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Swaps & Swaptions

by

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Abstract

The “plain vanilla” interest rate swap is one of most popular type of swaps and will be described in details in this paper. We will describe the mechanics of and valuation techniques of interest rate swaps. The relation between swaps and bonds and bootstrapping from swap curve are also discussed.

S*waptions* are option to enter into an interest rate swap. They can be therefore either *receiver swaptions*, or *payer swaptions*. Swaptions are illustrated with examples and valuation of swaption by the Black’76 model is demonstrated.

Table of Contents

1. Introduction

Swaps are important OCT interest rate derivatives. The swap market is huge and growing by the day. A swap is shortly explained as an agreement between two companies, called counterparties, to exchange cash flows in the future according to a predetermined formula.

There are two basic types of swaps: currency and interest rate. A currency swaps is an agreement to deliver one currency against another. The most common type of interest rate swap is called the "plain vanilla" interest rate swap. With this swap one party agrees to pay interest payments at a pre-specified fixed rate on a notional principal for several years. In return, it receives interest payments at a floating rate on the same notional principal for the same period of time. There are also interest rate swaps with some variations to this standard type (plain vanilla). Besides what mentioned above, many other types of swaps are also traded, for example, equity swaps, commodity swap, and other exotic instruments. We concentrate on plain vanilla interest rate swap here, and unless stated otherwise we will shortly call it as interest rate swap or just swap.

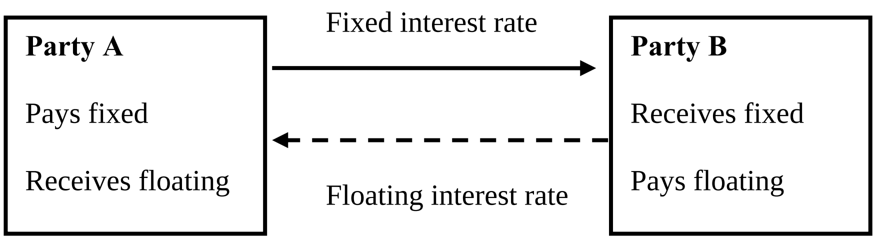
Swaptions are popular interest rate options. They are options to enter into an interest rate swap. The fixed rate and the time period of the interest rate swap as measured from the option expiration date are specified in the option contract. Market makers that offer interest rate swap contracts to their corporate clients are often prepared to sell them swaptions or buy swaptions from them. When the market quotes on a swaption, it generally quotes on the fixed-rate part of the swap. Swaptions can be therefore either *receiver swaptions*, where the holder has the right to receive fixed and pay floating rates, namely floating-for-fixed, or *payer swaptions* where the holder has right to pay fixed and receive floating rates, also called fixed-for-floating. As interest rate fall, the holder of a receiver swaption benefits whereas the holder of a payer swaption suffers and vice versa.

To simplify the matters, we will ignore the impact of day count convention throughout the paper.

2. Interest rate swaps

As already mentioned, in an interest rate swap, one party agrees to pay interest payments at a pre-specified fixed rate on a notional principal for several years. In return, it receives interest payments at a floating rate, based on the same notional principal and for the same period of time. As Figure 1 illustrates.

Figure 1: The parties to an interest rate swap



Since the principal is only "notional", it will not be exchanged at the maturity. The floating-rate payment is linked to some short-term interest rate such as the 3-month or 6-month LIBOR rates. Most swaps are arranged so that their value is zero at initiation, which is done by choosing a "right" fixed rate.

