated using a method called final-year averaging. This means that we record the level of the FTSE 100 Index at the close of trading on the 14th day of each month (or if that day is not a business day, the next business day) over the final year of the investment period. These monthly points are then averaged to produce the Final Index Level. We then calculate the percentage difference between the Initial Index Level and the Final Index Level and multiply the percentage rise amount by 106%.

- If this results in a negative figure (i.e. if there happens to have been a fall in the FTSE 100 Index), you will still get your money back.
- If this results in a positive figure, we will calculate what percentage of your initial investment this represents. This will be added to your initial investment and returned to you.

The effects of averaging

We use averaging to calculate the Final Index Level during final year of the Plan. While this averaging can reduce the potential for investment gain when the Index rises in the final year, it can also lessen the effects of falls in the Index in the final year. The table below will help you understand how the returns are calculated at the end of the five-year period.

Amount invested	Rise or fall in the FTSE 100 Index (averaged)	What you could receive back after five years		
		Your Investment money back plus return		Total payment
£10,000	+40%	100%	42.4%	£14,240
£10,000	+30%	100%	31.8%	£13,180
£10,000	+20%	100%	21.2%	£12,120
£10,000	+10%	100%	10.6%	£11,060
£10,000	0% or negative	100%	0%	£10,000

Please note that the above figures are examples only and that changes in the FTSE 100 Index level have been chosen to demonstrate the potential returns under the Plan only. If you sell your investment before the end of the Plan, you might not receive back the full amount you originally invested.

Offer period	1 May 2006 to 30 June 2006.
Offer period for ISA and/or PEP transfers	1 May 2006 to 16 June 2006.
Start Date	14 July 2006.
Maturity Date	14 July 2011.
Interest End Date	7 July 2006.
Interest Rate	4.5% gross per annum.
Valuation Dates	The 10th business day and last business day in each month.
Index	The FTSE 100 Index.
Initial Index Level	The level of the Index as at close of business on 14 July 2006.
Final Index Level	The average of the closing levels of the Index calculated over the final year of the five-year term on successive monthly occasions, on the 14th of each month. Where the 14th day of the month is a day on which the Index is not published, the closing level on the next Business Day on which the Index is published will be used. The first date of the averaging period is 14 July 2010 and the final date is 14 July 2011. This averaging process will tend to restrict the potential for growth.
Interim Index Level	The closing level of the Index on 14 January 2009.
Investment Objective	To receive your capital back at the end of the Term of the Plan, plus an amount (rounded down to two decimal places) that is equal to 106% of the percentage amount (if any) by which the Final Index Level exceeds the Initial Index Level.
Investment Options	Direct investment: minimum of £4,000 up to £500,000 ISA investment: Mini ISA £4,000 Maxi ISA £4,000 to £7,000 ISA and/or PEP transfers: minimum of £4,000 up to £500,000 Investments in excess of £500,000 are accepted at the discretion of Woolwich Plan Managers.
Statements	These will be prepared as at 30 June and 31 December each year and will normally be sent out in July and January respectively.

Thus we input the following features to the program to valuate this contract:

- Underlying index: FTSE 100
- Bond term: 5 years
- Participation rate: 106%
- Annual guaranteed rate: 0%
- Indexing method: averaged over the final year
- Cap rate: No
- Number of simulation: 200

The price of the contract is 0.994019

As the minimum deposit to the contract is £4,000. We calculate the profit to the company,

£4000(1-0.994019) ~ £23.9 ~ 300 SEK

So we can see that it is similar to the previous one. Although the present value of the future payoff is higher than the previous one, the deposit to enter the contract is higher. Thus the company can still make a profit as the previous one if they can sell thousands of contracts to the clients.

6.3 NS&I GEB (issue 11)

The NS&I GEB (issue 11) is offered by the NS&I company. Its return linked to the performance of the FTSE 100 Index and guarantee initial capital. It is a five-year plan linked to the FTSE 100. Investors receive 112% of the growth in the FTSE. A minimum deposit of £1,000 is required for five years. At the end of this fixed term, investors are guaranteed to get the money back, plus a return of the investment. If the overall performance of the Index is negative or zero, investors will not receive a return on the investment. Here is the extracts from NS&I company.

