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The SABR Model

Calibrated for Swaption's Volatility Smile

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Abstract

Title: The SABR Model - Calibrated for Swaption's Volatility Smile

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Keywords: SABR, Volatility smile, Swaption, Stochastic volatility, Black-Scholes model.

Problem: The standard Black-Scholes framework cannot incorporate the volatility smiles usually observed in the markets. Instead, one must consider alternative stochastic volatility models such as the SABR. Little research about the suitability of the SABR model for Swedish market (swaption) data has been found.

Purpose: The purpose of this paper is to account for and to calibrate the SABR model for swaptions trading on the Swedish market. We intend to alter the calibration techniques and parameter values to examine which method is the most consistent with the market.

Method: In MATLAB, we investigate the model using two different minimization techniques to estimate the model's parameters. For both techniques, we also implement refinements of the original SABR model.

Results and Conclusion: The quality of the fit relies heavily on the underlying data. For the data used, we find superior fit for many different swaption smiles. In addition, little discrepancy in the quality of the fit between methods employed is found. We conclude that estimating the α parameter from at-the-money volatility produces slightly smaller errors than using minimization techniques to estimate all parameters. Using refinement techniques marginally increase the quality of the fit.

Sammanfattning

Titel: SABR Modellen – Kalibrerad för Swaptioner med Volatilitetsleende

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Nyckelord: SABR, Volatilitetsleende, Swaption, Stokastisk volatilitet, Black-Scholes modellen.

Problem: Det standardiserade Black-Scholes-ramverket kan inte inkorporera de volatilitetsleenden som vanligtvis observeras på marknaden. Istället så måste man överväga alternativa stokastiska volatilitets-modeller så som SABR-modellen. Lite forskning angående lämpligheten av SABR-modellen för svensk (swaptions) data har blivit utförd.

Syfte: Syftet med denna rapport är att beskriva och kalibrera SABR-modellen för swaptioner på den svenska marknaden. Vi avser att ändra kalibreringstekniker och parametervärden för att undersöka vilken metod som är mest förenlig med marknaden.

Metod: I MATLAB undersöker vi modellen genom att använda två olika minimiseringstekniker för att estimera SABR-modellens parametrar. För båda metoderna så implementerar vi även förfiningar av den ursprungliga SABR-modellen.

Resultat and Slutsats: Kvaliteten av passformen beror i stor grad på underliggande data. För använd data så hittar vi förstklassig passform för många olika swaptionsleenden. Vi finner liten skillnad i kvaliteten av passformen mellan metoder implementerade i denna rapport. Vi drar slutsatsen att estimering av α-parametern från ”at-the-money-volatilitet” producerar något mindre fel jämfört mot använda tekniker som estimerar alla parametrar. Användning av förfiningstekniker ger en marginell förbättrad kvalitet av passformen.

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List of Important Notations and Abbreviations

σ	Standard deviation of the underlying asset value, also known as volatility.
σ_B	Volatility produced by the SABR volatility model that should be the substitution to the constant volatility in the Black-76 pricing model for option.
σ_I	Implied volatility that is observed from market prices.
σ_{loc}	Volatility produced from the local volatility model.
$f(t_i, t_j)$	The forward rate between times t_i and t_j where $0 < j < i$.
$p(t_i, t_j)$	The discounting function between times t_i and t_j where $0 < j < i$.
$r(t_i)$	The spot rate at time t_i.
$R(0, T_S, T_E)$	Swap rate observed at time 0, that has a starting date T_S and expiring date T_S
$\log x$	Natural logarithm of x. Or $\log_e x$ or $\ln x$, where e ≈ 2.71828…
$\mathcal{N}(x)$	Cumulative normal distribution function of x.

ATM	At-The-Money
BPS	Basis Points
IRS	Interest Rate Swap
ITM	In-The-Money
OTC	Over The Counter
OTM	Out-of-The-Money
PDE	Partial Differential Equation
SSE	Sum of Squared Errors.

1 INTRODUCTION

1.1 Background

During the early 1970s, Black and Scholes (1973) and Merton (1973) developed what became a pricing model for European put- and call options. Usually known as the *Black-Scholes model* or the *Black-Scholes-Merton model*, their Nobel Prize awarded work has played a keen role in the field of pricing financial derivatives.

The Black-Scholes model was initially developed to fit a world where there are no arbitrage opportunities with unlimited possibilities to lend and borrow at a risk-free rate. A world where continuous (transaction costs free) trading occurs in non-dividend paying underlying. The underlying asset (usually a stock that can be shorted and/or traded in fraction) is assumed to follow a geometric Brownian motion (i.e. a stochastic process). Under these assumptions, the model can price European options with only five inputs. Four of which are observable on the marketplace (price of underlying, strike price, time to maturity and risk-free interest rate) and one that needs to be estimated or inferred (volatility). With its simplicity and the underlying risk-neutral valuation, the Black-Scholes model speaks to investors' independent on their attitudes towards risk (Hull, p. 289).

Over the last 40 years, financial markets have changed rapidly and now are very different compared to when Black, Scholes and Merton presented their ground-breaking work. Complex instruments are nowadays traded and traders are familiar with exotic options. Pricing of these innovative derivatives are usually a perplexity and the Black-Scholes model, that was develop to price plain vanilla (European) options, has now been rendered somewhat inadequate. In addition, variables that are assumed constant appear in fact to be random, thus jeopardizing the accuracy of models that do not account for these phenomena.

Prior to the Black Monday, 19th of October 1987, the only unidentified variable in the Black-Scholes model, the volatility, did not depend greatly on the strike price when being observed in the market. In the aftermath, a new pattern arose. Volatilities were now smaller for at-the-money (ATM) options rather than for dittos that were deep in-the-money (ITM) and/or out-of-the-money (OTM). The shape of the volatility curve came to be known as a *volatility skew* or *volatility smile*[1] and has since then been complicated to include in any pricing models.

Various attempts have been made to come up with models that handle for example, a stochastic interest rate or a model that tries to seize the volatility smile observed on the market. Black (1976) developed a formula for pricing forward contracts. His work later came to be known as the *Black model* (also known as the *Black-76 model*). In his paper "*Pricing with a smile"*, Dupire (1994), extends the Black-Scholes model, claiming it to be compatible with the volatility smiles. In the same time, Derman and Kani (1994) presented their work on lo-

[1] Skew usually refers to the slope of the curve while smile indicate the curvature.

cal volatility models. In addition, models allowing for discontinuity (aka *jump diffusion* models) have been developed (see e.g. Cox and Ross (1976); Bakshi, Cao and Chen (1997)).

Hagan et al. (2002) stated that local volatility models predicted the wrong dynamics for the implied volatility curve. The authors derived the *SABR model* that they claimed, *"captures the correct dynamics of the smile"* (Ibid, p. 84). Named after its parameters: Stochastic, Alpha, Beta, Rho, the SABR model has over the last decade gained vast popularity, especially in interest derivative markets.

1.2 Problem Statement

Without possible hedging opportunities, financial institutions cannot protect their clients' money. In order to hedge properly, a trader needs to be able to value instruments correctly. With real interest rates that are sometimes negative, pricing models that traders earlier have used might now appear obsolete. The Black-Scholes model breaks down and traders are nowadays switching toward models that can handle negative interest rates (e.g. the Normal Black model). Furthermore, some assumptions of the famous Black-Scholes model are quite unreasonable. Yet, despite its flaws, the model is still considered market standard and is commonly used as benchmark when evaluating new models.

If one would plot the implied volatility as a function of strike, he or she will most likely experience a parabola shaped curve instead of a straight line. This stands in great contrast to the assumptions by Black and Scholes (1973) that presumed constant volatilities with a one-to-one relationship between price of an option and the volatility of the underlying asset. Despite that traders usually quote options in implied volatilities, there still exist a problematic decision to choose what volatility to use in order to price an option.

As mentioned, attempts have been made in order to develop pricing models that handle the smile effects. One of the most praised one is the SABR model. It is a stochastic volatility model for forward LIBOR rates and is, in comparison to other volatility models, considered somewhat user friendly. Subsequently to Hagan et al.'s (2002) derivation of the SABR model, many papers evaluating its properties have been performed. Bartlett (2006) suggested a new set of risk formulas (updated Greeks) and West (2005) proposed a way for calibrating the model in illiquid markets. Later, Oblój (2008) showed a way of fine-tuning the calibration. Their suggestions can come to good usage when evaluating the aptness of the model in Hagan et al. (2002) and enhancing quality of the fit.

A swaption is an option to enter a swap. It gives the holder the possibility (but not the obligation) to enter into a swap at a certain future time. Swaptions are commonly priced using the Black-76 model, where the implied volatility is read from a volatility surface. Once again, choosing an appropriate measurement for the standard deviation of the derivative (due to the fact of the observed volatility smiles) is a testing task.

This problem is aggravated when one, for example, want to write an over-the-counter (OTC) derivative of a strike that cannot be read from a volatility surface. Will a calibration of the model create volatility surfaces that traders can use for pricing swaptions whose underlying have both short and long tenors? Are there various methods that are better to capture the smile effect that exists on the swaption market?

For the Swedish swaption market, little research has been published regarding the suitability of the SABR model. To evaluate the applicability of the SABR model will not only add to the current body on literature about the model in general, but it will also appraise country specific conditions, if any, in particular.

1.3 Review of Current Literature

In this part, we summarize preceding papers concerning mainly the empirical results of the SABR-model. Since papers investigating the performance of the model for swaptions are few in numbers, this section also includes papers relating to other financial instruments.

Hagan et al. (2002), the founders of the SABR model, could adequately show that local volatility models predicted the wrong dynamics (the opposite of what was expected) of the implied volatility curves when changes in underlying assets forward price occurred (see Section 2.7). Hagan et al. (2002, p. 93) were able to fit the implied volatility with good accuracy for various set of options and swaptions.

Later, Henrard (2005) compared the risk measurements for some swaptions pricing models. The author concluded that the normalized models performed better under the investigated period. Under his study, Henrard (2005) found that the delta of the models could differ up to ten percent. The author also saw a clear difference from models following a geometric Brownian motion to those with an arithmetic Brownian motion in their risk statistics. In a delta hedging contest, the Vasicek model (extended in Hull and White (1990)), outperformed the SABR model as well as the Black-76 model (Henrard 2005, p. 56).

West (2005) calibrated the SABR model for illiquid markets. The author claimed that the algorithm employed for finding the parameters (α, β, ν and ρ) made the results robust. Under an arbitrarily chosen constant beta of 0.7, West (2005 p. 383) found that rho and volvol (volatility of volatility) only change occasionally. He favoured a constant beta, which, in favour over a non-constant, reduced the hedging costs.

Rebonato, Pogudin and White (2008) tested the hedging performance of the SABR model and the LMM-SABR model (developed by Rebonato (2007)). The authors claimed that the SABR-model is well-specified and found support for correct and unbiased hedge ratios.[2] Despite positive results, the scholars addressed one flaw of the model, namely that it cannot

[2] A hedge ratio is the value of a position protected by a hedge compared with the size of the position itself.

incorporate jumps when they occur.[3] Similar results were found for the LMM extension of the model (Rebonato, Pogudin & White 2008, p. 99).

Wu (2011) investigated the pricing- and hedging performance for interest rate caps, using the SABR model. The results indicated good pricing correctness of the model as well as superior hedge ratios compared to the Black-76 model. Wu (2011, p. 11) pointed out that keeping the parameters ν and ρ simultaneously constant, produced larger pricing errors than when letting either one or both of them to vary (i.e. by recalibrating the model). Under his study, Wu (2011, p. 24) also found support that the altered Greeks by Bartlett (2006) outperformed those of Hagan et al. (2002).

Oblój (2008) presented a new refinement to the SABR model that tackled a small but persistent theoretical flaw of the original model. The author pointed out that the new correction term is consistent as $\beta \to 1$ and can thus eliminate the creation of wrong price in small strikes for large maturities region of the original model. Later in the thesis, this refinement will be reiterated and assessed alongside the model by Hagan et al. (2002).

Skov Hansen (2011) calibrated the SABR model for swaptions to fit with the observed market smiles. In his extensive thesis, the author found good fit for the model. Skov Hansen also pointed out that the delta risk is very dependent on parameterization used (different betas), and suggested a refinement to Hagan et al. (2002) in order to generate similar deltas regardless of beta employed.

Mercurio and Morini (2008) conveyed critique against the SABR and other local volatility models in the sense that the models did not show the desired behaviour in hedging. The authors claimed that the stochastic volatility models only worked in a desirable way when model-inconsistence hedging were applied (i.e. shifting the underlying ceteris paribus).

Regarding Swedish research, papers apropos of the SABR model are few in numbers. Boqvist and Sigurjonsson (2006) evaluated the SABR model for index options. They argued that the model sufficiently captures the volatility function of the market. On the other hand, the authors stressed that the model is strongly dependent on the quality of underlying data. Sjöstrand (2010) claimed in her paper that the SABR model slightly outperform the Black-Scholes model for European put options. However, the author only studied put options of one specific company under a very brief time.

[3] A “jump” occurs when a function is discontinuous for its entire domain.

1.4 The Aim of the Thesis

The aim of this paper is to describe the theory surrounding the SABR model and evaluate its suitability. We intend to alter the parameters in the model while employing two different methods of calibration to see how well it fits to authentic swaption data for the Swedish market.

1.5 Limitations

This paper is bounded to investigate swaptions that are traded on the Swedish market. The focus will be only on physical settled swaptions based on interest rate swaps (IRS). We assume that the counterparties involved in the swap contract agreed to exchange a fixed series of payment for a floating series of payment. The data provided to us is adjusted (such as interpolated). Thus, it does not necessarily correspond to accurate data if it could be observable on the marketplace.

Due to the nature of the model of investigation, some parameters will be chosen arbitrarily. These are selected based on our review of literature and should correspond to the most plausible. An account for intermediate values, hopefully, will only marginally affect the outcomes of this paper.

As with all pricing models, some assumptions about the characteristics of the market place need to be taken in to account and will be addressed further in later section.

In order to keep the paper concise, and to put a focus on the application, we will refrain from deeper mathematical derivation of areas outside the scope of this thesis. Furthermore, we leave out definitions of fundamental financial and mathematical concepts since we assume the plausible reader to be familiar with non-accounted terms. Due to hard, sometimes undefined, estimation methods, we will only present general theoretical illustration of risk terms and swaption pricing.

1.6 Data Selection, Assortment and Limitation

On the courtesy of Jan Röman and Swedbank AB, we have received market data for plain vanilla swaptions. The data is quoted in implied volatilities (Black-76) and was observed at the first of September 2013 using Swedbank's trading software Murex Mx3.

The set of data includes a volatility term structure (see Section 2.6) where ATM swaptions are quoted in Black-76 volatilities. There are eleven different maturities for the swaptions {1M,3M,6M,1Y,2Y,3Y,4Y,5Y,7Y,10Y,20Y} for ten different tenors of the underlying swap {1Y,2Y,3Y,4Y,5Y,7Y,10Y,12Y,15Y,20Y}. Every swaption is quoted for 17 strikes with a spread of 2 percentage points in each direction around the ATM rate. There are thus 1870 different theoretical prices that could be reproduced from the current data. For a full account of input data to our calibration, see Appendix 2.

The volatility surfaces have been interpolated and extrapolated for strikes and maturities of swaptions that are not traded. This is a standard industry procedure where the institution wants to cover the smile effect over an entire volatility surface for a specific derivative.

The data handed to us is already "fitted" to the market smile. This will consequently lead to biased results when we try to fit the SABR model to market data. Albeit this modification is of a small margin, we expect it will cause a smaller error of the smile fit. On the other hand, since we want to investigate the model from a more empirical point, we strive to use actual market data. Our result is then realistic and reflects the fallouts that actual traders get.

1.7 Disposition of the Paper

The rest of this paper is organized as follows: In Section 2, we will briefly recap the properties of the Black-Scholes model. There, a deeper understanding in specific financial instruments, especially caps/floors and swaptions will be presented. Section 3 will mainly be devoted to the SABR model as presented in Hagan et al. (2002). In Section 4, we will calibrate the model using data for the Swedish market. This Section also shows results and compares them under certain circumstances. The paper is concluded in Section 5. Finally, suggestions about further research on the topic are presented.

2 THEORETICAL FRAMEWORK

This Section starts by reviewing different interest rates and the properties of a swap and swaption contract. Later, we will recapitulate some theoretical aspects that can assist the understanding of the SABR model. Finally, the ground of motivation from which the SABR is developed.

2.1 Interest Rates

Here, we will briefly define different interest rates that act as fundamentals for the valuation of swaps and swaptions. As a general source, we refer to Röman (2012).

Spot rate: Spot rate $r(t_i)$ is the percentage of the amount invested (say X) one gets at time t_i when invested X in a zero coupon bond at time t_0 (i.e. today). In our case, this spot rate is quoted quarterly and is realized through bootstrapping the zero coupon bond yield curve.