In short, there're four components of an interest rate swap,

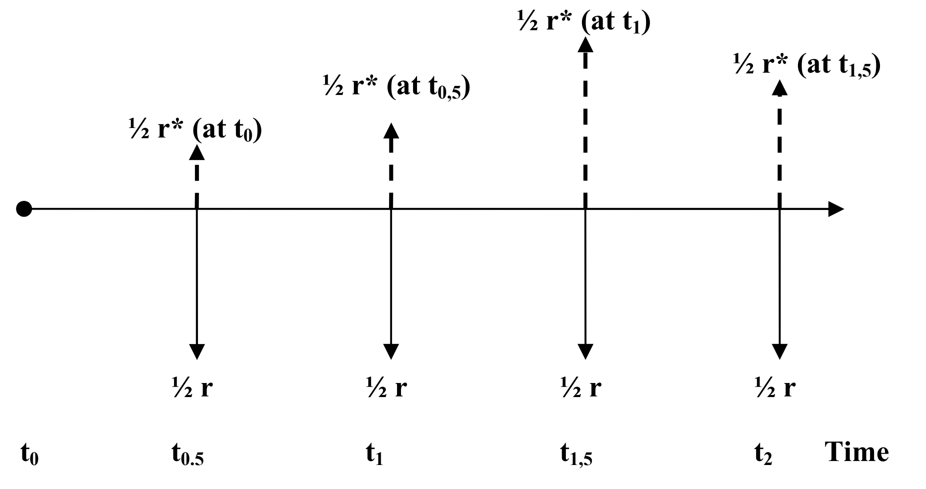
1. Notional principal amount
2. Interest rates for the parties (fixed rate and floating rate)
3. Payment frequency (semi-annual, quarterly)
4. Duration of the swap (also called tenor or maturity)

Example 1: Cash payments in the swap

Suppose we entered into a 2-year swap with semi-annual payments, the settlement date is 14^{th} Dec 2005, we will call it t_0. The annual floating rate is denoted as r* which is variable and tied to for example 6-month LIBOR. The annual fixed rate is denoted as r which is a fixed number. For simplicity, assume the principal is only one dollar.

Figure 2 illustrates the pattern of cash payments in the floating rate side and fixed rate side. The cash payments in the floating side and the fixed side are denoted by the dashed and straight lines respectively.

Figure 2: Diagram of the cash payments in the swap



There are two important things needed to be mentioned. First, r and r* are annualized interest rate, so each semi-annual interest payment is ½ r or ½ r *. Second, in the floating payments side, the first payment at ½ year from t_0 ($t_{0.5}$)equals to the 6-month floating interest rate quoted six month before (quoted t_0), which is ½ times t_0's r*. That means the first floating payment is known at the initiation of the swap contract. Similarly, the second floating rate payment at 1 year from t_0 is ½ times r*(at $t_{0.5}$)), which is the 6-month floating interest rate quoted ½ years from t_0, and so on.

3. The comparative-advantage argument of swaps

The fundamental purpose of swaps is to transform the character of a liability without liquidating the liability. Swaps are used for hedging and speculation purposes as well. However, a common explanation to the popularity of swaps is to exploit comparative advantage. Comparative advantage exists when two companies who want to borrow money are quoted fixed and floating rates such that by exchanging payments between themselves they benefit, at the same time benefiting the intermediary who puts the deal together, as Example 2 demonstrates.

Example 2: Comparative advantage of swaps[1]

Two companies A and B want to borrow $50 Million from a bank, to be paid back in two years. They are quoted the interest rates for borrowing at fixed and floating rates as in Table 1.

[1] Adapted from Paul wilmott, 2001, *Introduces Quantative Finance*, John Wiley & Sons, Chichester, p277,

Table 1: Borrowing rates for company A & B

	Fixed	Floating
A	7%	Six-month LIBOR + 30 bpd
B	8.2%	Six-month LIBOR + 100 bpd

Note that both must pay a premium over LIBOR to cover risk of default, which is perceived to be greater for company B. Ideally, company A wants to borrow at floating and B at fixed. If they each borrow directly then they pay the following:

Table 2: Borrowing rates with no swaps involved

A	Six-month LIBOR + 30 bpd
B	8.2% (fixed)

The total interest they are paying is:

Six-month LIBOR + 30 bps + 8.2% = Six-month LIBOR + 8.5%.