Market-leading offer
This new GEB offers a gross return equivalent to 112% of any growth in the FTSE 100 index over the five-year term, with no upper limit or cap on the return. The 112% figure means the investment will earn a return equivalent to all of the growth in the FTSE 100 index over the five-year term, plus an extra 12% of FTSE growth on top.

For example, if the FTSE 100 index rose by 20% over the five-year term, £10,000 invested would earn a gross return of £2,240 (customer receives £12,240) at the end of the five years – i.e. 22.4%. If the FTSE rose by 30% over five years, £10,000 would earn £3,360 (customer receives £13,360, i.e. 33.6%).

If the FTSE 100 index falls or fails to rise over the term, investors' initial investment is returned in full.

The 112% share in the FTSE 100 index makes it one of the most competitive guaranteed equity bonds of this type on the market at the moment, and the only one on the market with a sovereign guarantee.

Provider	NS&I	Bank of Ireland – Isle of Man	Bristol & West	Britannia Building Society
Name	Guaranteed Equity Bond	Guaranteed FTSE Bond 4-5 yr	Guaranteed FTSE Bond 5yr	Guaranteed Capital Bond 5yr
Participation rate	112%	105%	105%	110%
Minimum return	100%	100%	100%	100%
Term	5 years	5 years	5 years	5 years
Minimum investment	£1,000	£5,000	£2,500	£500
Averaging	Five days initial averaging and six months final daily averaging	Monthly readings over last year of investment	Monthly readings over last year of investment	Monthly readings over last year of investment
Charges	No	No	No	Included in terms

Source: Structured Retail Products (correct at 24 April 2006)

Trevor Bayley, National Savings and Investments chief executive, said: "This new Guaranteed Equity Bond, and our first for a year, is a great opportunity for any investor looking for a cash deposit where the returns are linked to the performance of the FTSE but will also beat that performance by a further 12%, as long as the FTSE rises over the five years.

"On top of that, all capital is guaranteed to be 100% secure, even if the FTSE falls, and you can invest from the comfort of your own home, either online or by phone. It couldn't be simpler."

Thus we input the following features to the program to valuate this contract:

- Underlying index: FTSE 100
- Bond term: 5 years
- Participation rate: 112%
- Annual guaranteed rate: 0%
- Indexing method: Five days initial averaging and six months final daily averaging
- Cap rate: No
- Number of simulation: 200

The price of the contract is 0.999038

As the minimum deposit to the contract is £1,000. We calculate the profit to the company,

£1000(1-0.999038) ~ £0.9 ~ 13 SEK

So we can see that it is the best one to the investors among all the GEBs. However, the company have to sell hundreds of thousands of contracts to maintain a profit to the company itself. As this company, NS&I, is less famous and smaller than the previous two company, Abbey and Barclay, it should provide some contracts with high competitive in the market. Thus the present value of this contract is much higher than previous two. This is their strategy to survive between the big companies and expand their market. However, at the same time, investors have to bear some default risk to buy the contract from less famous company.

6.4 Hints to investors and company

After going through the three current traded contracts in the market, we have a general idea to deal with the GEBs. Average design is the most common type in the market and investors should bear in mind that averaging can work to be an advantage if the index plummets in the last few months of the term. However, investors will lose if the market has climbed steadily but the final level is an average over the last 12 months. Also, it seems that the annual guaranteed interest is not common in the UK market. Investors should be aware that the higher the participation rate and cap rate is favourable, but be careful the minimum deposit to enter the contracts. For the company to synthesize the GEBs, they can make a profit although the GEBs carry no explicit charges provided that they can sell a lot of the contracts to the clients. From the previous example, we can see that the company earn about 300 SEK for each contracts. It likes a kind of commission to the sellers.

Anyway, the GEBs do provide an ideal return to the investors when the return from fixed-income market is low. It provides another long-term alternative to the investors between equity and fixed-income market.

The optimisation of the features in the contracts between the investors and the company is worth to study in the future. The balance between the profitability to the investors and the company is one of the possible future research directions. It can provide suggestions to both the company and investors to synthesize and invest wisely.