Forward rate: Forward rate $f(t_i, t_j)$ is the percentage of X invested in zero coupon bond at time t_0 which one will get for the time between t_j and t_i in the future. This rate can be plotted through the spot rates curve. The relationship is as follows

$$(1 + r(t_1))^{t_1} \cdot (1 + f(t_2, t_1))^{t_2 - t_1} = \left(1 + r(t_2)\right)^{t_2}$$

Where after rearranging the terms can be stated as

$$f(t_2, t_1) = \left(\frac{(1 + r(t_2))^{t_2}}{(1 + r(t_1))^{t_1}}\right)^{\frac{1}{t_2 - t_1}} - 1$$

We see that the forward rate is more clearly described as the relationship between two spot rates. Another way to represent the forward rate is through the discount function $p(t_i, t_j)$ where $0 < i < j$, we have

$$p(0, t_1)p(t_1, t_2) = p(0, t_2) \Rightarrow p(t_1, t_2) = \frac{p(0, t_2)}{p(0, t_1)} = \frac{p(t_2)}{p(t_1)}$$

It is common to use continuous compounding interest rate. In this case the forward curve can be discounted by using e^{-rt}, it then becomes

$$e^{-r(t_1)t_1} e^{-f(t_2,t_1)(t_2 - t_1)} = e^{-r(t_2)t_2} \Rightarrow e^{-f(t_2,t_1)(t_2 - t_1)} = \frac{e^{-r(t_2)t_2}}{e^{-r(t_1)t_1}}$$

If we continue to simplify the above expression for forward rate, we arrive at a very neat result below

$$\Rightarrow f(t_2, t_1)(t_2 - t_1) = r(t_2)t_2 - r(t_1)t_1$$

$$\Rightarrow f(t_2, t_1) = \frac{r(t_2)t_2 - r(t_1)t_1}{(t_2 - t_1)} = \frac{r(t_2)t_2 + r(t_2)t_1 - r(t_2)t_1 - r(t_1)t_1}{(t_2 - t_1)}$$

$$= \frac{r(t_2)(t_2 - t_1) - t_1\big(r(t_2) + r(t_1)\big)}{(t_2 - t_1)} = \frac{r(t_2)(t_2 - t_1)}{(t_2 - t_1)} - \frac{t_1\big(r(t_2) + r(t_1)\big)}{(t_2 - t_1)}$$

$$= r(t_2) - \frac{t_1\big(r(t_2) + r(t_1)\big)}{(t_2 - t_1)}$$

Spot rates and forward rates for longer time (>3 years) are usually extracted by bootstrapping methods using swaps and zero coupon bonds and with known yield as constructors. For shorter periods, the liquid Forward Rate Agreements (FRAs) and short-term cash deposits are favoured. Interpolating and/or extrapolating techniques are commonly used to find the estimates for the missing rates so that traders get a (smooth) curve including all conceivable tenors. An example is found in Figure 1. There also exist many other techniques that can be employed such as the Newton-Raphson algorithm, or Nelson-Siegel (Svensson) parameterization along with a vast amount of interpolation procedures. For a good exposition, we invite the reader to Röman (2012).

Figure 1 - Bootstrapped Curve Example

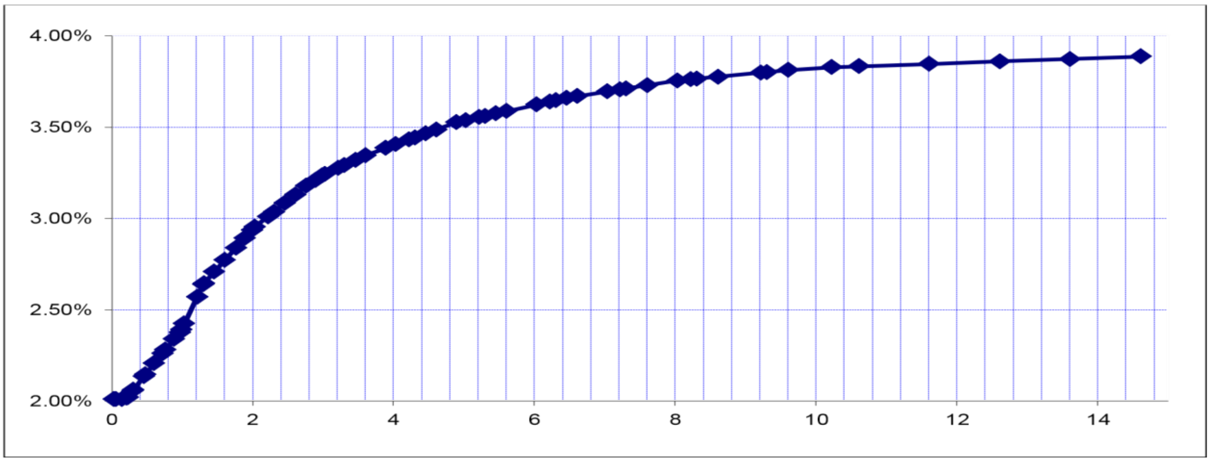


Figure 1 shows an example of how a bootstrapped curve could look like. The Figure shows the bootstrapped yield curve for Swedish bond data at 2006-04-24. Source: Röman (2012, p. 112).

Swap rate: Swap rate is the fixed interest rate that causes the swap starting value to be zero, for the Swedish market it is an interbank rate. This rate could be expressed as risk-free interest rate plus interest risk premium for the swap. In this paper, it is denoted as $R(0, T_S, T_E)$ and will be derived in the next section.

Discounting rate: Discounting rate is the rate used for discounting cash flows. In certain contexts, this equal to government's zero-coupon bond rate, which is usually a representation of the risk-free interest rate.

xIBOR rates: A xIBOR rate is the x InterBank Offered Rate where x refers to the body that fixes the rate[4]. It is rate in which a selection of banks is willing to lend to each other. The xIBOR rates are carefully monitored by traders. They act as indicators of the level of demand and supply on the financial markets.

[4] Two examples are LIBOR (London InterBank Offered Rate) and STIBOR (Stockholm Interbank Offered Rate)

2.2 Swaps and Swaptions

2.2.1 Swap

Swap is a financial derivative that allows an exchange of a series of payment for a different series of payment. The most common type of a swap is an IRS where the payments are depending on an underlying interest rate (e.g. a xIBOR rate).[5] These types of payments are referred to as having a fixed leg position (entering a payer swap) or a floating leg position (enter a receiver swap). Payments for these contracts usually occur on a semi-annual basis. The swap market is of a vast size where the value of traded contracts on interest rate far exceeds those of other instruments (such as commodities and equity). In order to grasp the usefulness of a swap we give a simple example.

Suppose that there are two companies A and B with different credit ranking and/or financial position. They both need to generate a loan of £5 million from their current banks. Company A can borrow the money using two options: it can pay either LIBOR or a fix rate at 6%.

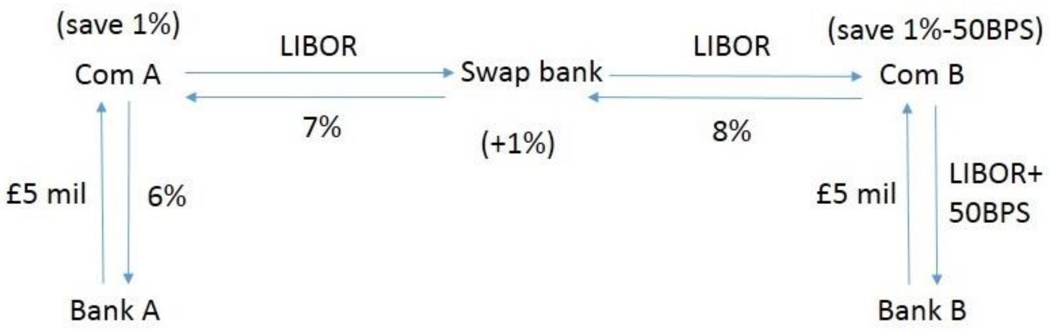
Company B has the possibility to borrow the money for LIBOR rate plus 50 BPS[6] or at the fix rate of 9%. Assume that, for whatever reason, these companies have different view about the future trend of interest rates; company A prefers to pay a floating rate and company B would like to pay the fix rate. In this case, an intermediate institution, "a swap bank", can aid both companies and benefit itself by issuing a swap. [7]

The swap bank can issue a swap in which company A will pay it the LIBOR rate and company B will receive the LIBOR rate, company B will then pay a fix rate of 8% and company A will receive 7% of its loan from the swap bank. The swap bank can then keep 1% for itself. By doing that, company A will get the fixed rate loan from its bank and pay back the money only with the rate of LIBOR – 1%. Company B will get the floating rate loan from its bank also and will have to pay a fixed rate of 8% plus 50 BPS. Bank A will get 6% and bank B will get the LIBOR + 50 BPS. Thanks to the present of the swap contract, all parties involved have gained equally or more than the initial scenario.

The swap market, in this way of functioning, depends on the comparative advantages companies bear when subjected to different interest rates (due to unlike borrowing conditions) and are commonly observable in reality. The diagram below gives an illustration of cash flows between the three parties.

[5] Other type of swaps are e.g. currency-, commodity and credit default swaps.
[6] Where 100 basis points (BPS) equals 1%.
[7] Obviously, a swaption can be sat up directly between two firms although using an intermediate institution will bear the risk if either of the companies defaults.

Figure 2 - Swap Market Cash Flow Illustration



In Figure 2, company A accepts to pay the bank 6% and company B accepts to pay Libor + 50 BPS. With a swap contract between the firms and the swap bank, company A ends up paying Libor – 1% while company B pays 8.5%. Under this setup, both companies are better off financially compared to if they initially would accept a floating rate loan (company A) and fixed rate loan (company B).

According to Skov Hansen (2011), the present value (PV) of a floating leg can be calculated as the sum of discounted forward rate payments. This forward rate can be extracted from the spot rate's yield curve by using some suitable bootstrapping technique, which was discussed in Section 2.1.

$$\boldsymbol{PV}^{float} = \sum_{i=S+1}^{E} \delta_i^{float} f(0, T_{i-1}, T_i) p(0, T_i)$$

Concerning a receiver swap, it is rather straightforward to calculate the present value of a payer swap since the payments are known in advance. The PV of the fixed leg is obtained by discounting fix rate payments over the entire tenor of the swap with E payment periods.

$$\boldsymbol{PV}^{fix} = \sum_{i=S+1}^{E} \delta_i^{fix} K p(0, T_i)$$

Where

δ_i^{float} is tenor of the floating leg (in years)

δ_i^{fix} is tenor of the fix leg (in years), could be different from that of the floating leg.

$f(0, T_{i-1}, T_i)$ is the future forward rate from T_{i-1} to T_i observed at time 0.

K is the known fix rate of payment.

$p(0, T_i)$ is the discount rate from T_i to today.

If we know let $R(0, T_S, T_E)$ to be the fixed rate which is set so that the present value of the swap contract is zero. That is only possible when $PV^{float} = PV^{fix}$. We have the following

$$\sum_{i=S+1}^{E} \delta_i^{float} f(0, T_{i-1}, T_i) p(0, T_i) = \sum_{i=S+1}^{E} \delta_i^{fix} R(0, T_S, T_E) p(0, T_i)$$

$$\Leftrightarrow \frac{\sum_{i=S+1}^{E} \delta_i^{float} f(0, T_{i-1}, T_i) p(0, T_i)}{\sum_{i=S+1}^{E} \delta_i^{fix} R(0, T_S, T_E) p(0, T_i)} = 1$$

Once solving for the swap rate, we get

$$R(0, T_S, T_E) = \frac{\sum_{i=S+1}^{E} \delta_i^{float} f(0, T_{i-1}, T_i) p(0, T_i)}{\sum_{i=S+1}^{E} \delta_i^{fix} p(0, T_i)}$$

One can also express the swap rate in terms of discounting factors to get a simpler expression of $R(0, T_S, T_E)$. As described in Röman (2012), we can describe the forward rate with respect to discounting factors.

$$p(0, T_i) = p(0, T_{i-1}) p(T_{i-1}, T_i) \Rightarrow \ p(0, T_i) = p(0, T_{i-1}) \frac{1}{1 + \delta_i^{float} f(0, T_{i-1}, T_i)}$$

$$\Rightarrow f(0, T_{i-1}, T_i) = \frac{1}{\delta_i^{float}} \frac{p(0, T_{i-1}) - p(0, T_i)}{p(0, T_i)}$$

Then the value of a floating leg can be expressed as

$$\sum_{i=S+1}^{E} \delta_i^{float} f(0, T_{i-1}, T_i) p(0, T_i) = \sum_{i=S+1}^{E} \delta_i^{float} \frac{1}{\delta_i^{float}} \frac{p(0, T_{i-1}) - p(0, T_i)}{p(0, T_i)} p(0, T_i)$$

$$= \sum_{i=S+1}^{E} [p(0, T_{i-1}) - p(0, T_i)] = p(0) - p(T_i)$$

$$= 1 - p(T)$$

So the swap rate is

$$R(0, T_S, T_E) = \frac{1 - p(T)}{\sum_{i=S+1}^{E} \delta_i^{fix} p(0, T_i)}$$

We can thus see that the swap rate can be expressed as a relationship between the floating rate and the fixed rate multiplied by a discounting rate. This rate, which equates the value of the fixed and floating rate, is also called swap rate or par swap rate. After the crisis, swap valuation has changed to a very complicated process. Banks have issues to determine the right forward and discounting rate that should be used to price this instrument. This has led to an inconsistency problem in swap valuation, which will not be further discussed in our paper. We only have ambition to present here the fundamental properties of a swap to assist reader for the next section.

2.2.2 Swaption

A swaption is an option on a swap that serves as a right but not an obligation to enter into a swap at a specified future date. Swaptions are frequently used by banks and other financial institutes with the purpose of hedging cash flow against the exposure to random events such as fluctuations of interest rates or currency rates, but are not common instruments for private investors. The most common swaptions are the so-called plain vanilla swaptions. It is of a European type with an IRS (which is accounted for in the previous section) as underlying asset.[8]

Settlement of swaptions can occur either by physical settlement where actual exchanges of cash flows on the underlying swap takes place, or by cash settlements where the value of the underlying swap is paid at the time of exercise.

As with a swap, there are two sides of every swaptions. They are known as payer- and receiver swaptions where each party's position is named relative to the fixed leg. The person entering a payer swaption get the right to enter a swap where he or she pays a fix leg and receive a floating leg. The holder of a receiver swaption can, at a future date, enter into receiver swap where he or she receives a fixed leg and pay the floating.

A swaption is usually denoted as a nYmY swaption where m is the tenor of the underlying swap and n is the time to maturity of the swaptions. A 1Y10Y ("one into ten" or "in-one-for-ten") payer swaptions gives the holder the option to enter a 10-year swap (paying fixed leg, receiving floating) in one (1) year.

Extending the previous valuation of a swap, one now needs to extend the notation to account for the right to exercise. The value of a physically settled payer swaption observed at time t,

$$PS_t^{phys} = A(t, T_S, T_E)[(R(t, T_S, T_E) - K)^+]$$

where the term

$$A(t, T_S, T_E) = \sum_{i=s+1}^{E} \delta_i^{fix} p(t, T_i)$$

is also called the annuity of the swap.[9]

Swaptions are mainly traded OTC with absent regulations and are usually quoted in Black-76 (log-normal) volatility. They can also be quoted in price (e.g. USD) or in other forms of implied volatilities (normal volatility). If quoted in implied volatility, one then plug it into the "correct" model to get the dollar equivalent price.

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[8] There also exist e.g. American and Bermudian swaptions that are of a more complex type.

[9] In this context the + denotes the maximum value between the calculated price and zero.

2.3 Martingales

So far, we have gained understanding about the characteristics and the present value of swaptions. In the following sections, we continue to show how to price a swaption in the Black-Scholes framework, where the market is assumed free of arbitrage. In order to fulfil that assumption, we first need to look at the definition of a martingale and the martingale representation Theorem.

A martingale is an integrable stochastic process that represents the notion of fair game in mathematics.[10] A martingale implies that at a particular time t in a sequence of random variables, given all the knowledge of past results, the expected outcome at time s is the same as the result at the current time t, where 0 < t < s. We have the following,

Definition 1. *Continuous-time martingale*

A continuous-time stochastic process {X(t)} defined on the probability space (Ω,F,P) with respect to filtration $\{F_t\}$ where F_t represents the information generated by X(t) on the time interval [0, T], is a martingale if

$$(1).\ \ E[|X(t)|] < \infty\ for\ each\ t\ \in [0, T], and$$

$$(2).\ \ E_t[X(s)] = X(t), t < s \leq T,$$

where $E_t[X(s)]$ denotes the conditional expectation of $X(s)$ given the information F_t.

Definition of Martingale is referred to Kijima (2003).

Theorem 1. *Martingale representation Theorem*

Let W_t be a standard Brownian motion, m_t be a martingale process adapted to filtration $\{F_t\}$ where $t\ \in [0, T]$, then there exists a uniquely determined $F_t -$ adapter stochastic process $c(t,*)_s$ such that

$$m_t = m_0 + \int_0^t c(t,*)_s dW_s \quad or\ (equivalently) \quad dm_t = c(t,*)_t dW_t$$

Martingale representation Theorem is referred to Björk (2004). Here the uncertain growth of a martingale process m_t is equal to the Brownian motion development multiplied by some process $c(t,*)_s$. For example, $c(t,*)$ could be a function of S, K, r, σ, t.