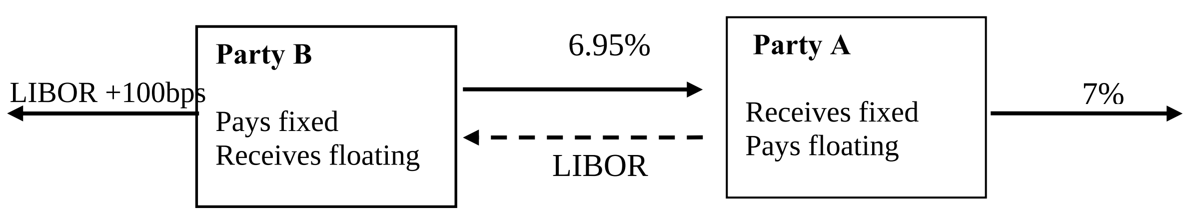
If only they could get together they'd only be paying:

Six-month LIBOR + 100 bps + 7% = Six-month LIBOR + 8%

That's a saving of 0.5%.

Let's suppose that A borrows fixed and B floating, even though that's not what they want. A is therefore currently paying 7% and B six-month LIBOR plus 1%.Their total interest payments are six-month LIBOR plus 8%. Suppose now they enter into a swap in which A pays LIBOR to B and B pays 6.95% to A, in other words, they have swapped interest payments. Figure 3 illustrates the effect of the swap.

Figure 3: Swap-agreement between A & B when the rates in Table 1 apply



A now has three sets of cash flows:

1. It pays LIBOR to B
2. It pays 7% to the bank
3. It receives 6.95% from B

A therefore has a net floating payment of LIBOR plus 5 bps. Not only is this floating, as A originality wanted, but it is 25 bps better than if they had borrowed directly at the floating rate. There's still another 25 bps missing and, of course, B gets this.

B has three sets of cash flows.

1. It pays 6.95% to A
2. It pays LIBOR +100 bps to the bank
3. It receives LOBOR from B

Those cash flows net out at 7.95%, which is fixed, as required and 25 bps less than the original deal.

To generalize, let's do the same calculation with x instead of 6.95. A is currently paying 7% and B six-month LIBOR plus 1%. They enter into a swap in which A pays LIBOR to B and B pays x% to A. Again they have swapped interest payments.

Looked at from A's perspective they are paying 7% and LIBOR while receiving x%, a net floating payment of LIBOR plus 7 - x%. Now we want A to benefit by 25 bps over the original deal, this is half of the 50 bps advantage. So...

LIBOR + 7 -x + 0.25 = LIBOR + 0.3, i.e. x = 6.95%.

Not only does A now get floating, as originally wanted, but it is 25 bps better than if they had borrowed directly at the floating rate. There's still another 25 bps missing and, of course, B gets this. B pays LIBOR plus 100 bps and also 6.95% to A while receiving LIBOR from A. This nets out at 7.95%, which is fixed, as required and 25 bps less than the original deal.

In practice the two counterparties would deal through an intermediary who would take a piece of the action.

4. The relation between swaps and bonds[2]

There are two sides to a swap, the fixed-rate side and the floating-rate side. The fixed interest payments, since they are all known in terms of actual dollar amount, can be seen as the sum of zero-coupon bonds. Let $Z(t;T_i)$ denotes the zero coupon bond price at time t and a maturity of T_i. If the fixed rate of interest is r_s then the fixed payments add up to

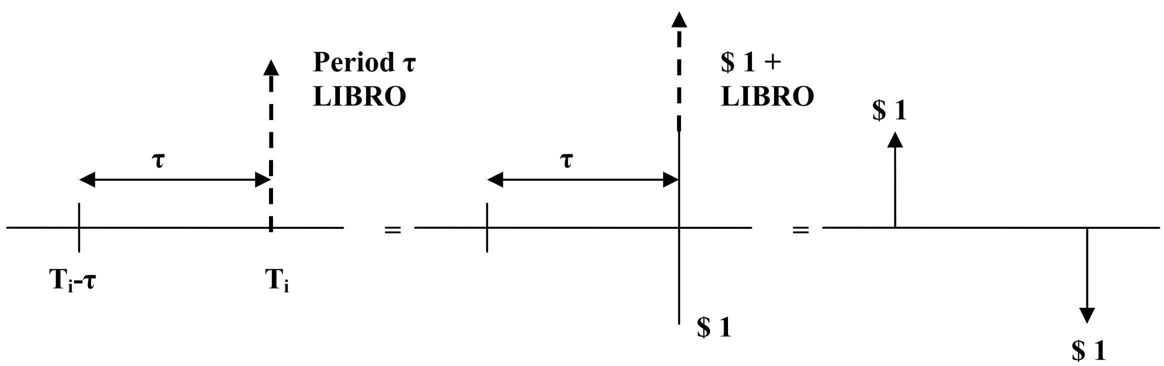
[2] p280,Paul wilmott, 2001, *Introduces Quantative Finance*, John Wiley & Sons, Chichester