Chapter 7
Conclusion

Quantitative finance is one of the fastest growing areas in the modern banking and corporate world. The main reason behind this phenomenon has been the success of sophisticated quantitative methodologies in helping professionals to manage financial risks. Pricing, hedging and risk management of financial instruments is an indispensable part in the subject of quantitative finance. It is no doubt that more and more new financial instruments with complex payoff structure will be synthesized by financial engineers to fulfil the needs of different investors in the future. For example, Interest Rate and Credit Hybrid, FX Hybrids, Inflation-linked hybrid are popular in the present financial markets. To correctly price and hedge these kinds of complex financial instruments, Black-Scholes model is for sure not enough to explain. New mathematical model and simulation are needed to price these new instruments accurately and efficiently.

In this thesis, we focus on the pricing of Guaranteed equity bonds (GEBs), which is one of the most popular financial instruments in the current markets. Apart from the vanilla contracts (point-to-point), we also studied the exotic contracts. (average, lookback, ratchet, barrier). We explained the features of GEBs in details, from participation rate, cap rate, guarantee interest to indexing methods.

Then the contracts were valuated by Monte Carlo simulation under stochastic interest rate and volatility modelling, or constant interest rate and volatility modelling. The method is robust, and truly describes the mechanism of GEBs in the real financial market because the parameters in the pricing models are calibrated to the observed market data. However, a long running time of simulation is a major drawback of this technique.

By studying and varying the features of the GEBs contracts, we can understand the structure of these financial instruments more easily. Investors should be aware that high participation rate, cap rate and guarantee interest is favourable, but be careful of the indexing method.

In order to investigate the profitability to the investors and the company, we also calculated the model price of GEBs by our program. We have focused on three types of contracts, which are commonly traded in the market. One is the Abbey Capital Guaranteed UK Equity Bond; another is the Barclays' Five-year Protected FTSE Plan; the third one is the NS&I GEB (issue 11).

We found that the present value of the previous two was similar, although with different features but same indexing method. Moreover, from the company point of view, the profit is also similar because of different minimum deposit requirement. The company can still make a profit if they can sell thousands of contracts to the clients even the GEBs carry no explicit charges. For the third contract, the present value is much higher than the previous two. That means it is more worth to hold it. However, the company can still maintain a profit provided they can sell hundreds of thousands of contracts. As the GEB market is growing rapidly nowadays, we believed that the profit will be optimistic to the company in the future.

When the return from the bank account is low or the equity market is volatile, the GEBs do provide a not bad and stable return to the investors, because it can protect the initial capital plus some extra return. It provides both the good sides between the equity and fixed income market.

Future research direction may include the optimisation of the profitability between the investors and the company, new indexing method for cheaper synthesis. It will be expected that the research results will facilitate the GEBs market and provide suggestions to both the company and investors to synthesize and invest wisely.

Finally, the field of quantitative finance is still experiencing rapid development, not only in theoretical design and synthesis of new financial instruments with better payoff structure and lower risk, but also in the method of valuation. It will be expected that a closer interdisciplinary collaboration between financial engineers and mathematicians is required in the future.

References

[1] Craig McCann and Dengpan Luo, An Overview of Equity-Indexed Annuities, Securities Litigation and Consulting Group, Inc, (2006)

[2] J. Aase Nielsen and Klaus Sandmann, Equity-linked life insurance – a model with stochastic interest rates, Discussion Paper No. B-291, September 1994

[3] Vlad Sachelarie, Improvements On The Equity Indexed Annuity Market, PhD thesis of Graduate School of The Ohio State University, (2002)

[4] X. Sheldon Lin and Ken Seng Tan, Valuation of equity-indexed annuities under stochastic interest rates, North American Actuarial Journal, **7**, Number 3

[5] Hangsuck Lee, Pricing equity-indexed annuities with path-dependent options, Insurance: Mathematics and Economics, **33**, 677–690, (2003)

[6] Hangsuck Lee, Pricing equity-indexed annuities embedded with exotic options, Contingencies, 34-37, (January/February, 2002)

[7] In Jae Myung, Tutorial on maximum likelihood estimation, Journal of Mathematical Psychology, **47**, 90–100, (2003)