2.4 Arbitrage Free Theory

The First Fundamental Theorem of asset pricing ensures with necessary and sufficient condition that the market is free of arbitrage and complete. The following theorem is extracted from Björk (2004).

[10] A fair game is a game in which each participant is not more likely to win than another player is.

Theorem 2. *Arbitrage Free Theorem*

The market is free of arbitrage if and only if there exists a risk-neutral probability measure Q such that the discounted price process $\left(\frac{V_t}{g_t}\right)$ is a martingale, given the time horizon T, a risky asset with V_t as its price process and a risk-free asset g_t.

Now consider a plain vanilla European swaption with a fixed strike rate R_{fix}, R_t is the swaption's forward rate at time t and R_0 is the forward swap rate as of today. As discussed in the earlier section, the value of a payer swaption can be expressed as

$$PS_t^{phys} = A(t, T_S, T_E)[(R_t - R_{fix})^+]$$

Here, in order to follow the Arbitrage Free theory, the factor R_t should be a martingale. Expressed mathematically we have

$$d\widehat{R_t} = c(t,*)dW, \qquad d\widehat{R_t} = R_0$$

Once again, dW is a Brownian motion and the factor $c(t,*)$ is a determinable function of time and other parameters while $\widehat{R_t}$ is the expected value of the swaption's forward rate. In other words, to assure that the market is free of arbitrage opportunity, the forward price process should be expressed as a random Brownian process multiplied by a scale factor $c(t,*)$. To go further in pricing this option, a suitable model for $c(t,*)$ should be postulated. This is the arbitrage free pricing theory where the Black-Scholes model based on.

2.5 Black-Scholes Model

In this section, we will give a brief presentation to the Black-Scholes model. For a full account, we invite the reader to e.g. Black and Scholes (1973) or Chapter 13 in Hull (2009).

2.5.1 Assumptions

In order for the Partial Differential Equation (2.5.2) below to hold, some assumptions need to be carried out.

- The stock price follow a lognormal distribution where the returns are normally distributed
- Short selling is allowed
- No transaction costs or taxes
- No dividends during the life of the derivative
- No arbitrage opportunities
- Continuous trading in securities
- Constant risk-free interest rate
- Constant volatility in the underlying asset

2.5.2 The Black-Scholes Equations and Formulas

Given the above assumption, where σ_B represents the assumed constant volatility, S is the asset price process given by a geometric Brownian motion, $dW \sim N(0,1)$ is a Wiener process with mean 0 and a variance $\sigma^2 = 1$. If μ is the stock's expected return we have the following

$$dS = \mu S dt + \sigma_B S dW \quad where\ S_0 = s \tag{2.5.1}$$

A derivation of the Black-Scholes Partial Differential Equation usually includes setting up a risk-free portfolio consisting of one derivative, which has value of f and one part of underlying asset. Another possibility is to use the capital asset pricing model (CAPM).[11] Regardless of derivation techniques, we land in the famous Black-Scholes Partial Differential Equation.

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{2.5.2}$$

The above Partial Differential Equation (PDE) has many solutions in regards to different derivatives with S being the underlying variable. To solve the equation for a particular derivative, the boundary conditions that specify values of the derivative at the boundaries of possible S and t have to be determined. For example, in case of a European call option, the key boundary condition is

$$f = (S_t - K)^+ \ when\ \ t = T$$

Equation (2.5.2) can also be understood in term of the Greeks. $\frac{\partial f}{\partial t}$ is the change in value of option with respect to time, in other word the Θ - Theta. $\frac{\partial f}{\partial S}$ is the change in value of option with respect to a change in underlying asset's price, can be denoted as Δ – delta. The last derivative $\frac{\partial^2 f}{\partial S^2}$ is the so-called Γ – gamma. It represents the rate of change of option's delta with respect to the underlying stock price. Therefore, we can rewrite Equation (2.5.2) as

$$\theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rf$$

This PDE has excluded the dependence of the option's value to risk preferences, which is the expected return on a stock, μ. Since an investor is only willing to pay for a highly risky asset if he believes that the expected return will be high, μ thus depends greatly on risk preference. This exclusion of the expected return of the underlying asset μ has consequently given the Black-Scholes model a great advantage.

From the above PDE, price of a call option, c, can be obtained to be[12]

[11] See for example Chapter 13 of Hull (2009) for a derivation.

[12] One approach is to define the contingent claim as $(S_t\text{-}K)^+$ and the use the Itô lemma on the process of underlying asset to reach a Feynman-Kac representation of the claim. Steps included are to integrate the value of the call option over a lower limit with respect to K, σ, r, T and S and finally to express the value of a plain vanilla European call option as shown above. See e.g. Röman (2012) for a deeper derivation.

$$c = S_0\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2) \tag{2.5.3}$$

While the price of the put, p, can be calculated by similar techniques or via the put-call-parity

$$p = Ke^{-rT}\mathcal{N}(-d_2) - S_0\mathcal{N}(-d_1) \tag{2.5.4}$$

where

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

and $\mathcal{N}(x)$ is the cumulative normal distributed function (explained further in Appendix 1). Observe that the option's value is now depending only on the maturity T, the risk-free interest rate r, variance σ^2 and the moneyness of the option S_0/K. Here, the only vague value to determine is the variance σ^2, which is usually calculated based on historical return data and is assumed constant through the option's life. Merton (1973) has shown that the call option's value is positively correlated with the value of T, r or σ^2. As these variables increase, the value of a call option is approaching its maximum, which is the stock price. Nowadays, simple call and put options are still widely valued using this model.

2.5.3 Extensions and Critique of the Black-Scholes Model

Black and Scholes or Merton did not trade upon their framework – at first. In 1993, Black and Merton founded Long Term Capital Management (LTCM) and attracted vast capital to their hedge found. Initially, it started out good with high returns during the first three years. In late 90s when Asian and Russian financial crises struck the markets, LTCM betted against it. The results were disastrous. The Federal Reserve eventually had to bail out the hedge fund in order to prevent a complete financial meltdown.[13]

Despite that the Black-Scholes model is considered as the benchmark for option pricing, it has received its vast share of critique. First, the distribution of the underlying prices is in fact not generally log-normally distributed. Traders instead assume a heavier left tail and a less heavy right tail (Hull 2009, p. 400). This is roughly the same as saying that the log returns are not normally distributed due to the fact of too many "extreme" movements that are creating kurtosis in the distribution. In addition, assets prices jumps frequently, thus violating the assumption of smooth movements. Instead, traders use volatility smiles to allow for the non-lognormality (Ibid). Secondly, the assumption of continuous and instant trading without any transactions costs is a falsification of reality and is thus a vast limitation to the Black-

[13] For a full account on the history behind the crash of LCTM we recommend the documentary *The Midas Formula: Trillion Dollar Debt* (Clark, 1999).

Scholes model. It is thus not so hard to understand how LCTM could fail when the markets do not act like theory.

The Black-76 Model

Black (1976) made some slight modifications to his and his colleagues preceding work. Instead of using the spot price of an underlying asset, S_0, the model now discounts a forward price, $\hat{F}$. This model is more suitable to price interest rate derivatives (i.e. bond options, interest caps/floors and swaptions). With notation earlier introduced, Black postulated that $c(t,*)$ is $\sigma_B \hat{F}(t)$. Under the Black-76 model, the theoretical value of a call and a put reads

$$d\hat{F} = \sigma_B \hat{F}(t)dW \qquad where \qquad \hat{F}(0) = F.$$

$$\mathrm{c}_{76} = e^{-rT}[\mathrm{F}\mathcal{N}(\mathrm{d}_1) - K\mathcal{N}(\mathrm{d}_2)]$$

$$\mathrm{p}_{76} = e^{-rT}[\mathrm{K}\mathcal{N}(-\mathrm{d}_2) - \mathrm{F}\mathcal{N}(-\mathrm{d}_1)]$$

$$\mathrm{d}_1 = \frac{\log(F/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$\mathrm{d}_2 = \frac{\log(F/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = \mathrm{d}_1 - \sigma\sqrt{T}$$

Swaption under the Black-76 model

Among other derivatives instruments, swaptions can also be included in this pricing model as a more complicated type of option. Based on the definitions given earlier, value of a payer swaption (PS) can be seen as a call option on the swap rate $R(t, T_S, T_E)$ with strike price K. Using the Black-76, we get

$$PS = A(t, T_S, T_E)[R(t, T_S, T_E)\mathcal{N}(d_1) - K\mathcal{N}(d_2)]$$

Where

$$d_1 = \frac{log\left[\frac{R(t, T_S, T_E)}{K}\right] + \frac{1}{2}\sigma_{S,E}^2(T_S - t)}{\sigma_{S,E}\sqrt{T_S - t}},$$

$$d_2 = d_1 - \sigma_{S,E}\sqrt{T_S - t}$$

Explicitly, if we consider the discount function in detail, as done in Röman (2012) we can express the value of a payer and receiver swaption (RS) as following

$$PS = \frac{1 - \frac{1}{(1 + F/m)^{\tau m}}}{F} e^{-rT}[F\mathcal{N}(d_1) - K\mathcal{N}(d_2)]$$

$$RS = \frac{1 - \frac{1}{(1 + F/m)^{\tau m}}}{F} e^{-rT}[K\mathcal{N}(-d_2) - F\mathcal{N}(-d_1)]$$

$$d_1 = \frac{\log(F/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$$

Where

τ = The time between swaption maturity and underlying swap maturity – tenor.

m = Compounding swap rate per year.

F = Forward rate of the underlying swap $R(t, T_S, T_E)$, also called par swap or swap rate.

K = Strike rate of the swaption.

r = Risk-free interest rate.

T = Time to swaption expiration in years.

σ = Volatility of the forward starting swap rate.

The Normal Black Scholes Model

Another extension to the Black-Scholes framework is the *Normal Black Scholes Model* developed by Iwasawa (2001). Iwasawa pointed out that some traders believe that the traded assets follow a normal distribution rather than a lognormal ditto, as assumed in in Black-Scholes model. In the model by Iwasawa (2001), the underlying asset is allowed to take on negative values. We have the following theoretical prices of a call and put[14]

$$c_{normal} = e^{-r(T-t)}\left[(\mathrm{F} - \mathrm{K})\mathcal{N}(\mathrm{d}_1) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-d_1^2/2}\right] \tag{2.5.5}$$

$$p_{normal} = e^{-r(T-t)}\left[(\mathrm{K} - \mathrm{F})\mathcal{N}(-\mathrm{d}_1) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-d_1^2/2}\right] \tag{2.5.6}$$

Where $d_1 = \frac{F-K}{\sigma\sqrt{T-t}}$

It is thus possible to realize that by inspecting the formulas for d_1, Equation (2.5.5) and Equation (2.5.6), the strike price, K as well as the current future price F can take on any value, including negative values. With huge amounts of OTC traded derivatives where underlying instruments can be negative (such as derivatives on any real interest rates), the Normal Black Model could then act as a suitable alternative where the Black-Scholes model breaks down.

To summarize, the Black-Scholes option-pricing model has its advantages (e.g. its simplicity) but the framework is built upon a set of assumptions that are strongly questionable. Hence, many attempts have been made to come up with either new alternatives or expansions and/or adjustments the Black-Scholes model. These models are based on fewer assumptions.

[14] Iwasawa (2001) also derives the normal process under a bounded negative assumption where the underlying prices cannot fall below a certain level.

2.5.4 Greeks under the Black-Scholes Model

In order to derive the Greeks in the SABR model later, we should first look at this set of the basic Greeks. However, we will not take into account all Greeks but only the most important two, which are Delta and Vega. For other Greeks such as Theta, Gamma and Rho, the reader could refer to additional readings in e.g. Chapter 17 of Hull (2009).

Delta

The delta of an option is the rate of change in the price of the option caused by a change in the price of the underlying asset. Under the Black-Scholes model, the delta of a call option is usually denoted

$$\Delta = \frac{\partial c}{\partial S}$$

It is possible to express the delta of a call or a put from the cumulative normal distribution functions where d_1 is defined as in Section 2.5.2. For European options we have

$$\Delta_{call} = \mathcal{N}(\mathrm{d}_1)$$

$$\Delta_{put} = \mathcal{N}(\mathrm{d}_1) - 1$$

Delta for a call option is always positive and approaches one for options that are deep ITM.

$$\Delta_{call} \in [0,1]$$

While delta for a put is bounded to the closed interval

$$\Delta_{put} \in [-1,0]$$

For investors, the concept of delta hedging is of great importance. If a portfolio is set up so that it has a delta equal zero, it is known to be delta neutral. This implies that the value of the portfolio stays the same when a change in underlying occurs. In order to maintain a delta neutral portfolio, it must be rebalanced on a frequency basis. Actually, if one wants to have a delta neutral portfolio, he or she must use dynamic hedging (thus continuously rebalancing the portfolio). This would obviously invoke great transaction costs so the concept of continuous hedging is only appropriate in theory. In practice, delta hedging for financial institutions is performed on a daily basis (Hull 2009, p. 377).

With a delta neutral portfolio, one can see that the Equation (2.5.2) is reduced to depend only on theta and gamma, as shown below

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

Vega

Vega[15] of an option is associated with a change in the option price caused by a change in the volatility of the underlying asset. The Vega of a call option in the Black-Scholes model is thus

$$\Lambda = \frac{\partial c}{\partial \sigma}$$

For European options, vega is calculated as

$$\Lambda_{call,put} = S_0\sqrt{T}\mathcal{N}'(d_1)$$

$$\Lambda_{call,put} > 0$$

Where d_1 is defined as in Section 2.5.2. From the First Fundamental Theorem of Calculus, $\mathcal{N}'(x)$ is the probability density function for a standard normal random variable.[16] Vega is always positive for both put- call options, and is at greatest where the option is ATM.

Since volatility, σ is an assumed constant in the Black-Scholes world, the concept of a change is counterintuitive. However, for stochastic volatility models (such as the SABR model) where the volatility is non-constant, vega is more appealing. A vega neutral portfolio (i.e. when Λ = 0) would then keep its value if a change in volatility occurred.

2.6 Volatility Smiles, Skews and Surfaces

Volatility smile is the plot of the implied volatility of an option and its strike price. It is not until after the stock market crash in October 1987 that the pattern of the plot appeared as a smile. Before the crash, implied volatility depended on strike to a much lesser extent. Rubinstein (1994) suggested that this is because traders are more concerned of another crash and thus, price their options accordingly. This phenomenon is also known as "crash-o-phobia" and has been supported with several empirical observations. History shows that declines in the S&P 500 have resulted in steeper volatility skew. On the other hand, increases tend to become less steep (Hull 2009, p. 395).

In the Black-Scholes model, it is impossible to solve for the implied volatility explicitly. Instead, e.g. the Newton-Raphson method or the method by Chance (1996), must be used to find the implied σ so that the theoretical price agrees with the observed market price. Since the data we are using to calibrate the SABR model is already quoted in implied (Black-76) volatility, we will not discuss further the method use to solve for volatility.

Once one is able to obtain implied volatilities for a set of different strikes, the volatility smile can be plotted. Figure 3 is the result for the actual data of 1M15Y swaption.

[15] Vega is actually not a Greek letter. Yet it is still referred to as one of the Greeks within the field of finance and we will adopt that lingo throughout this paper. We denote Vega with the capital Lambda.
[16] See appendix 1 for formulae for $\mathcal{N}'(x)$.

Figure 3 - 1M15Y Swaption Smile

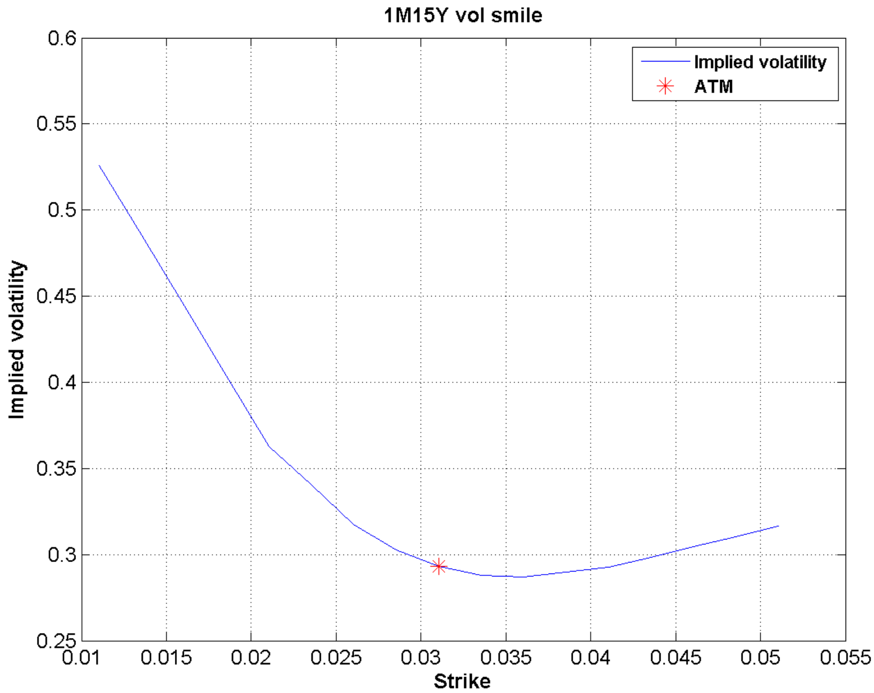


Figure 3 shows the implied volatility for a 1M15Y swaption as a function of the strike. Simple linear regression is done by MATLAB between the 17 observations. The ATM rate was circa 3.10%, on first of September 2013, which is indicated by the red star. From inspecting the smile, we see that the implied volatility resembles a parabola-shaped form with a minimum value of approximately 29%. We can also see that the smile is asymmetrical but clearly that the market prices this derivative with different implied volatilities depending on different strikes. To experience a perfect shaped smile on the market is extremely implausible, the smile in Figure 3 should be considered as a “good” actual smile.