(1) $$PV_{fix}(t) = r_s \sum_{i=1}^{N} Z(t;T_i).$$

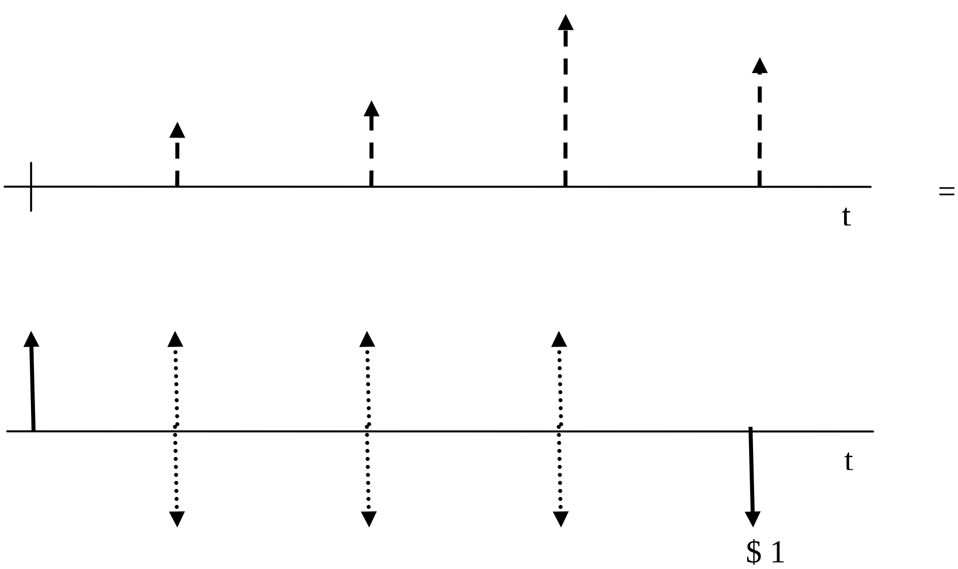
This is the value today, time *t*, of all the fixed-rate payments. Here there are *N* payments, one at each T_i. Of course, this is multiplied by the notional principal, but assume as in Example 1 that we have scaled this to one.

To see the simple relationship between the floating leg and zero-coupon bonds we draw some schematic diagrams and compare the cashflows. A single floating leg payment (dashed line) is shown in Figure 4.

<u>*Figure* 4: A single floating leg in an swap & the replicating portfolio</u>



<u>*Figure* 5: All floating legs in a swap</u>



At time T_i there is payment of r_τ of the notional principal, where r_τ is the period τ rate of LIBOR, set at time $T_{i-\tau}$. We add and subtract $1 at time T_i to get the second diagram. The first and the second diagrams obviously have the same present value. Now recall the precise definition of LIBOR. It is the interest rate paid on a fixed-term deposit. Thus the $1 + r_τ at time T_i is the same as $1 at time $T_{i-\tau}$. This gives the third diagram. It follows that the single floating rate payment is equivalent to two zero-coupon bonds. A single floating leg of a swap at time T_i is exactly equal to a deposit of $1 at time $T_{i-\tau}$ and a withdrawal of $1 at time τ .

Now add up all the floating legs as shown in Figure 5, note the cancellation of all $ 1 (cashflows represented by dotted lines) except for the first and last. This shows that the floating side of the swap has value

(2) $$PV_{fl}(t) = 1 - Z(t;T_N)$$

Bring the fixed and floating sides together to find that the value of the swap, to the receiver of the fixed side, is

(3) $$V_s(t) = r_s \sum_{i=1}^{N} Z(t;T_i) - 1 + Z(t;T_N)$$

This result is model-independent. This relationship is independent of any mathematical model for bonds or swaps.