Appendix

```
Option Explicit

Dim n, t, pr, no_sim, dt, rho, sqrrho, ba, sumc1, sumc2, sumc3, sumc4, sumc5
Dim sumc6, cap, gua, ini_index, i, j, r1, sig, s, z1, z2, zstar, dr1, dsig
Dim ds, c1, pvc1, averagec2, c2, pvc2, highestc3, c3, pvc3, c4, pvc4, c5
Dim pvc5, c6, pvc6, a, b, c, kappa, theta, lambda

'------------------------------------------------------------------------------

Sub check()
  Dim a(7) As Boolean

  If IsNumeric(Cells(3, 9)) Then
    If Cells(3, 9) < 0 Then
      MsgBox "please enter a non-negative value for participation rate"
    Else
      a(0) = True
  Else
    MsgBox "please enter a numeric number for participation rate"
  End If

  If IsNumeric(Cells(4, 9)) Then
    If Cells(4, 9) < 1 Then
      MsgBox "The cap rate should be larger than 1"
    Else
      a(1) = True
  Else
    MsgBox "please enter a numeric number for cap rate"
  End If

  If IsNumeric(Cells(5, 9)) Then
    If Cells(5, 9) < 1 Then
      MsgBox "The guaranteed annual return should be larger than 1"
    Else
      a(2) = True
  Else
    MsgBox "please enter a numeric number for guaranteed annual return"
  End If

  If IsNumeric(Cells(6, 9)) Then
    If Cells(6, 9) < 0 Then
      MsgBox "please enter a value larger than 0 for the number of simulation"
    Else
      a(3) = True
  Else
    MsgBox "please enter a numeric number for number of simulation"
  End If

  If IsNumeric(Cells(7, 9)) Then
    If Cells(7, 9) < 0 Then
      MsgBox "please enter a value larger than 0 for the initial index level"
    Else a(4) = True
  Else
    MsgBox "please enter a numeric number for initial index level"
```

```
  End If

  If IsNumeric(Cells(8, 9)) Then
    If Cells(8, 9) < 0 Then
      MsgBox "please enter a value larger than 0 for the barrier level"
    Else
      a(5) = True
  Else
    MsgBox "please enter a numeric number for barrier level"
  End If

  If IsNumeric(Cells(9, 9)) Then
    If Cells(9, 9) < 0 Then
      MsgBox "please enter a value larger than 0 for the initial interest rate"
    Else
      a(6) = True
  Else
    MsgBox "please enter a numeric number for initial interest rate"
  End If

  If IsNumeric(Cells(10, 9)) Then
    If Cells(10, 9) < 0 Then
      MsgBox "please enter a value larger than 0 for the initial volatility"
    Else
      a(7) = True
  Else
    MsgBox "please enter a numeric number for initial volatility"
  End If

  If a(0) = True And
     a(1) = True And
     a(2) = True And
     a(3) = True And
     a(4) = True And
     a(5) = True And
     a(6) = True And
     a(7) = True Then sim
End Sub

'------------------------------------------------------------------------------

Sub addcomment()
  Worksheets("Sheet1").Range("i3").addcomment
    ("Typical value from 0.5 to 1.1")
  Worksheets("Sheet1").Range("i4").addcomment
    ("Typical value from 1.2 to 1.5")
  Worksheets("Sheet1").Range("i5").addcomment
    ("Typical value from 1.03 to 1.05")
  Worksheets("Sheet1").Range("i6").addcomment
    ("10,100,1000,10000")
  Worksheets("Sheet1").Range("i7").addcomment
    ("Initial index level at the start date of the contract")
  Worksheets("Sheet1").Range("i8").addcomment
    ("This is required to valuate barrier type contracts")
  Worksheets("Sheet1").Range("i9").addcomment
    ("Initial interest rate at the start date of the contract")