The basic definition of a volatility smile is, as mentioned, a parabola shaped curve where ITM and OTM money options have a higher implied volatility than ATM options. [17] There are also tweaks to the curve where the smile is of a different shape. First, if ITM calls and OTM puts are traded with a higher implied volatility, this pattern is known as volatility smirk or reverse skew. This is illustrated in Figure 4.

[17] Due to no-arbitrage arguments, European puts and calls with the same underlying, strike price and maturity date will have the same implied volatility and thus create identical smiles (Hull 2009, p. 389).

Figure 4 - Reverse Skew Smile

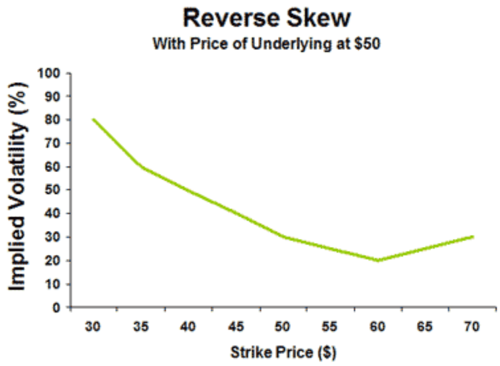


Figure 5 - Forward Skew Smile

Figure 4 (left) and Figure 5 (right) illustrates a reverse skew and a forward skew respectively.
(Source: http://www.theoptionsguide.com/volatility-smile.aspx)

A reverse skew that appears on the market is due to the fear traders possess against a new (vast) crash. The implied probability distribution of e.g. equity options with a reverse skew tend to have heavier left tails (less heavy right tails) than lognormal distribution (Hull 2009, p. 394). On the other hand, if one experiences the opposite with higher implied volatilities for ITM puts and OTM calls, the pattern arising is known as a forward skew (as is illustrated in Figure 5). A forward skew pattern can arise in e.g. commodities markets where the expectations about declined future harvests due to drought, frost or any non-controllable factor that will make the traders to drive up the demand for OTM calls.

By taking a second look at Figure 3, we can see that the smile of the 1M15Y swaption would resemble a reverse skew. This should not be surprising since one might expect a large decline in the interest rate due to some extreme event, e.g. a larger recession.

If one plots the implied volatility as a function of both the strike price and time to maturity, the resulting 3D-surface is known as the implied volatility surface or volatility cube. This is illustrated in Figure 6.[18] For swaptions, an alternative is to use ATM options with various tenors and times to expire to visualize the volatility term structure. An example can be found in Figure 8. Any procedure to construct a volatility cube will require some kind of "intelligent interpolation" (Lesniewski 2008, p. 13). In the examples below, MATLAB has done this automatically.

[18] An alternative would be to plot the tenor of the underlying swap for a different set of strikes while keeping the time to expiry constant.

Figure 6 - 15Y Volatility Surface

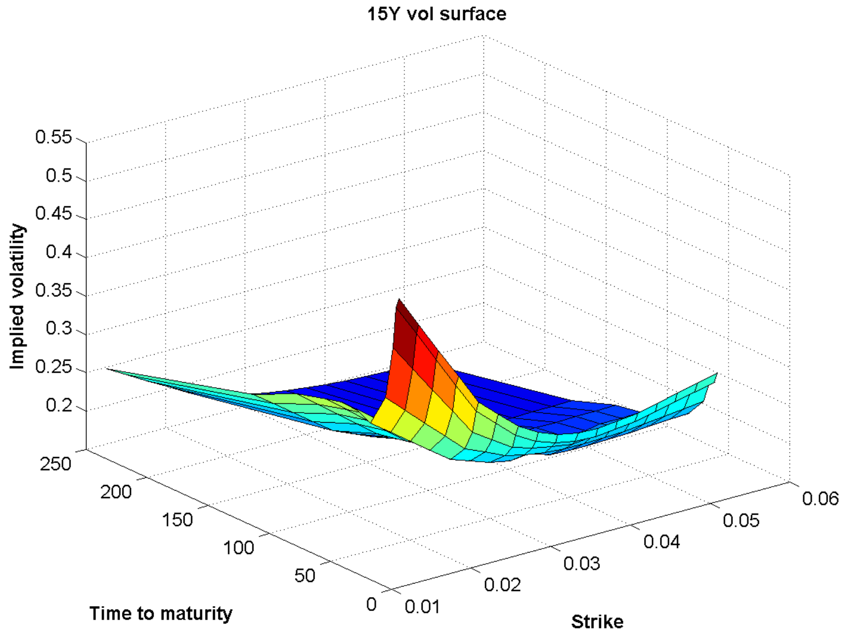


Figure 5 generated by MATLAB shows a 15-year volatility surface as of first of September 2013. The at-the-money swap rate is approximately 3.10%. Maximum implied volatility is 52.6% while the minimum equals 15.55%.

One can see from Figure 6 that the implied volatility is much greater for expiries that are short into the future while swaptions that expires later on in the future are generally traded with a much lower implied volatility. Looking at the shortest time to maturity, the smile corresponds to the volatility smile of a 1M15Y swaption that was plotted in Figure 3. It could be observed that individual smiles for each possible time of expiry take on a reverse skew pattern.

We can now visually conclude that the assumption about a constant volatility for all possible strikes is very farfetched. However, the longer is time to maturity the flatter are the curves. For swaptions with an expiry of more than 10 years, we would only experience a small smirk if the volatility surface were plotted. This could be seen by looking at the 15-year volatility surface from another angle, see Figure 7. Here we see how flat the surface actually is where it is rather flat over its domain.

Figure 7 - 15Y Volatility Surface (different angle)

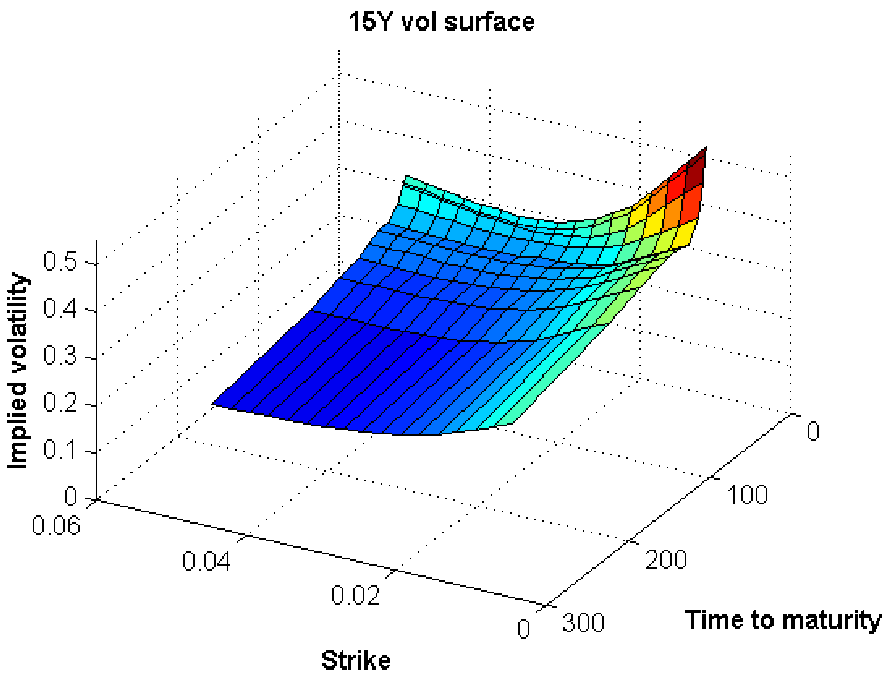


Figure 8 illustrates the volatility cube for ATM swaptions. Here we note that the surface is very unsmooth with many local maximum and minimum points. However, the general pattern is that ATM swaptions with a short time to expiration are traded at a higher implied volatility compared to ATM swaptions that can be exercised in more than 10 years.

Figure 8 - ATM Swaption Volatilities

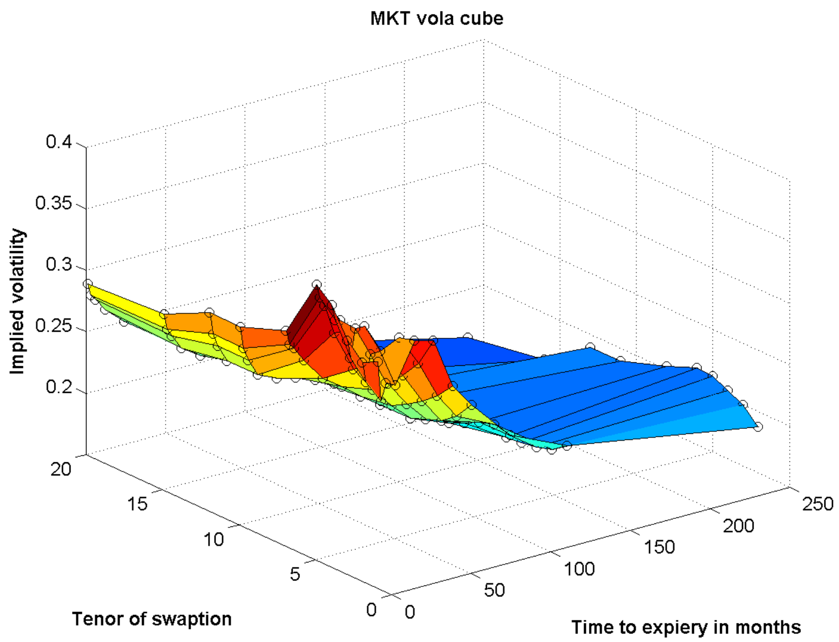


Figure 8 is created by MATLAB using ATM swaption volatilities (see Appendix 2) at the 1st of September 2013. The maximum point is 37.4% while the minimum is 16.1%.

2.7 Local Volatility Models

Before the creation of SABR model, the local volatility model by Derman and Kani (1994) was an attempt made to create single, self-consistent model to extract volatilities for any strikes. According to the authors, perhaps the most direct and simple way to tackle the constant volatility in the Black–Scholes is by replacing Equation (2.5.1) with

$$\frac{dF}{F} = \mu(t)dt + \sigma_{loc}(F,t)dW$$

Where $\mu(t)$ is the risk neutral drift depending on time and $\sigma_{loc}(F,t)$ is the local volatility function depending on the forward price F and time t. Instead of a constant volatility, now $\sigma_{loc}(F,t)$ will be deduced numerically from the smile. Using binomial option pricing technique, a local volatility surface is created that causes the binomial tree's option prices to be consistent with the market ones. This model is arbitrage-free, preference-free, self-consistent and it avoids additional factors to the Black-Scholes. From $\sigma_{loc}(F,t)$ obtained after calibration, it calculates correct market price of options (calls and puts) for all strikes and exercise dates. Therefore, this model is a natural and easy way to value option only from observable market data.

In the initial setup, the following stock price process must be valid

$$F_i = p_i S_{i+1} + (1 - p_i)S_i$$

$$or \quad p_i = \frac{F_i - S_i}{S_{i+1} - S_i}$$

Where F_i is a known forward price, p_i is the unknown transition probability to the upper node $i+1$, $(1-p_i)$ is the transition probability to the lower node i , S_{i+1} is value of the stock at node $i+1$, S_i is stock value at node i. Because the implied tree is measured in risk-neutral condition, the expected value of the stock price $p_i S_{i+1} + (1-p_i)S_i$ must be equal to its known forward price F_i. At each node of the tree, a call option with strike price K, maturity t_{n+1} is valued, using Arrow-Debreu[19] stock prices λ_j as follow

$$C(K, t_{n+1}) = e^{-r\Delta t} \sum_{j+1}^{n} \left[\lambda_j p_j + \lambda_{j+1}(1 - p_{j+1})\right](S_{j+1} - K)^+$$

This formula is the common risk-neutral option valuation, where at every node, the option's price is the difference between stock's price and forward price or zero multiply with the expected value of Arrow-Debreu stock prices$\left[\lambda_j p_j + \lambda_{j+1}(1 - p_{j+1})\right]$. Every node is then summed and discounted back to current time zero. Next is to determine stock price on the upper node

[19] Arrow-Debreu pricing model is based on equilibrium theory of supply and demand. In this model, the first Arrow-Debreu price is 1 and the next upper/lower price is $\lambda_a = \frac{\lambda_0 p}{r}$ where p is the transition probability to upper/lower price.

$$S_{i+1} = \frac{S[e^{r\Delta t}C(S, t_{n+1}) + \lambda_i S - \Sigma.]}{\lambda_i F_i - e^{r\Delta t}C(S, t_{n+1}) + \Sigma.}$$

Where $\Sigma. = \sum_{j=1}^{i-1} \lambda_j (s_i - F_j)$

This sum is equal to zero (Derman and Kani, 1994). The stock price for the lower node is

$$S_i = \frac{S^2}{S_{i+1}}$$

S is the centre node at the previous level, according to the logarithmic Cox-Ross-Rubinstein centering condition the author chose, the above relationship holds.

Finally, implied volatility at node i, σ_i is given as

$$\sigma_i = \sqrt{p_i(1-p_i)}\log(S_{i+1}/S_i)$$

This implied volatility σ_i has been calculated based on the risk-free transition probability and the logarithm difference of the possible values (either up or down) of S at each time step. Where S is deduced directly from the market value of the option with strike K and time to expiry t_{n+1}.

For a full account of how the model works and its derivations, readers can refer to Derman and Kani (1994). Hagan (2002, p. 87) pointed out that the model, unfortunately, *"predicts the wrong dynamics of the implied volatility curve, which leads to inaccurate and often unstable hedge"*. To examine the dynamics of this model, first, we have to simplify the initial setup by omitting the time variable and that will leave us with

$$\frac{d\hat{F}}{F} = \sigma_{loc}(\hat{F})\hat{F}dW \qquad where \qquad \hat{F}(0) = f$$

Using singular perturbation technique to analyse this model, the market implied volatility to apply in the Black-76's formula $\sigma_I(K, f)$ to price option could be explained as

$$\sigma_I(K,f) = \sigma_{loc}\left(\frac{1}{2}[f+K]\right)\left[1 + \frac{1}{24}\frac{\frac{d^2\sigma_{loc}}{df^2}\left(\frac{1}{2}[f+K]\right)}{\sigma_{loc}\left(\frac{1}{2}[f+K]\right)}(f-K)^2 + h(K,f)\right]$$

According to evaluation of the equation's right hand side by Hagan et al. (2002), the volatility depends largely on the first term. The second term gives little adjustment to the result and the omitted terms notated with $h(K, f)$ only account for less than 1% of the first term. Therefore, the market implied volatility that is also understood as

$$\sigma_I(K,f) = \sigma_{loc}\left(\frac{1}{2}[f+K]\right)$$

Suppose f_0 is today's forward price, $\sigma_I^0(K)$ is the implied volatility curve observed from the market for some strike K at time 0 (today). The local volatility after calibration to fit the market is

$$\sigma_{loc}(f) = \sigma_I^0(2f - f_0) \tag{2.7.1}$$

Because, for the observed implied volatility to be consistent with (2.7.1), we have the following

$$\sigma_I^0(2f - f_0) = \sigma_{loc}\left(\frac{1}{2}[f_0 + |2f - f_0|]\right) = \sigma_{loc}\left(\frac{1}{2}2f\right) = \sigma_{loc}(f)$$

Here we can use the absolute value condition because f is by definition a forward price so it should not be negative. This Having obtained $\sigma_{loc}(f)$, now let the forward price shift to a new value of f, the new implied volatility predicted by the model is

$$\sigma_I(K, f) = \sigma_{loc}\left(\frac{1}{2}[f + K]\right) = \sigma_I^0\left(2\frac{f + K}{2} - f_0\right) = \sigma_I^0(K + |f - f_0|)$$

arrival formula of the local volatility tells us some unusual properties of the model. Intuitively, when the forward f increases, volatility curve is expected to shift to the right hand side. However, the local volatility will shift the smile to the opposite hand side. A graphical illustration found below in Figure 9.