5. The swap curve & Bootstrapping with swaps

Swap rates & swap curve

As we said swap is usually arranged so it has zero value to either party at initialization, in other words, the present values of the fixed leg and the floating leg have the same value, netting out to zero. Such a swap is often called par swap. The fixed rate which is chosen to make a swap at par is call swap rate or par swap rate. So we can obtain the swap rate by setting expression (3) to 0,

$$V_s(t) = r_s \sum_{i=1}^{N} Z(t;T_i) - 1 + Z(t;T_N) = 0$$

i.e.

(4) $$r_s = \frac{1 - Z(t;T_N)}{\sum_{i=1}^{N} Z(t;T_i)}$$

These rates are quoted at various maturities and can make up the swap curve.

Bootstrapping with swaps

Swaps are now so liquid that in USA the swap curve has emerged as an alternative to Treasuries as a benchmark for USD interest rates at maturities exceeding a year. In practice it is easy to obtain the swap rates $r_s(T_i)$ for many maturities T_i, the prices of zero-coupon bonds can then be calculated using the formula (4), in this way we obtain yield curve from swap curve instead of the other way around.

The bootstrapping process goes as following.

1. Obtain $Z(t;T_1)$ by solving

$$r_s(T_1) = \frac{1 - Z(t;T_1)}{Z(t;T_1)} \Rightarrow Z(t;T_1) = \frac{1}{1 + r_s(T_1)}$$

2. Since we now have $Z(t;T_1)$, we can calcualte $Z(t;T_2)$ by solving

$$Z(t:T_2) = \frac{1 - r_s(T_2)Z(t;T_1)}{1 + r_s(T_2)}$$

…

In general,

(5) $$Z(t:T_{k+1}) = \frac{1 - r_s(T_{k+1})\sum_{i=1}^{k} Z(t;T_i)}{1 + r_s(T_{k+1})}$$

6. Valuation of swaps[3]

We now move on to discuss the valuation of interest rate swaps. As discussed earlier, a swap is often worth to zero when it is first initiated. However, after it has been in existence for time, its value may become positive or negative. One valuation approach is to value the swap in terms of fixed-rate bond, and floating-rate bond. This illustrates other point of view of the relationship between swap and bond.

We know the principal payments are not exchanged, however, if we assume that they are both received and paid at the end of the swap, the value of the swap is not changed. Therefore, assume company A is now paying floating to B and receive fixed from. The swap for A can then be

[3] Hull, J,C, (2006), Options, Futures and Other Derivatives, 6ed, Pearson Prentice Hall, Upper Saddle River, New Jersey, P161-162

regarded as a long position in a fixed-rate bond and a short position in a floating-rate bond, so that

(6) $V_s = B_{fix} - B_{fl}$

Where V_s is the value of the swap, B_{fl} is the value of the floating-rate bond (corresponding to payments that are made), and B_{fix} is the value of the fixed-rate bond (corresponding to payments that are received). Similarly, from the point of view of B, the swap is a long position in a floating-rate bond and a short position in a fixed-rate bond, so that the value of the swap is

(6) $V_s = B_{fl} - B_{fix}$

The value of the fixed rate bond, B_{fix} can be determined as a standard coupon-paying bond. To value the floating-rate bond, we note that the bond is worth the notional principal immediately after an interest payment. This is because at this time the bond is a "fair deal" where the borrower pays LIBOR for each subsequent accrual period.

Suppose that the notional principal is L, the next exchange of payments is at time T_i, and the floating payment that will be made at time T_i (as discussed in Example 1, it was determined at the last payment data) is k_i. Immediately after the payment $B_{fl} = L$ as just explained. It follows that immediately before the payment $B_{fl} = L + k_i$. The floating-rate bond can therefore be regarded as an instrument providing a single cash flow of $L + k_i$ at time $T_{i.}$ Discounting this, the value of the floating-rate bond today is

(7) $(L + k_i)\exp(-r(T_i))$,

where r is the LIBOR for a maturity of T_i.