  Worksheets("Sheet1").Range("i10").addcomment
```

```
    ("Initial volatility at the start date of the contract, a good estimate is
      from the implied volatility")
End Sub

'------------------------------------------------------------------------------

Sub sim()
  n   = 1250         ' assume 1 year = 250 trading days
  t   = 5
  pr  = Cells(3, 9)
  cap = Cells(4, 9)
  gua = Cells(5, 9)
  no_sim = Cells(6, 9)
  dt  = t / n
  ini_index   = Cells(7, 9)
  Cells(2, 3) = ini_index
  Cells(2, 2) = Cells(9, 9)
  Cells(2, 4) = Cells(10, 9)

  ba     = Cells(8, 9)
  sumc1  = 0
  sumc2  = 0
  sumc3  = 0
  sumc4  = 0
  sumc5  = 0
  sumc6  = 0
  kappa  = 0.01
  theta  = 0.079847
  lambda = 0.245675

  a = kappa * theta
  b = (1 - kappa) * (1 - lambda)
  c = (1 - kappa) * lambda

  For j = 1 To no_sim
    r1  = Cells(9, 9)
    sig = Cells(10, 9)
    s   = ini_index

    For i = 1 To n
      Cells(i + 2, 1) = i * dt
      z1 = randn()
      z2 = randn()

      s = s * Exp((r1 - sig ^ 2 / 2) * dt + sig * Sqr(dt) * z2)
      Cells(i + 2, 3) = s

      dr1 = 0.43652 * (0.05 - r1) * dt + 0.09 * Sqr(r1) * z1 * Sqr(dt)
      r1  = r1 + dr1
      Cells(i + 2, 2) = r1

      sig = Sqr(a + b * ((Log(s / Cells(i + 1, 3)) - (r1 - 0.5 * sig ^ 2) * dt) /
            Sqr(dt)) ^ 2 + c * sig ^ 2)
      Cells(i + 2, 4) = sig

    Next i
```

```
    c1 = Application.Max(Application.Min(1 + pr * (Cells(n + 2, 3) /
         ini_index - 1), cap), gua ^ t)

    pvc1 = Exp(-Cells(n + 2, 2) * t) * c1
    Cells(j + 2, 13) = pvc1
    sumc1 = sumc1 + pvc1

    averagec2 = average(t)
    c2 = Application.Max(Application.Min(1 + pr * (averagec2 / ini_index - 1),
         cap), gua ^ t)

    pvc2  = Exp(-Cells(n + 2, 2) * t) * c2
    Cells(j + 2, 14) = pvc2
    sumc2 = sumc2 + pvc2

    highestc3 = highest(t)
    c3 = Application.Max(Application.Min(1 + pr * (highestc3 / ini_index - 1),
         cap), gua ^ t)

    pvc3  = Exp(-Cells(n + 2, 2) * t) * c3
    Cells(j + 2, 15) = pvc3
    sumc3 = sumc3 + pvc3

    c4    = ratchet(t)
    pvc4  = Exp(-Cells(n + 2, 2) * t) * c4
    Cells(j + 2, 16) = pvc4
    sumc4 = sumc4 + pvc4

    c5    = barrier_ratchet(t, ba)
    pvc5  = Exp(-Cells(n + 2, 2) * t) * c5
    Cells(j + 2, 17) = pvc5
    sumc5 = sumc5 + pvc5

    c6    = barrier(t, ba, n)
    pvc6  = Exp(-Cells(n + 2, 2) * t) * c6
    Cells(j + 2, 18) = pvc6
    sumc6 = sumc6 + pvc6
  Next j

  Cells(no_sim + 4, 13) = sumc1 / no_sim
  Cells(no_sim + 4, 14) = sumc2 / no_sim
  Cells(no_sim + 4, 15) = sumc3 / no_sim
  Cells(no_sim + 4, 16) = sumc4 / no_sim
  Cells(no_sim + 4, 17) = sumc5 / no_sim
  Cells(no_sim + 4, 18) = sumc6 / no_sim
End Sub

'------------------------------------------------------------------------------

Function barrier(t, ba, n) 'up and in
  Dim cross

  cross = 0

  For i = 0 To t * 250
    If Cells(2 + i, 3) > ba Then
      cross = 1
```

```
    Else
      cross = 0
    End If
  Next i

  If cross = 1 Then
    barrier = Application.Max(Application.Min(1 + pr * (Cells(n + 2, 3) /
              ini_index - 1), cap), gua ^ t)
  Else
    barrier = gua ^ t
  End If
End Function

'------------------------------------------------------------------------------

Function barrier_ratchet(t, ba) 'up and in
  Dim cross, total

  cross = 0

  For i = 0 To t * 250
    If Cells(2 + i, 3) > ba Then