Figure 9 - Dynamics from Local Volatility Models

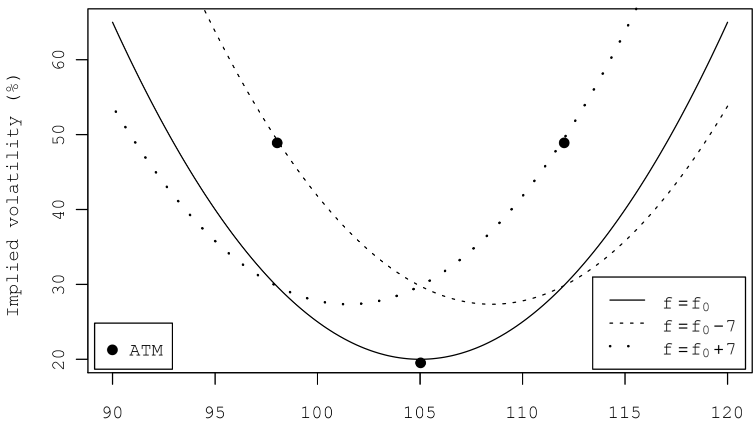


Figure 9 shows the unexpected (opposite) shifts in the implied volatilities that are the results when the forward prices are altered under a local volatility model. Source: Skov Hansen (2011, p. 33).

This inconsistency has also affected on the calculated risks from this model. Delta and vega hedging falls due to the wrong dynamics predicted. Therefore, unfortunately, this model cannot be applied correctly with reality. However, the model has produced future distributions of the stock prices through market quoted option's price. Derman and Kani (1994) believed that this model can be very useful in case of barrier options, where the striking barrier probability is especially sensitive to the implied volatility smile.

3 THE SABR MODEL

In this section, given the scenario of the local volatility model, the Black-76 model, definitions of swaption and different interest rates, we will present the SABR model as well as the properties of its parameters. In addition, the original Greeks under the model accompanied with an updated set by Barlett (2006) and finally the adjustment of the SABR will also be mentioned.

3.1 The Original Formula

From Hagan et al. (2002), the original SABR model can be described as

$$d\hat{F} = \hat{\alpha}\hat{F}^{\beta} dW_1 \tag{3.1.1}$$
$$\hat{F}(0) = f$$
$$d\hat{\alpha} = v\hat{\alpha} dW_2$$
$$\hat{\alpha}(0) = \alpha$$
$$dW_1 dW_2 = \rho dt$$

In which, the forward price process is denoted as $\hat{F}$, the volatility $\hat{\alpha}$ is a stochastic process. Both W_1 and W_2 are Brownian motions without drift which are correlated by a coefficient ρ. Therefore, in this model, the volatility process is allowed to be random through the development of $\hat{\alpha}$, which is scaled up by including the factor volvol, v. This extra randomness has solved the problem of constant volatility, which is an unrealistic assumption of the Black-Scholes model.

Recall that the price of a European option with strike price K, time to maturity T, by Black-76 formula is

$$c_{76} = e^{-rT}[f\mathcal{N}(d_1) - K\mathcal{N}(d_2)]$$
$$p_{76} = c_{76} + e^{-rT}[K - f]$$

where the put price is retrieved from the put-call parity and

$$d_{1,2} = \frac{log(f/K) \pm \frac{1}{2}\sigma_B^2 T}{\sigma_B}$$

We can directly observe f, K, r and T while the implied volatility, σ_B is solved by using singular a perturbation technique in Hagan et al. (2002). The formula for σ_B is as follows

$$\sigma_B(K,f) = \frac{\alpha}{(fK)^{(1-\beta)/2}\left[1+\frac{(1-\beta)^2}{24}log^2 f/K+\frac{\frac{(1-\beta)^4}{1920}log^4 f}{K}+g(K,f)\right]}\left(\frac{z}{x(z)}\right) \quad (3.1.2)$$
$$\left[1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta v\alpha}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}v^2\right]T+e(K,f)\right.$$

Where z and $x(z)$ are defined as

$$z=\frac{v}{\alpha}(fK)^{(1-\beta)/2}\log(f/K)$$

$$x(z)=log\left[\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right]$$

In the explicit Formula (3.1.2), the volatility is a function of the strike price K and the current forward price f for a specific date of expiry, T. The terms that we denote as $g(K,f)$ and $e(K,f)$ are complex but taken a very small margin of the total result (Hagan et al. 2002), so they will be ignored in our calculation. When one sets f = K into Equation (3.1.2) the formula for the ATM volatility, σ_{ATM}, is being reduced to

$$\sigma_{ATM}=\sigma_B(f,f)=\frac{\alpha}{f^{(1-\beta)}}\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}}+\frac{1}{4}\frac{\rho\beta v\alpha}{f^{(1-\beta)}}+\frac{2-3\rho^2}{24}v^2\right]T+e(K,f)\right.$$

In order to extract the volatility $\sigma_B(K,f)$, parameters α, β, v, ρ need to be calibrated using the observable market data for implied volatility at every strike. The current forward price f and strike price K are given market condition. After calibration, the model can produced estimated value for volatility that are valid in the near future and captures the dynamic of the smile. These values can be plugged back in the Black-76's formula, for any strikes, to arrive at the option's theoretical price. Hagan et al. (2002) predicted that the calibrated volatility function could give correct value up to six months. However, institutions usually recalibrate it on a frequent basis, usually every day.

3.2 Dynamics of the Parameters

In this section, we investigate the main parameters of the SABR model. Focus will be drawn upon the effect of these parameters on the model's dynamics, their reasonable range and how they alter the shape of the predicted smile. Below we present a numerical example of a fully calibrated model to market data. We will adjust each parameter while keeping the others constant and observe the result in the smile's shape.

β is the exponent of the forward rate and it represents the belief traders possess about the distribution of the underlying asset.[20] Its range should be $\beta \in [0,1]$. Because, if $\beta < 0$, Equation (3.1.1) suggests that an increase in value of the forward process produces a relative decrease in the change of the of price process (Skov Hansen, 2011). Moreover, if $\beta > 1$, an increase in the forward process will result in a greater change of the forward process, which is greater than the volatility times the current forward price. These two cases are very disagreeable and they explain the reason for the upper and lower bound of the parameter.

Figure 10 - Change in Beta

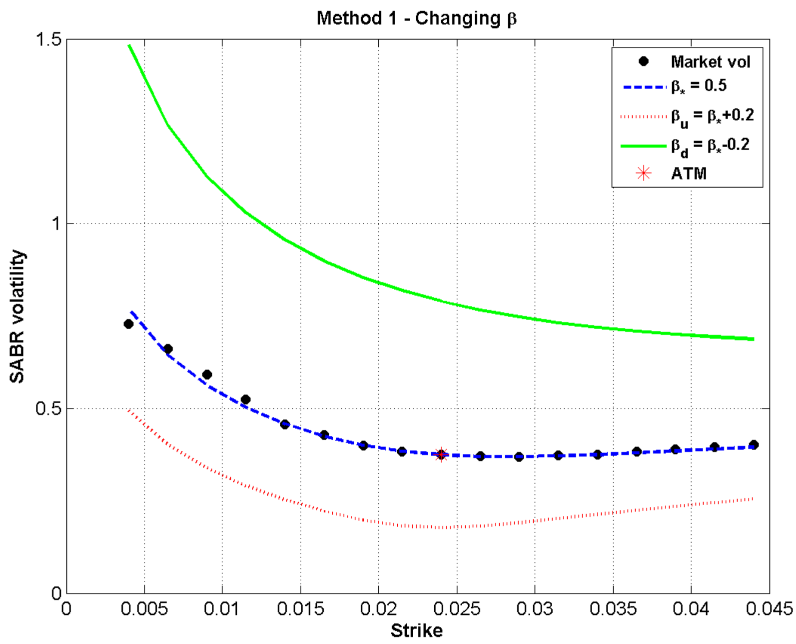


Figure 10 shows the SABR volatility smile for a 1M5Y swaption calibrated using Method 1 (see Section 3.4) with β = 0.5. Resulting parameters are ρ ≈ 0.084, ν ≈ 0.674 and α ≈ 0.058. The SSE for the fit when β = 0.5 is approximately 0.015. Beta is later increased/decreased to 0.7/0.3 while keeping the rest of the parameters constant. The red star indicates the ATM volatility of 37.4% for a swaption with strike of 2.4%.

After calibrating the model with $\beta = 0.5$, we shift the value to 0.3 and 0.7. Observe that the curve in Figure 10 shifts upward for a decreased beta and downward for an increased beta, ceteris paribus. Of course, there is no longer any fit to the market volatility since beta is changed subsequently to the calibration. In addition, we experienced a rather big effect on the curvature of the smile for a change in beta where the left hand side of ATM point is more effected than the right ditto. The higher the beta, the flatter is the curve. Our findings are similar to those of Skov Hansen (2011) while Hagan et al. (2002) do not evaluate this matter.

[20] See Section 3.4 for a deeper description.

V or volvol is the volatility of the volatility. As shown in Figure 11, a change in volvol would have an impact on the smile's curvature. This parameter should be directly calibrated to best fit by using market data with the constraint that ν≥0. Because volatility can never be negative and hence it is reasonable to argue that the volatility of volatility also must be postivive.

Figure 11 - Change in Volvol

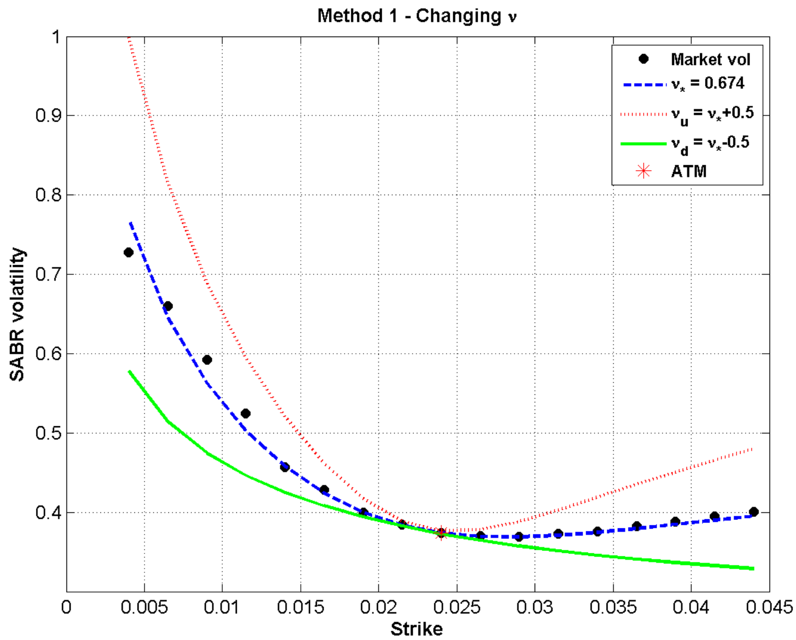


Figure 11 shows the SABR volatility smile for a 1M5Y swaption calibrated using Method 1 (see Section 3.4) with β = 0.5. Resulting parameters are ρ ≈ 0.084, ν ≈ 0.674 and α ≈ 0.058. The SSE for the fit when ν = 0.674 is approximately 0.015. Volvol is later increased/decreased to 1.174/0.174 while keeping the rest of the parameters constant. The red star indicates the ATM volatility of 37.4% for a swaption with strike of 2.4%.

After calibration using predetermined $\beta = 0.5$, we get $\nu \approx 0.674$. With an increase and decrese in ν by 0.5, we obtain a more convex and a flater smile around ATM point respectivly. One can see, in agreement with Hagan et al. (2002), that the volvol controls how much smile the curve exhibits, where an increase in ν would increase the smile effect of the curve, ceteris paribus.

ρ will rotate the curve around the ATM point when increased. As in previous examples, we calibrate the SABR model with the assumption that $\beta = 0.5$. The best fit possible under this scenario returns $\rho = 0.084$. Then lower and higher ρ by 0.25, we get the following graph.Theoretically, ρ is the correlation between two Brownian motion and is therefore bounded to $\rho \in [-1,1]$.

Figure 12 - Change in Rho

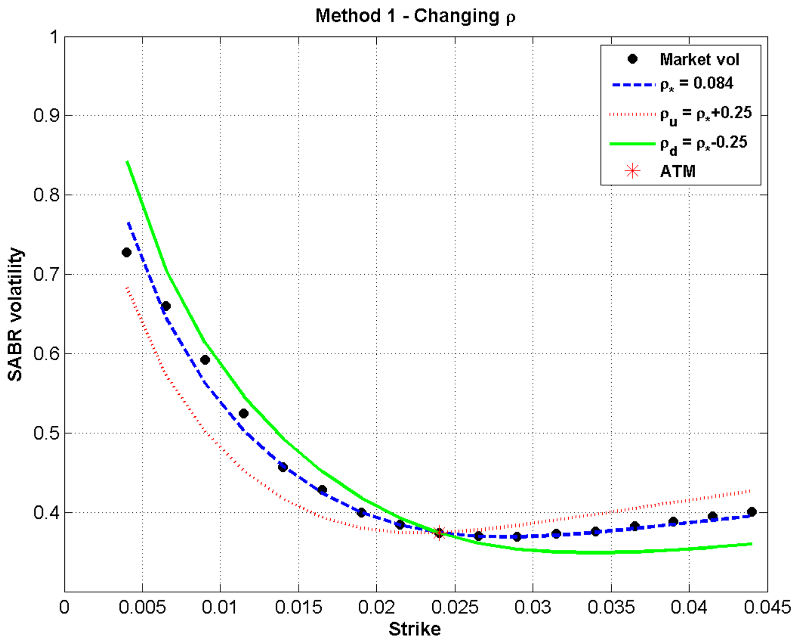


Figure 12 shows the SABR volatility smile for a 1M5Y swaption calibrated using Method 1 (see Section 3.4) with β = 0.5. Resulting parameters are ρ ≈ 0.084, ν ≈ 0.674 and α ≈ 0.058. The SSE for the fit when ρ = 0.084 is approximately 0.015. Rho is later increased/decreased to 0.334/-0.166 while keeping the rest of the parameters constant. The red star indicates the ATM volatility of 37.4% for a swaption with strike of 2.4%.

Hagan et al. (2002) claim that rho controls the skew of the curve. This is verified by looking at Figure 12 to see what happens when we increase/decrease the parameter. One can see that an increased ρ to circa 0.334 rotates the curve counter clockwise, creating a flatter smile. On the contrary, a decrease (ρ = -0.166) would lead to a clockwise rotation of the SABR curve and hence a steeper smile.

α is different to β,ν and ρ since it is a stochastic parameter. An increase in this parameter will lead to an upward shift of the entire smile while a decrease will result in an downward shift. This observation can be made after setting α_0 to be the initial volatility where the entire stochastic process would begin from. As a result, α should govern the vertical location of the smile rather than the smile's shape. Also for this reason α>0 since we can not experince non-positive volatility. From our calibration corresponding with the earlier figures, we receive $\alpha \approx 0.058$. Adding and subtracting 0.01 to this value we attain the curves in Figure 13.

Figure 13 - Change in Alpha

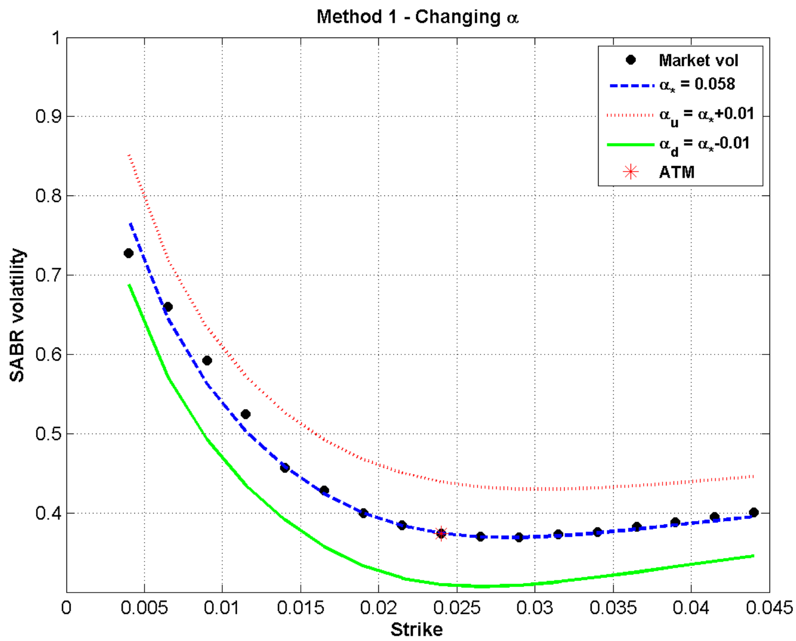


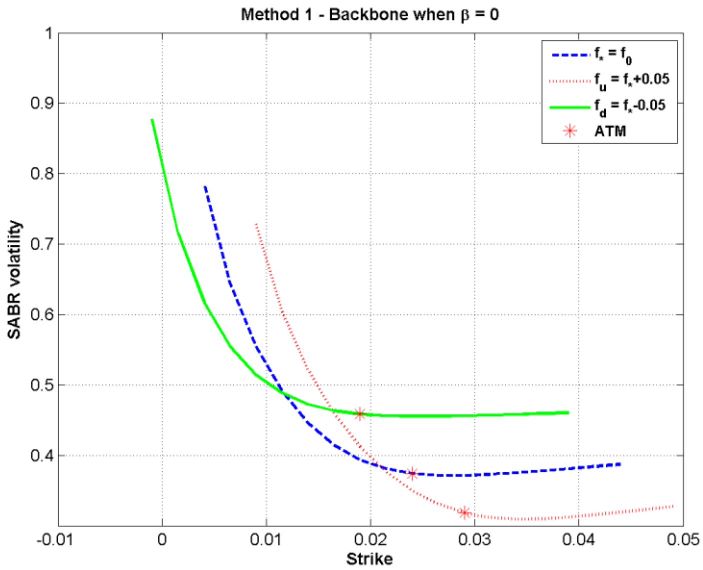
Figure 13 shows the SABR volatility smile for a 1M5Y swaption calibrated using Method 1 (see Section 3.4) with β = 0.5. Resulting parameters are ρ ≈ 0.084, ν ≈ 0.674 and α ≈ 0.058. The SSE for the fit when α ≈ 0.058 is approximately 0.015. Alpha is later increased/decreased to 0.068/0.048 while keeping the rest of the parameters constant. The red star indicates the ATM volatility of 37.4% for a swaption with strike of 2.4%.