7. Swaptions

We now discuss swaptions on interest rate swaps. We know a swaption gives the holder the right but not the obligation to enter into an interest rate swap at a specific date in the future, at a particular fixed rate and for a specified term. The components of a swaption are

1. Notional
2. Maturity of the option
3. Strike rate
4. Payer or receiver
5. Type: American, European or Bermudan[4]
6. Maturity of the swap (tenor)

[4] Bermudan Swaptions give the buyer the right to exercise on specific dates during the option period.

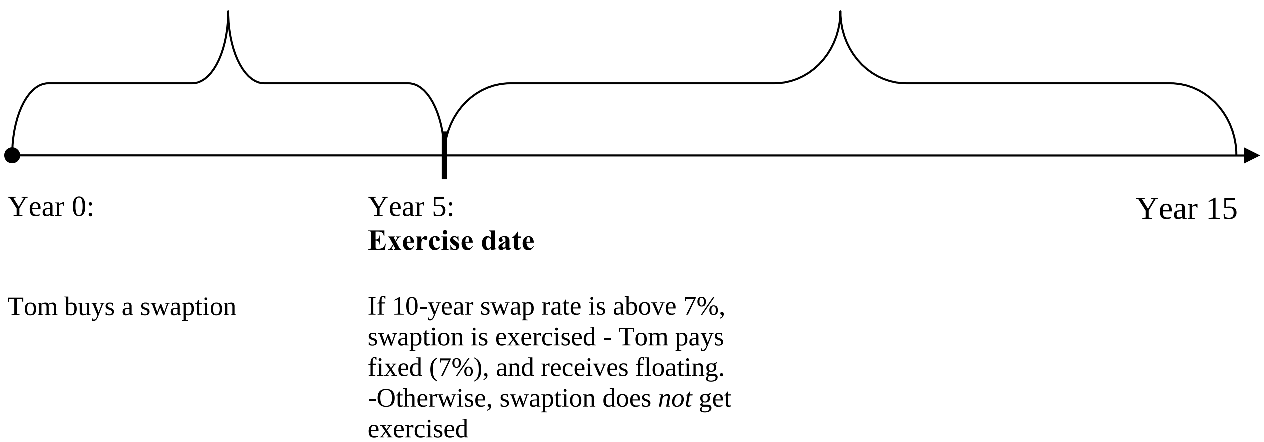
7. Frequency of settlement of the swap
8. Floating rate

Take the payer swaptions as example, the holder has an option to enter into a swap paying fixed and receiving floating. She or he obtains benefit of the pre-set strike rate if the market rates are higher, with the flexibility to enter into the current market swap rate if they are lower. The opposite is true for a receiver swaption holder. A swaption is exercised if the strike rate of the swap is more favorable than the prevailing market swap rate, it expires worthless otherwise. If exercised, the holder then enters into an interest rate swap as determined in the swaption contract.

Example 3: A European in-5-for-10 payer swaption with strike rate 7%

The decision that the holder Tom will have to face in five years is illustrated below:

Figure 6 Decision of the payer swaption holder



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8. Valuation of Swaptions

Swaptions are almost always of European types. European swaptions can be priced using Black’76 option pricing model. For this purpose, the underlying is a forward on a swap instead of a swap. We first go through a quick review of Black’76.

Black’76 Option Pricing Model[5]

It is known that the evolution of many commodity prices does not show non-randomness. Because of such non-randomness, many spot commodity prices cannot be modelled with a

[5] *Black(1976) Option Pricing Formula*, Riskglossary.com, http://www.riskglossary.com/articles/black_1976.htm

geometric Brownian motion, and the Black-Scholes (1973) or Merton (1973) models for options on stocks do not apply.

In 1976, Fischer Black published a paper addressing this problem. His solution was to model forward prices as opposed to spot prices. Forward prices do not exhibit the same non-randomness of spot prices. Consider a forward price for delivery shortly after a harvest of an agricultural product. Prior to the harvest, the spot price may be high, reflecting depleted supplies of the product, but the forward price will not be high. Because it is for delivery after the harvest, it will be low in anticipation of a drop in prices following the harvest. While it is not reasonable to model the spot price with a Brownian motion, it may be reasonable to model the forward price with one.