      cross = 1
    Else
      cross = 0
    End If
  Next i

  total = 1

  For i = 1 To t
    If cross = 1 Then
      total = Application.Max(total * (Application.Max
             (Application.Min(1 + pr * (Cells(2 + 250 * i, 3) /
             Cells(2 + 250 * (i - 1), 3) - 1), cap), 1)), gua ^ t)
    Else
      total = total * gua
    End If
  Next i

  barrier_ratchet = total
End Function

'------------------------------------------------------------------------------

Function ratchet(t)
  Dim total

  total = 1

  For i = 1 To t
    total = Application.Max(total * (Application.Max
           (Application.Min(1 + pr * (Cells(2 + 250 * i, 3) /
           Cells(2 + 250 * (i - 1), 3) - 1), cap), 1)), gua ^ t)
  Next i

  ratchet = total
```

```
End Function

'-----------------------------------------------------------------------------

Function average(t)
  Dim total

  total = 0

  For i = 0 To t * 250
    total = total + Cells(2 + i, 3)
  Next i

  average = total / (t * 250 + 1)
End Function

'-----------------------------------------------------------------------------

Function highest(t)
  Dim maxi

  maxi = Cells(2, 3)

  For i = 0 To t * 250
    maxi = Application.Max(maxi, Cells(2 + i, 3))
  Next i

  highest = maxi
End Function

'-----------------------------------------------------------------------------

Sub sim_bs()
  n = 1250         ' assume 1 year=250 trading days
  t = 5
  pr = Cells(3, 9)
  cap = Cells(4, 9)
  gua = Cells(5, 9)
  no_sim = Cells(6, 9)
  dt = t / n
  ini_index = Cells(7, 9)
  Cells(2, 5) = ini_index
  ba = Cells(8, 9)
  r = Cells(9, 9)
  sig = Cells(10, 9)
  sumc1 = 0
  sumc2 = 0
  sumc3 = 0
  sumc4 = 0
  sumc5 = 0
  sumc6 = 0

  For j = 1 To no_sim
    s_bs = ini_index
    For i = 1 To n
      Cells(i + 2, 1) = i * dt
      z1 = randn()
```

```
      ds1 = r * s_bs * dt + sig * s_bs * z1 * Sqr(dt)
      s_bs = s_bs + ds1
      Cells(i + 2, 5) = s_bs
    Next i
    c1 = Application.Max(Application.Min(1 + pr * (Cells(n + 2, 5) /
        ini_index - 1), cap), gua ^ t)

    pvc1 = Exp(-r * t) * c1
    Cells(j + 2, 20) = pvc1
    sumc1 = sumc1 + pvc1

    averagec2 = average(t)
    c2 = Application.Max(Application.Min(1 + pr * (averagec2 / ini_index - 1),
         cap), gua ^ t)
    pvc2 = Exp(-r * t) * c2
    Cells(j + 2, 21) = pvc2
    sumc2 = sumc2 + pvc2

    highestc3 = highest(t)
    c3 = Application.Max(Application.Min(1 + pr * (highestc3 / ini_index - 1),
         cap), gua ^ t)
    pvc3 = Exp(-r * t) * c3
    Cells(j + 2, 22) = pvc3
    sumc3 = sumc3 + pvc3

    c4 = ratchet(t)
    pvc4 = Exp(-r * t) * c4
    Cells(j + 2, 23) = pvc4
    sumc4 = sumc4 + pvc4

    c5 = barrier_ratchet(t, ba)
    pvc5 = Exp(-r * t) * c5
    Cells(j + 2, 24) = pvc5
    sumc5 = sumc5 + pvc5

    c6 = barrier(t, ba, n)
    pvc6 = Exp(-r * t) * c6
    Cells(j + 2, 25) = pvc6
    sumc6 = sumc6 + pvc6
  Next j

  Cells(no_sim + 4, 20) = sumc1 / no_sim
  Cells(no_sim + 4, 21) = sumc2 / no_sim
  Cells(no_sim + 4, 22) = sumc3 / no_sim
  Cells(no_sim + 4, 23) = sumc4 / no_sim
  Cells(no_sim + 4, 24) = sumc5 / no_sim
  Cells(no_sim + 4, 25) = sumc6 / no_sim
End Sub

'------------------------------------------------------------------------------

Function randn()
  randn = Application.NormSInv(Rnd())
End Function
```