In Figure 13, we can see that the curvature of the SABR curves seem to remain constant when alpha is increased and decreased. That a change in alpha will keep the shape of the smile is in agreement with e.g. Skov Hansen (2011, p.39).

3.3 The Backbone

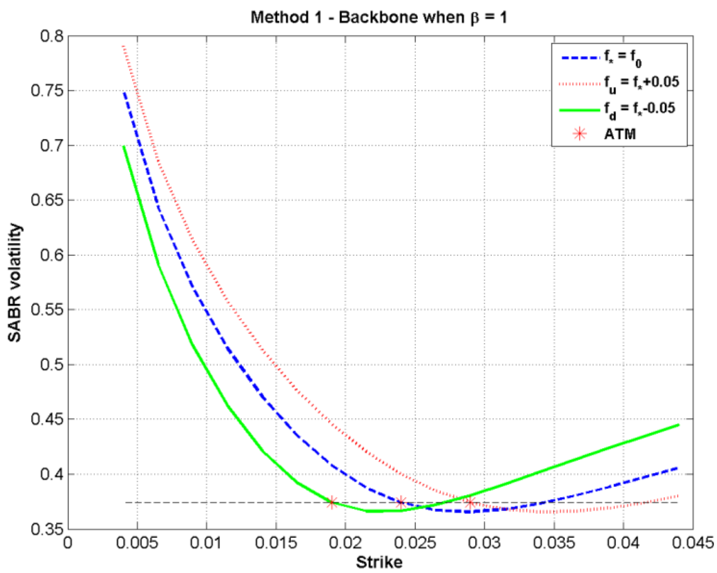
From Hagan et al. (2002), the so-called backbone is the curve that is traced out from ATM volatility when the forward price, f, changes. This backbone is observed to be dependent almost entirely on the β used as the exponent of the price process. Initially, we calibrate the model with β = 0 for a 1M5Y swaption with and ATM rate of 2.4%. If we instead let the forward rate shift to 2.9% and 1.9% respectively, we can infer from Figure 14 that the ATM volatility shift accordingly.

Figure 14 - Backbone with β = 0



If one instead calibrates the model with β = 1 and then change the forward rate by 50 BPS the result will be different. Both an increase and a decrease in f would only shift the curve on the horizontal axis while keeping the ATM volatility constant, as seen in Figure 15.

Figure 15 - Backbone with β = 1



3.4 Parameter Approximation

There are many ways one can calibrate the SABR model, however, the general idea is to minimize the gap between observed and predicted implied volatility fitted by the SABR model for each corresponding strike. This technique is also known as the Least Square Method. Using mathematical language, the problem's objective function can be formulated as

$$\min_{\nu,\alpha_0,\rho,\beta} \sum_i (\sigma_I - \sigma_B(\nu, \alpha_0, \rho, \beta; K_i, f))^2$$

Where σ_I is the implied market volatility (found by some appropriate method) and $\sigma_B(\nu, \alpha_0, \rho, \beta; K_i, f)$ is the SABR volatility as a function of SABR parameters given the strike K and an ATM forward price f. The Sum of the Squared Errors (SSE) above will then be minimized with the constraints of the model's parameters, i.e.

$$\rho \in [-1,1]$$

$$\beta \in [0,1]$$

$$\nu \geq 0$$

$$\alpha_0 > 0$$

To solve the above optimization problem with the given restrictions, one can put different weights to each parameters according to distinct characteristic of the market under consideration. For example, in an illiquid market, a trader might want to put more weights to traded instruments than equal weights in order to produce best possible fit. In our calibration, given the market data from Murex Mx3, we will only apply equal weights.

3.4.1 β

According Hagan et al. (2002), β is the first choice for a pre-estimating parameter, for it can reflex the prior belief on the forward process of the underlying asset. Otherwise, if fitted with the market smiles, it would only be fitting the market noises.[21] The authors pointed out that various β values do not give substantial difference in quality of the fit. However, the choice of β can affect the Greeks (Skov Hansen, 2011). Below we will discuss three common choices of beta, which occurs when we let $\beta \epsilon \left\{0, \frac{1}{2}, 1\right\}$.

[21] This would imply that we instead are interested to minimize $\min_{\nu,\alpha_0,\rho} \sum_i (\sigma_I - \sigma_B(\nu, \alpha_0, \rho, \beta; K_i, f))^2$ where we let $\rho \in [-1,1]$, $\nu \geq 0$ and $\alpha_0 > 0$.

$\boldsymbol{\beta} = \mathbf{0} - Stochastic\ Gaussian\ (Normal)\ Model$

$$d\hat{F} = \hat{\alpha} dW_1$$

When beta equals zero, the forward process becomes stochastic normally distributed with mean zero and a lognormal distributed standard deviation. With a symmetric break-even point, this is an effective choice for managing risk. This choice is also suitable for trading market like Yen, Krona and interest rates, where the forwards f can be negative or near zero. However, for most normal cases of forward price, this beta is not the most preferable one as stated by Skov Hansen (2011).

$\boldsymbol{\beta} = \frac{\mathbf{1}}{\mathbf{2}} - \ Stochastic\ CIR\ model$

$$d\hat{F} = \hat{\alpha} \hat{F}^{\frac{1}{2}} dW_1$$

This choice usually concerns with US interest rate desks that often use CIR models. The stochastic CIR model takes its name from Cox, Ingersoll and Ross (Cox, Ingersoll & Ross, 1985) model of short-term interest rates. With setting beta equals to one-half, our current level of the price process is under a square root. This proponent will prevent the forward price to be negative, which is in contrast to the previous case where beta equals to zero.

$\boldsymbol{\beta} = \mathbf{1} - Stochastic\ Lognormal\ Model$

$$d\hat{F} = \hat{\alpha} \hat{F}^1 dW_1$$

In this case, the forward process is a lognormal process, it is almost similar to the Black and Scholes's setup, where the stock price follows a Brownian motion. The only difference is that Black and Scholes assumed constant volatility and this stochastic model sets the volatility process to be a stochastic process as well. If beta is chosen to be one, one should also believe that the market's backbone is horizontal (as can be seen in Section 3.3). This case also resembles the case beta is one-half because the lognormal process will also prevent negative forward rate.

However, β can also be estimated as any other parameters in the model or it can be deduced from the observed backbone. Taking logarithm of Equation (3.1.2) produces

$$log\sigma_{ATM} = log\alpha - (1 - \beta) logf$$

From this, we can use linear regression of natural logarithm of observed ATM volatility and natural logarithm of forward rates in order to estimate the slope of the line above. β is then the slope + 1.

3.4.2 ρ, ν and α

After realizing beta, we are left with three remaining parameters to estimate α, ρ and ν. In general, there are two well-known ways to do so.

The first method (denoted in this paper as Method 1) is recommended by Hagan et al. (2002) and can be seen as the more convenient one. It uses ATM volatility to infer the parameter α so we only need to estimate ρ and ν. Given the expression for σ_{ATM} in earlier section, we have

$$\sigma_{ATM} = \frac{\alpha_0}{f^{(1-\beta)}}\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha_0^2}{f^{(2-2\beta)}} + \frac{\rho\beta\nu\alpha_0}{4f^{(1-\beta)}} + \frac{2-3\rho^2}{24}\nu^2\right]T\right\} \quad (3.4.1)$$

$$\Leftrightarrow 0 = A\alpha_0^3 + B\alpha_0^2 + C\alpha_0 - \sigma_{ATM}f^{(1-\beta)}$$

Where

$$A = \left[\frac{(1-\beta)^2 T}{24f^{(2-2\beta)}}\right], B = \left[\frac{\rho\beta\nu T}{4f^{(1-\beta)}}\right] \; and \; C = \left[1 + \frac{2-3\rho^2}{24}\nu^2 T\right]$$

West (2005) suggested that this cubic function could have more than one real root. In that case, the smallest positive root should be selected to best capture the smile effect. Now the minimization problem's objective function becomes

$$\min_{\nu,\rho} \sum_i (\sigma_I - \sigma_B(\nu, \rho, \alpha_0(\rho, \nu, \sigma_{ATM}); K_i, f, \beta))^2$$

However, this estimation will take more time to produce the final result than the second method. Since every iteration has been added an extra step of estimating α through ρ and ν. In detail, the procedure is

i. Assign initial values to ρ and ν.
ii. Solve for α_0 through Equation (3.4.1) that use inputs β, ρ and ν.
iii. Insert α_0, β, ρ and ν into Equation (3.1.2) to calculate σ_B for every strike.
iv. Minimize the objective function above to get a new set of ρ and ν.
v. Repeat (ii) and (iii) to get a new set of parameters and σ_B.
vi. Plug the new σ_B into the objective function, then compare the objective function's value with a convergence criteria. Move on to the next iteration until the algorithm converge to a level of tolerance.

The second method (denoted in this paper as Method 2) uses common techniques of optimization such as Newton-Raphson Method for finding roots or minimizes the sum of squared of errors (SSE) (see Appendix 4) to solve the stated general minimization problem for all parameters[22] and arrive at a set of parameters that gives the smallest error. Both

[22] Assuming that β is predetermined prior to the calibration of the model.

methods are very intuitive and may be done with a computer software that allows such implementation.

3.5 Greeks under the SABR Model

3.5.1 Original Set

In this section, we briefly describe the previous Greeks (delta and vega) under the SABR framework. Consider an option with a forward rate f, strike K and time to expiry T expressed in year. Under the model, the value of this option measured with the Black-76 formula is

$$V = B(f, K, \sigma(K; f, \alpha, T), T)$$

Where, $\sigma(K; f, \alpha, T)$ is the volatility extracted from SABR and $B(f, K, \sigma, T)$ is the Black-76 pricing formula. From the original paper by Hagan et al. (2002), delta is the change in the current value of the option when the current value of the forward is shifted, while keeping other parameter and the level of alpha fixed. However, in addition to the Black-76 model, the underlying asset price is now a function of volatility. Therefore, by applying the chain rule, we find the SABR delta as

$$f_t = f_s + \partial f$$

$$\alpha_t = \alpha_s$$

$$\Delta = \frac{\partial B}{\partial f} + \frac{\partial B}{\partial \sigma}\frac{\partial \sigma}{\partial f}$$

$where\ 0 < s < t$, f_t is the forward price at time t and f_s is the forward price at time s, α_t is the volatility at time t and α_s is the volatility at time s and ∂f is the change in f.

Similarly, vega risk is calculated as the change in price of the option with respect to a change in volatility of the underlying asset's alpha, where $\partial \alpha$ is the change in α,

$$f_t = f_s$$

$$\alpha_t = \alpha_s + \partial \alpha$$

$$\Lambda = \frac{\partial B}{\partial \sigma}\frac{\partial \sigma}{\partial \alpha}$$

However, these risk measurements are not the optimal ones under the SABR model. Hence, we need to turn to better alternatives.

3.5.2 An Updated Set of Greeks

Bartlett (2006) claimed that the delta and vega risk could be hedged more precisely by adding new terms to the risk measurements of Hagan et al. (2002). Under the Greeks below, the author claimed that delta risk is less sensitive to the beta-exponent chosen in the SABR model.

Barlett argued that because in the SABR model alpha and f is correlated, when f changes, in average, alpha changes too. Therefore, he postulated a more realistic way to calculate delta risk.

$$f_t = f_s + \partial f$$

$$\alpha_t = \alpha_s + \delta_f \alpha$$

The average change in alpha caused by the change in f is denoted as $\delta_f\alpha$. In order to calculate alpha, Barlett (2006) rewrote the SABR dynamics in term of two uncorrelated Brownian motions dW_t and dZ_t,

$$df_t = \alpha_t f_t^{\beta} dW_t,$$

$$d\alpha_t = \nu\alpha_t \left(\rho dW_t + \sqrt{1-\rho^2} dZ_t\right)$$

If we rearrange the first expression as

$$dW_t = df_t(\alpha_t f_t^{\beta})^{-1},$$

then insert it into the second expression, we get

$$d\alpha_t = \frac{\rho\nu}{f_t^{\beta}} df_t + \nu\alpha_t\sqrt{1-\rho^2} dZ_t$$

Now, apparently, the change in alpha is affected by the change in two independent terms. The first term is the change in f and the second term is the unsystematic change in alpha. With this approach, we can write the average change in alpha due to the forward f as

$$\delta_f\alpha = \frac{\rho\nu}{f_t^{\beta}}\, df_t$$

The change in the option value is now

$$\Delta V = \frac{\partial B}{\partial f} + \frac{\partial B}{\partial \sigma}\left(\frac{\partial \sigma}{\partial f} + \frac{\partial \sigma}{\partial \alpha}\frac{\rho\nu}{f^{\beta}}\right)\Delta f$$

Therefore, the delta risk is now

$$\Delta = \frac{\partial B}{\partial f} + \frac{\partial B}{\partial \sigma}\left(\frac{\partial \sigma}{\partial f} + \frac{\partial \sigma}{\partial \alpha}\frac{\rho\nu}{f^{\beta}}\right)$$

This new definition of delta has included the average change in volatility alpha caused by changes in the underlying forward rate by adding the new term

$$\frac{\partial B}{\partial \sigma}\frac{\partial \sigma}{\partial \alpha}\frac{\rho v}{f^{\beta}}$$

In the same manner, vega risk should also be calculated from the following scenario

$$f_t = f_s + \delta_{\alpha} f$$

$$\alpha_t = \alpha_s + \partial \alpha$$

Similarly, where $\delta_{\alpha}\mathrm{f}$ is the average change in f caused by a change in alpha, we can also find that

$$\delta_{\alpha} f = \frac{\rho f^{\beta}}{\nu} d\alpha_t$$

Thus, the vega risk is now

$$\Lambda = \frac{\partial B}{\partial \sigma}\left(\frac{\partial \sigma}{\partial \alpha} + \frac{\partial \sigma}{\partial f}\frac{\rho f^{\beta}}{v}\right)$$

We will stop at deriving risks at this point. For more complicated derivatives such as swaption, it is difficult to measure delta and vega correctly. The reason is that, generally, delta of a swaption is defined as the change in swaption price in term of a change in the underlying swap price. In this case, the swap value must be determined and this is not an easy task to achieve in reality. Since different institutes have diverse techniques to approaching value of a swap, using different interest rate curves. Even with an appropriate method of calculating the a swap values, the task of calculating Greeks still appears problematic when one must decide how to shift the price of the underlying asset (e.g. the swap). One option is the shift the entire yield curve by, for example, 1 BPS and calculate new (theoretical) prices, or to use some applicable scaling factor. However, one will most likely never experience a parallel shift in the curve used to price the derivative. Instead, a shift occurring in one or several time buckets can be used. Consequently, to calculate risk is a delicate work that involves ambiguous decisions. Therefore, this section will not be further developed here, nor as a part of the software application for swaption's risks under the SABR model.

3.6 Refinement of the SABR Model

It can be observed that the original formula for volatility by Hagan et al. (2002) breaks down when the strike is small and maturity is long. In this section, we will simply state the refinement to that problem, which was summarized by Oblój (2008). We refer to Oblój (2008), and Chapter 3 and Chapter 7 in Gatheral (2006) for background and additional reading.

The implied volatility surface $\sigma_B(x,T)$ with maturity T and $x = \log(F/K)$ can be approximated using Taylor expansion, as the following

$$\sigma_B(x,T) \approx I_B^0(x)(1 + I_H^1(x)T),$$

$$where \quad I_H^1(x) = \frac{(\beta-1)^2}{24}\frac{\alpha^2}{(sK)^{1-\beta}} + \frac{1}{4}\frac{\rho v \alpha \beta}{(sK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}v^2$$

And four cases for $I_B^0(x)$

Case 1: $x = 0$

$$I_B^0(0) = \alpha K^{\beta-1}$$

Case 2: $v = 0$

$$I_B^0(x) = \frac{x\alpha(1-\beta)}{f^{1-\beta} - K^{1-\beta}}$$

Case 3: $\beta = 1$

$$I_B^0(x) = \frac{vx}{\log \frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}}$$

$$where \quad z = \frac{vx}{\alpha}$$

Case 4: $\beta < 1$

$$I_B^0(x) = \frac{vx}{\log \frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}}$$

$$where\ z = \frac{v(f^{1-\beta} - K^{1-\beta})}{\alpha(1-\beta)}$$

The parameters α, β, v, ρ are obtained after calibration of the SABR model and K, f, T are market data. This way, the volatility $\sigma_B(x,T)$ is calculated with regard to one of these four cases with only trivial operations and it is then plugged back into the Black-76's formula in order to get the price of an option. This adjusted model, even though, theoretically more reasonable than the original one, produce only small differences we will show in Section 4.