Black's (1976) option pricing formula reflects this solution, modelling a forward price as an underlying in place of a spot price. The model is not only used for modelling European options on physical commodities, forwards or futures, but also for pricing swaptions, interest rate caps and floors. The model is popularly known as **Black '76** or simply **Black's model**. The formula can be derived directly from Black-Scholes model with $F = e^{-rT} S$,

(8) $$P_{call} = e^{-rT}(F \cdot N(d_1) - K \cdot N(d_2))$$

(9) $$P_{put} = e^{-rT}(F \cdot N(-d_2) - K \cdot N(-d_1))$$

Where

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)T}{\sigma\sqrt{T}}; \quad d_2 = d_1 - \sigma\sqrt{T}$$

Valuation of European Swaptions using Black'76[6]

In practice, the Black-76 value is multiplied by a factor adjusting for the tenor of the swaption. This is the practitioner's benchmark swaption model. The model is arbitrage-free under the assumption of a lognormal forward swap rate. Let *c* denote a payer swaption, and *p* is a receiver swaption. We then have

(10) $$c = \frac{1 - \frac{1}{(1 - F/m)^{\tau m}}}{F} e^{-rT}\left[F \cdot N(d_1) - K \cdot N(d_2)\right]$$

(11) $$p = \frac{1 - \frac{1}{(1 - F/m)^{\tau m}}}{F} e^{-rT}\left[K \cdot N(-d_2) - F \cdot N(-d_1)\right]$$

[6] This section (inlcuding Example 4) is adapted from the Lecture Notes p 238.

Where

τ = Tenor of swap in years.
F = Forward rate of the underlying swap.
K = Strike rate of the swaption.
r = Risk-free interest rate.
T = Time to expiration in years.
σ = Volatility of the forward-starting swap rate.
m = Compoundings per year in swap rate.

Example 4: Valuation of A payer swaption

Suppose we have a two-year payer swaption on a four-year swap with semi-annual compounding. The forward swap rate of 7% starts two years from now and ends six years from now. The strike is 7.5%, the risk-free interest rate is 6%, and the volatility of the forward starting swap rate is 20% per annum..

It follows that, in this example F = 0.07, K = 0.075, T = 2, σ = 0.20, so that

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(0.07/0.075) + (0.20^2/2)2}{0.20\sqrt{2}} = -0.0125,$$
$$d_2 = d_1 - 0.20\sqrt{2} = -0.3853$$

And since F=0.07, m = 2, τ = 4.0

$$\frac{1 - \frac{1}{(1 - F/m)^{\tau m}}}{F} = 3.4368$$

With a semi-annual forward swap rate, given r = 0.06, the up-front value of the payer swaption in per cent of the notional is,

$$c = 3.4368 \times e^{--0.06\times 2}\left[0.07 \cdot N(-0.1025) - 0.075 \cdot N(-0.3853)\right] = 1.7964\%$$

9. Conclusion

The fundamental purpose of swaps is to transform the character of a liability without liquidating the liability. Swaps are used for hedging and speculation purposes as well. However, a common explanation to the popularity of swaps is to exploit comparative advantage.

There are two sides to a swap, the fixed-rate side and the floating-rate side. The fixed interest payments, since they are all known in terms of actual dollar amount, can be seen as the sum of zero-coupon bonds. For the floating side, the present value is the notional principal minus the present value of the principal repayment at the maturity.

Swaps are usually arranged so that it has no value at initiation, this is realized by a careful choice of fixed rate. Such fixed rates are called swap rates or par swap rates. Those rates for various maturities make up the swap curve. The yield curve can be bootstrapped from swap curve since swaps are so liquid and are available for a big range of maturities.

Although swaps worth zero at initiation, after it has been in existence for time, its value may become positive or negative. One valuation approach is to valuate the swap in terms of fixed-rate bond, and floating-rate bond.

A swaption gives its holder to right to enter into a swap. It is exercised if the strike rate of the swap is more favourable than the prevailing market swap rate, it expires worthless otherwise. If exercised, the holder then enters into an interest rate swap as determined in the swaption contract.

Swaptions are valued using the Black'76 option pricing formula. For this purpose, the underlying is a forward on a swap.

9. References

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