4 CALIBRATION AND RESULTS

Using the data set described earlier, in this section, we will calibrate the SABR model with MATLAB in several ways. First, we will set three prior values to beta (0, 1/2 and 1), then use different methods to calibrate the remaining parameters. Secondly, we will estimate all parameter values using only the given market smile.

In MATLAB, we partly use functions written by of Fabrice Douglas Rouah (see www.volopta.com). We also write our own program for implementing the SABR model where we call many of the external functions. We use the "fminsearch" that implements the Nelder-Mead algorithm to find parameters that minimize our SSE. Our written program imports all the data from Mx3 to convert it to suitable volatility matrices. Later it also plots the volatility smiles, surfaces and cubes that we have presented earlier in this paper. The entire process from importing data from an excel-sheet to receiving all our findings takes approximately 15 seconds using a standard computer. Segments of our and Fabrice Douglas Rouah's code can be found in Appendix 3.

4.1 Different Values of Beta

Using both Method 1 and Method 2 without refinements, we calibrate the model for three different values of beta, namely $\left\{0, \frac{1}{2}, 1\right\}$. Results are presented in Table 1 and Table 2. The result shows that the discrepancy in term of error[23] is insignificant across all beta under both methods.

Table 1 - Method 1 Estimated for Different Beta

beta	rho	volvol	alpha	error
0	0.447103	0.651421	0.008953	0.024757
0.5	0.08426	0.673686	0.057738	0.014563
1	-0.24621	0.783108	0.373115	0.014482

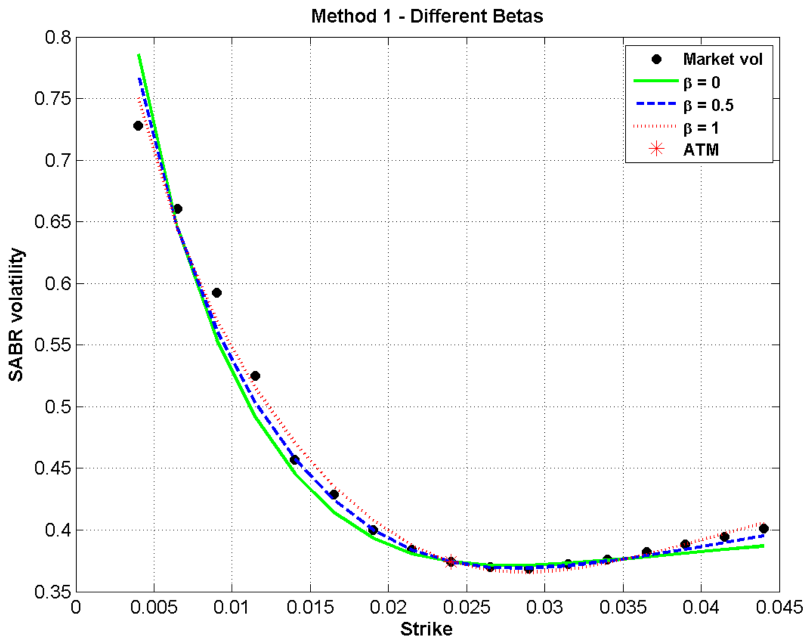
Table 2 - Method 2 Estimated for Different Beta

beta	rho	volvol	alpha	error
0	0.475036	0.587904	0.009232	0.027657
0.5	0.077919	0.641193	0.058745	0.018593
1	-0.24778	0.780557	0.373697	0.014509

[23] The error term is define as $\sum_i (\sigma_I - \sigma_B(\nu, \rho, \alpha_0(\rho, \nu, \sigma_{ATM}); K_i, f, \beta))^2$, in other words, it is the sum of the squared differences between the observed market implied volatility and the estimated volatility by the SABR model.

Although the different is small, we conclude that $\beta = \frac{1}{2}$ gives the best fit for Method 1 while β = 1 produces the smallest error under the second calibration. On the other hand, when $\beta = 0$, one experiences the largest error, approximately 0.01 higher than for other values of beta. Below are figures illustrating the fit to market data.

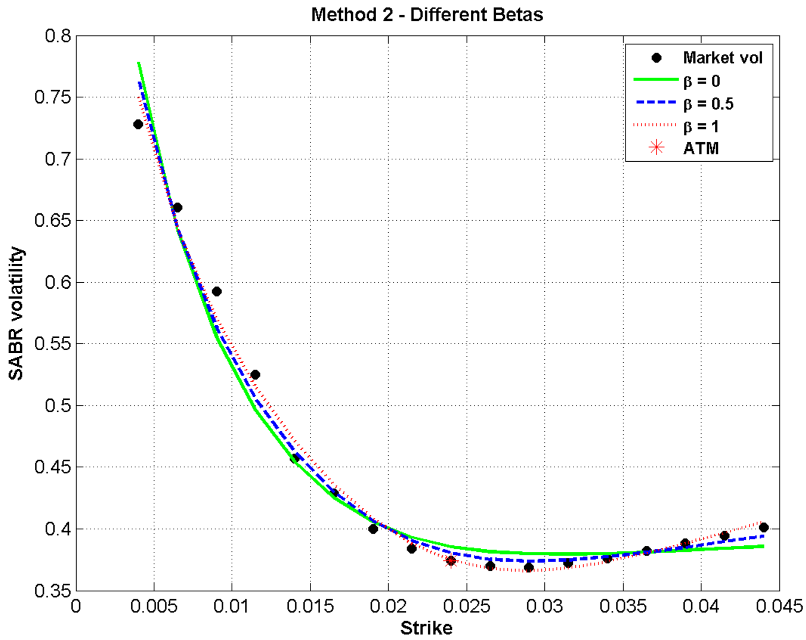
Figure 16 - Calibration with Different Beta (Method 1)



In Figure 16, a 1M5Y swaption is calibrated using Method 1. Parameter estimation can be found in Table 1. The red star indicates the ATM volatility of 37.4% for a swaption with strike of 2.4%.

As we can see from Figure 16, all the SABR curves go through the ATM point of the swaption. This is also the point of intersection for the three curves. This is due to the fact that we only minimize the sum of the squared errors for two parameters (i.e. ν and ρ). We also can conclude that the fit is equally good regardless of the predetermined value of beta.

Figure 17 - Calibration with Different Beta (Method 2)



In Figure 17, a 1M5Y swaption is calibrated using Method 2. Parameter estimation can be found in Table 1. The red star indicates the ATM volatility of 37.4% for a swaption with strike of 2.4%.

In Figure 17, Method 2 is employed where the sum of squared errors is minimized for three predetermined values of β. In contrast to Figure 16, the produced curves do no longer go through true the ATM market volatility (with the exception of when one is chosen as the beta exponent).

Comparing Method 1 and Method 2, we see that both methods produce spectacular fit to the market data. It is reasonable to believe that the chosen method should be based on whether the traders want to fit the SABR curve through the point of ATM volatility. Trader should fit the model so that it goes through ATM because more swaptions (and other derivatives) are traded at or around ATM. Obviously, the work effort to minimize SSE and solving eq. (3.4.1) take more time to implement. Although, in our opinion, this is a small price to pay in order to have a model that can fit ATM instruments.

The choice of which beta exponent to use is a puzzle to solve. Fortunately, we experience good fit regardless of which value we use when calibrating the model. We claim that, in agreement with Hagan et al. (2002), the beta value does not have such a great impact of the quality of the fit. It is quite appealing for traders to drop the assumption about a lognormal distributed underlying asset that is crucial in the Black-Scholes world and instead pick beta based on true beliefs.

4.2 Method 1 vs. Method 2 using Refinements

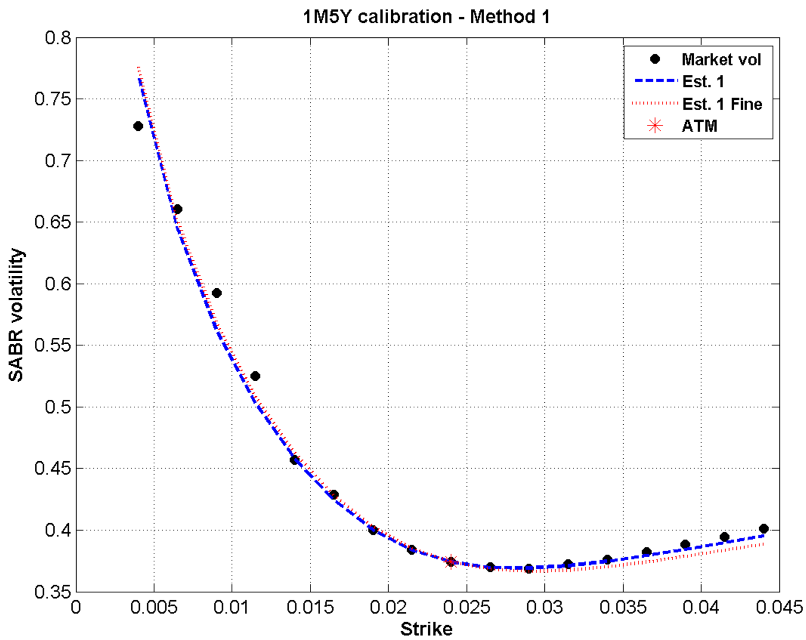
From the previous observation, we choose $\beta = 0.5$ to compare the quality of fit between two methods with their refinement suggested by Oblój (2008). Table 3 shows the calibrated parameters and errors.

Table 3 - Different Methods Calibrated for when Beta is 0.5

Calibration	beta	rho	volvol	alpha	error
Method 1	0.5	0.08426	0.673686	0.057738	0.014563
Method 2	0.5	0.077919	0.641193	0.058745	0.018593
Method 1F	0.5	0.046258	0.662306	0.057749	0.014231
Method 2F	0.5	0.041704	0.636768	0.058619	0.015944

As we can see, Method 1 with refinement has the lowest error while Method 2 has the highest. In average, refinement in both method shows equally fit when both Method 1 and Method 2 fit well to the market data without noticeable differences. The parameters are also very similar. The two figures below are illustrations of the calibration.

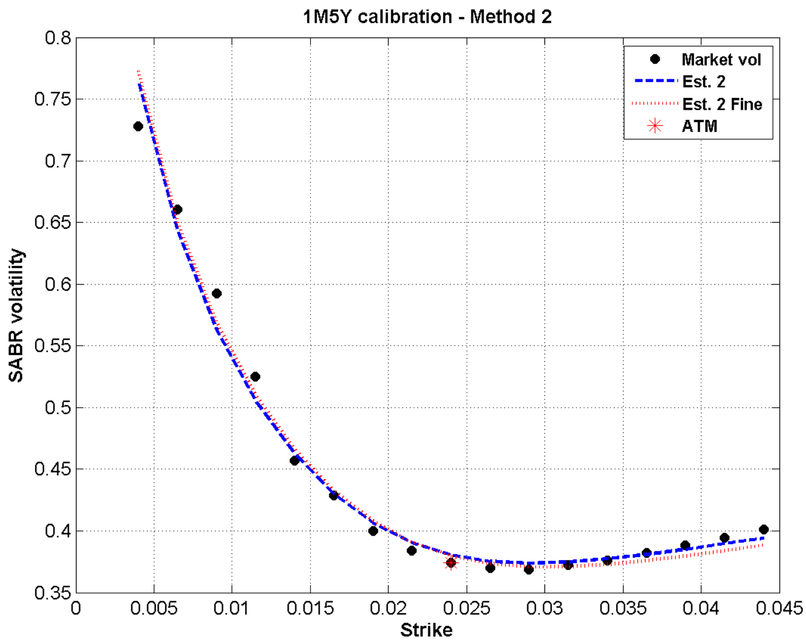
Figure 18 - Fine-tuned Smile (Method 1)



In Figure 18, a 1M5Y swaption is calibrated using method 1 with and without fine-tuning. Estimated parameters can be found in Table 3. The star indicates where ATM volatility is 37.4% for a swaption with strike of 2.4%.

One can once again see from Figure 18 that Method 1 calibrates the model by inferring alpha from ATM volatility. The fit of the two curves are almost identical for the entire domain of the curve and only seem to diverge slightly for large strikes.

Figure 19 - Fine-tuned Smile (Method 2)



In Figure 19, a 1M5Y swaption is calibrated using Method 2 with and without fine-tuning. Estimated parameters can be found in Table 3. The star indicates where ATM volatility is 37.4% for a swaption with strike of 2.4%.

In Figure 19, one can see that Method 2 produces almost identical curves with and without fine-tuning. Once again, the SABR curves only seem to differ for larger strikes. As mentioned previously, if SSE is used to minimize the errors for α, ρ and ν, the best fitting curves do not intersect the point of ATM volatility.

Despite the fact that the Oblój (2008) refinement produces smaller errors, they are in fact rather cumbersome to apply. If the extra quality required is not outweighed by the effort it takes to implement refinement, any techniques should be considered adequate. However, we argue that for our example above, the positive effect received by applying Oblój (2008) is dwarfed in comparison to those that potentially could be received by altering between Method 1 and Method 2.

4.3 Calibration for Long and Short Tenors

As seen from the previous results, the fit of the SABR model to a 1M5Y swaption is spectacular regardless of method used or the selected beta. We continue by looking at some extreme scenarios where the tenor of the underlying swap is very short/long and the time to maturity is also very short/long. We want to investigate the following swaptions: 1M20Y, 1M4Y, 20Y4Y and 20Y20Y.

Under the assumption that beta for the underlying asset is one half, we calibrate the four different swaptions using Method 1. The results are summarized in Table 4.

Table 4 - Different Swaptions Calibrated with Method 1

Swaption	beta	rho	volvol	alpha	error
1M20Y	0.5	-0.03445	0.764233	0.05122	0.005479
1M4Y	0.5	0.089563	0.541858	0.053886	0.042698
20Y4Y	0.5	1	0.061883	0.031644	0.015523
20Y20Y	0.5	-0.1098	0.208613	0.026842	0.005055

Figure 20 - 1M20Y Calibration

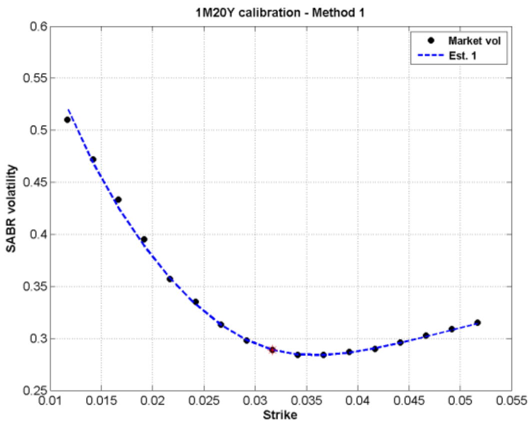
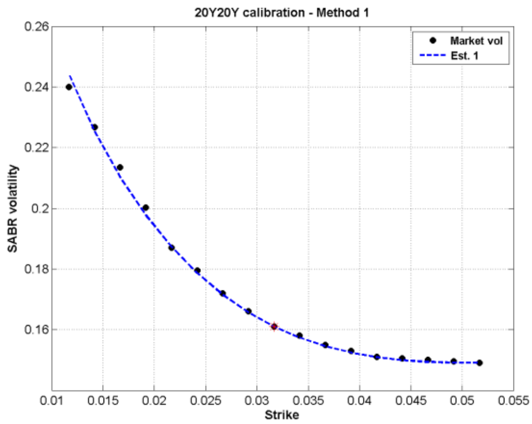


Figure 21 - 20Y20Y Calibration



In Figure 20 (left) and Figure 21 (right) the SABR model is calibrated for a 1M20Y and a 20Y20Y swaption (ATM rate 3.17%) with β = 0.5. Market ATM volatilities are approximately 20.9% and 16.1% respectively. Calculated parameters for ρ, ν, α and SSE are found in Table 4.

Figure 20 and Figure 21 show remarkable fit to the market data. In those figures, we compare two swaptions from the same volatility surface. One that expires in one month and one in 20 years. Investigating the parameters, we see that alpha is higher for the swaption that expires shorter which is consistent of what we can expect by looking at any swaption volatility surface. The negative value of ρ does not change by a great amount while ν is higher for the swaption that expires in one month.

Figure 22 - 1M4Y Calibration

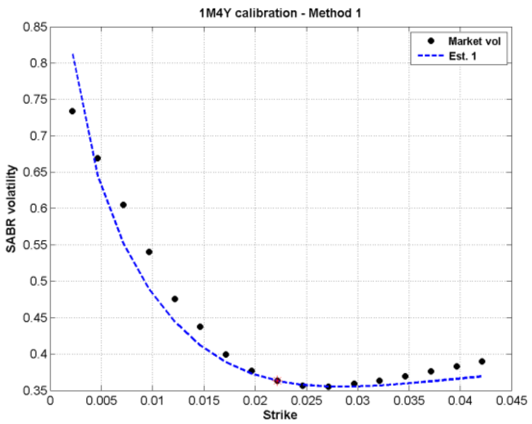
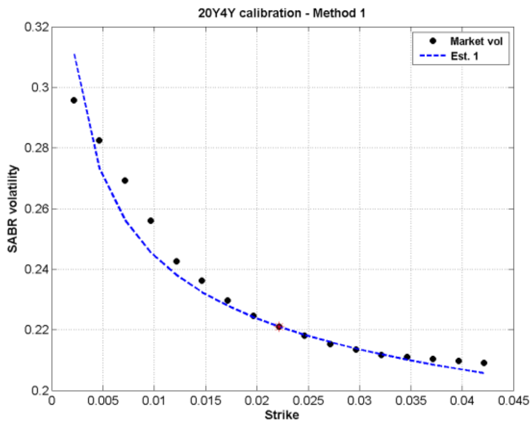


Figure 23 - 20Y4Y Calibration



In Figure 22 (left) and Figure 23 (right) the SABR model is calibrated for a 1M4Y and a 20Y4Y swaption (ATM rate 2.21%) with β = 0.5. Market ATM volatilities are approximately 36.3% and 22.1% respectively. Calculated parameters for ρ, ν, α and SSE are found in Table 4.

We now turn the attention to a swaption with a short tenor of underlying swap. After calibration, we end up with curves that do not show the superior quality of fit we have observed earlier. The errors for the 1M4Y- and 20Y4Y swaptions are many times greater than those of the 1M20Y and 20Y20Y swaption when using the exact same techniques of calibrating the SABR model. Comparing the estimated values for rho, one can also see that it is very different when calibrated to fit the two different smiles from the same volatility surface. The same is true for volvol and alpha.

It appears trickier to fit swaptions that have a shorter tenor than those with a long tenor regardless of when they expire. Of course, this could be a result of a poorly chosen calibration technique where for example refinements or another beta value would generate a much better fit. On the other hand, it can also be a proof of how complex and dynamic the market could be. There can thus exist tenors with smiles that the SABR model cannot capture correctly.

Above, we calibrated the SABR model for swaptions with underlying strikes of four years. We chose the tenor of four years to represent a swaption with a short-term underlying swap. In fact, many swaps have shorter tenors than four years. However, we did not choose them because in order to be able to calibrate the model around a spread of 400 BPS, a one-, two- or three-year swaption would have to be calibrated for negative strikes. The SABR model breaks down and we can no longer calibrate the curve under methods presented in this paper. In defence of Hagan et al. (2002) who presented their work during a time when interest rates were still somewhat high, no one could expect scenarios where an issue about negative strikes would be addressed. In order to resolve this issue a method that accounts for negative strikes must be developed.[24]

4.4 Additional Remarks

We have so far showed very good fit to market data. We do not argue that that SABR model in any way fail to capture the effect of the smile seen in the markets. Our findings presented and discussed above are clear proofs that the SABR model is efficient to calibrate itself to market data. Method employed, with or without refinements, also seem to have a small impact on the outcome.

However, the quality of the data must always be questioned. In this report, we have used market data provided to us by one of Sweden’s major banks. The data received is already interpolated/extrapolated for several strikes and tenors where traded instruments do not exist. As mentioned earlier, this is standard procedure. Although, if one tries to calibrate the model to data that is already altered to “fit the market”, one could almost expect to see the result we provide in previous sections.

[24] Such as extension of the SABR model to fit with the Normal Black model (see Section 2.5.4).

In this paper, we only study instantaneous movements in time to calibrate the SABR model. The SABR volatility calculated from the calibration ought to be plugged back into the Black-76 model to receive a theoretical price for any strike. Although, this is only valid at that certain point in time for when the calibration occurred and does not answers any questions regarding if the calibration will last for a longer time. Calibration of the model only takes a couple of seconds so recalibration of a frequent basis could easily be done. Hagan et al. (2002) claimed that calibration lasts for a long time given that alpha is recalibrated daily. It could be argued that the model should be recalibrated more frequently than that.

Implied volatility is an interesting concept. Up until today, any version of the Black-Scholes model is mainly used to find the implied volatility of an instrument. People are thus *"using the wrong number in the wrong formula to get the right price"* (Rebonato, 2007). Despite that the SABR model is not an actual pricing model it has gained vast popularity among traders. It is easy to calibrate and the result seem to correspond good to market data. Since we have not investigated any alternative models, little can be said about them and whether they succeed the SABR model. However if the market standard would be to use the SABR model in order to price certain interest rate derivatives, would the market adjust to the model such as it did to the Black-Scholes model during the 70s? What impacts and what kind of harm could this possible cause? Even if certain models appear very appealing, one must always remember that the market is dynamic and will always change, often unexpectedly. To put to strong credence in one model is thus very risky, regardless of its power to correctly capture smile dynamics and create good hedge ratios.

5 CONCLUSION

The SABR model developed by Hagan et al. (2002) is a stochastic volatility model that attempts to capture volatility smiles in derivative markets. In this paper, we account for and calibrate the SABR model for swaptions. By using two different methods (with and without refinement), we found the SABR model accurately capture the volatility smiles on the markets. Under our investigation, we experienced reverse skews rather than forward skews for all swaptions. On the other hand, for longer tenors, we do not experience any major smile effects for swaptions when the volatility curves are rather flat in general.

We find no major discrepancy between the two methods of calibration. However, we suggest the method where α is inferred from ATM volatility for conveniences and slightly better result. While using refinements, the fit to the market data is increased marginally. Despite a slightly better fit, we argue that a “better” value for β, which is more in line with the true distribution of the underlying asset, should be implemented rather than any refinements.

For future conceivable studies, as a natural continuation of this paper, a study to calculate swaptions, caps or floors prices, as well as the “Greeks” and how they are affected by changing the parameters under the SABR should be carried out. In addition, a research about volatility surfaces would be an interesting undertaking. Moreover, a deeper study of a dynamic SABR model would constructively add to the body of literature, see Appendix B, where Hagan et al. (2002, p. 103) presented the dynamic of SABR model. This would lead to one being able to calibrate the SABR model for entire volatility surfaces. Finally, as earlier mentioned, an extension of the SABR model that is consistent for negative strikes could be an interesting alternative for further studies.

6 APPENDICES

Appendix 1 – Cumulative Normal Distribution Function

To implement for example Equation (2.5.3) and Equation (2.5.4), there lies a problem calculating the cumulative normal distribution function, $\mathcal{N}(\mathrm{x})$. One can use inbuilt functions in e.g. Excel (*NORMSDIST*) or MATLAB (*normcdf*). Without computational assistance, a polynomial approximation for $\mathcal{N}(\mathrm{x})$ is given by

$$\mathcal{N}(\mathrm{x}) = \begin{cases} 1 - \mathcal{N}'(\mathrm{x})(a_1k + a_2k^2 + a_3k^3 + a_4k^4 + a_5k^5), & x \geq 0 \\ 1 - \mathcal{N}(-\mathrm{x}), & x < 0 \end{cases}$$

Where

$$k = \frac{1}{1 + \gamma x}$$

$$\gamma = 0.2316419$$

$$a_1 = 0.319381530$$

$$a_2 = -0.356563782$$

$$a_3 = 1.781477937$$

$$a_4 = -1.821255978$$

$$a_5 = 1.330274429$$

$$\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

For further reference, see Chapter 13 in Hull (2009).

Appendix 2 – Selected Data

Table A1 - Selected Data

ATM VOLATILI-TIES											
Tenor/Maturity	**1M**	**3M**	**6M**	**1Y**	**2Y**	**3Y**	**4Y**	**5Y**	**7Y**	**10Y**	**20Y**
1Y	33.4	30.1	31.5	31.1	34.3	33.9	29.8	27.8	24.5	22.4	19.7
2Y	32.8	31.1	33.3	33.3	34	31.5	28.4	26.3	24.8	21.5	20.9
3Y	34	32.5	34.8	34.8	32	29.9	27.4	25.4	24.4	21.1	21.5
4Y	36.3	35	34.9	33.9	30.6	28.7	26.6	24.9	23.8	20.9	22.1
5Y	37.4	36.3	35.5	33.1	29.7	27.6	25.9	24.4	23.2	20.8	22.2
7Y	32.5	31.3	30.9	29.7	27.8	26.7	25.5	24.4	22.9	20.8	21.2
10Y	31.1	29.3	28.2	26.8	26.1	25.7	25.1	24.4	22.9	21.1	19.9
12Y	31.1	29.3	28.2	26.8	26.1	25.7	25.1	24.4	22.9	21.1	19.9
15Y	29.3	28	27.3	26.1	25.1	24.8	24.1	23	22.3	20.3	17.2
20Y	28.9	27.9	27.4	26.4	25	24.6	23.8	22.8	21.9	19.8	16.1
4Y swaption											
Strike/Maturity	**1M**	**3M**	**6M**	**1Y**	**2Y**	**3Y**	**4Y**	**5Y**	**7Y**	**10Y**	**20Y**
0.213859081	37.0984	37.0984	28.6	24.9333	14.1049	11.0987	9.9779	8.86667	7.96642	7.10018	7.46706
0.463859081	30.6482	30.6482	23.6167	20.0667	11.4791	9.07393	8.16222	7.25833	6.50812	5.76683	6.14199
0.713859081	24.1979	24.1979	18.6333	15.2	8.85319	7.04913	6.34654	5.65	5.04982	4.43349	4.81693
0.963859081	17.7476	17.7476	13.65	10.3333	6.22733	5.02434	4.53085	4.04167	3.59151	3.10014	3.49186
1.213859081	11.2973	11.2973	8.66667	5.46667	3.60146	2.99954	2.71517	2.43333	2.13321	1.76679	2.16679
1.463859081	7.46483	7.46483	5.76667	3.68333	2.4177	2.01633	1.84071	1.66667	1.41656	1.16679	1.51674
1.713859081	3.63239	3.63239	2.86667	1.9	1.23394	1.03312	0.96624	0.9	0.69991	0.56679	0.8667
1.963859081	1.39964	1.39964	1.1	0.76667	0.53358	0.39991	0.38319	0.36667	0.26661	0.23339	0.3667
2.213859081	0	0	0	0	0	0	0	0	0	0	0
2.463859081	-0.6331	-0.6331	-0.6333	-0.5	-0.4336	-0.2999	-0.2665	-0.2333	-0.2	-0.1666	-0.3
2.713859081	-0.7664	-0.7664	-0.8	-0.7667	-0.6338	-0.4998	-0.3995	-0.3	-0.3333	-0.2	-0.5667
2.963859081	-0.3831	-0.3831	-0.55	-0.8167	-0.7672	-0.6164	-0.4742	-0.3333	-0.3166	-0.15	-0.75
3.213859081	0.00018	0.00018	-0.3	-0.8667	-0.9007	-0.733	-0.549	-0.3667	-0.2999	-0.1	-0.9334
3.463859081	0.66681	0.66681	0.08333	-0.7083	-0.8592	-0.6747	-0.345	-0.2833	-0.2082	0.025	-1.0001
3.713859081	1.33344	1.33344	0.46667	-0.55	-0.8176	-0.6163	-0.3408	-0.2	-0.1166	0.15	-1.0667
3.963859081	2.00007	2.00007	0.85	-0.3917	-0.776	-0.5579	-0.3366	-0.1167	-0.0249	0.275	-1.1334
4.213859081	2.6667	2.6667	1.23333	-0.2333	-0.7345	-0.4995	-0.3323	-0.0333	0.06679	0.4	-1.2001
5Y swaption											
Strike/Maturity	**1M**	**3M**	**6M**	**1Y**	**2Y**	**3Y**	**4Y**	**5Y**	**7Y**	**10Y**	**20Y**
0.399829154	35.4	35.4	27.1	17	11.4	9.7	9.04733	8.4	7.7	7	7.9
0.649829154	28.625	28.625	21.875	13.725	9.25	7.9	7.37285	6.85	6.275	5.675	6.5
0.899829154	21.85	21.85	16.65	10.45	7.1	6.1	5.69836	5.3	4.85	4.35	5.1
1.149829154	15.075	15.075	11.425	7.175	4.95	4.3	4.02387	3.75	3.425	3.025	3.7
1.399829154	8.3	8.3	6.2	3.9	2.8	2.5	2.34938	2.2	2	1.7	2.3
1.649829154	5.45	5.45	4.1	2.6	1.85	1.65	1.57469	1.5	1.3	1.1	1.6
1.899829154	2.6	2.6	2	1.3	0.9	0.8	0.8	0.8	0.6	0.5	0.9
2.149829154	1	1	0.7	0.5	0.4	0.3	0.3	0.3	0.2	0.2	0.4
2.399829154	0	0	0	0	0	0	0	0	0	0	0
2.649829154	-0.4	-0.4	-0.4	-0.3	-0.3	-0.2	-0.2	-0.2	-0.2	-0.2	-0.3
2.899829154	-0.5	-0.5	-0.4	-0.4	-0.4	-0.3	-0.2498	-0.2	-0.3	-0.2	-0.6
3.149829154	-0.15	-0.15	-0.05	-0.35	-0.45	-0.35	-0.2747	-0.2	-0.25	-0.15	-0.8
3.399829154	0.2	0.2	0.3	-0.3	-0.5	-0.4	-0.2996	-0.2	-0.2	-0.1	-1

3.649829154	0.825	0.825	0.65	-0.1	-0.4	-0.3	0	-0.1	-0.1	0.025	-1.075
3.899829154	1.45	1.45	1	0.1	-0.3	-0.2	0	0	0	0.15	-1.15
4.149829154	2.075	2.075	1.35	0.3	-0.2	-0.1	0	0.1	0.1	0.275	-1.225
4.399829154	2.7	2.7	1.7	0.5	-0.1	0	0	0.2	0.2	0.4	-1.3
15Y swaptions											
Strike/Maturity	**1M**	**3M**	**6M**	**1Y**	**2Y**	**3Y**	**4Y**	**5Y**	**7Y**	**10Y**	**20Y**
1.104474218	23.2997	23.2997	18.6	12.8	9.6	10.85	11.5529	12.25	11.7999	11.1	8.79975
1.354474218	19.2247	19.2247	15.1375	10.425	7.7	8.92503	9.54628	10.1625	9.79996	9.2125	7.3123
1.604474218	15.1498	15.1498	11.675	8.05	5.8	7.00003	7.5397	8.075	7.79997	7.325	5.82484
1.854474218	11.0749	11.0749	8.2125	5.675	3.9	5.07502	5.53311	5.9875	5.79999	5.4375	4.33739
2.104474218	6.99995	6.99995	4.75	3.3	2	3.15001	3.52653	3.9	3.8	3.55	2.84993
2.354474218	4.72497	4.72497	3.025	2.175	1.125	2.15001	2.42612	2.7	2.65	2.45	1.99996
2.604474218	2.44999	2.44999	1.3	1.05	0.25	1.15001	1.32571	1.5	1.5	1.35	1.14999
2.854474218	0.94999	0.94999	0.55	0.4	-0.3	0.5	0.60041	0.7	0.7	0.6	0.5
3.104474218	0	0	0	0	0	0	0	0	0	0	0
3.354474218	-0.5	-0.5	-0.35	-0.25	-0.9	-0.3	-0.4004	-0.5	-0.4	-0.45	-0.4
3.604474218	-0.6	-0.6	-0.1	-0.25	-1	-0.45	-0.6508	-0.85	-0.75	-0.75	-0.75
3.854474218	-0.325	-0.325	0.3	-0.15	-0.975	-0.5	-0.751	-1	-0.9	-0.925	-0.9999
4.104474218	-0.05	-0.05	0.7	-0.05	-0.95	-0.55	-0.8512	-1.15	-1.05	-1.1	-1.2499
4.354474218	0.55005	0.55005	1.3125	0.2375	-0.8	-0.425	-0.4193	-1.1	-1.0125	-1.0625	-1.3499
4.604474218	1.15005	1.15005	1.925	0.525	-0.65	-0.3	-0.3882	-1.05	-0.975	-1.025	-1.4499
4.854474218	1.75006	1.75006	2.5375	0.8125	-0.5	-0.175	-0.357	-1	-0.9375	-0.9875	-1.5499
5.104474218	2.35007	2.35007	3.15	1.1	-0.35	-0.05	-0.3258	-0.95	-0.9	-0.95	-1.6499
20Y swaptions											
Strike/Maturity	**1M**	**3M**	**6M**	**1Y**	**2Y**	**3Y**	**4Y**	**5Y**	**7Y**	**10Y**	**20Y**
1.166501312	22.1	22.1	17.5	12.6	8.5	10.7	11.5033	12.3	12	11.3	7.9
1.416501312	18.275	18.275	14.225	10.275	6.675	8.8	9.50287	10.2	9.95	9.375	6.575
1.666501312	14.45	14.45	10.95	7.95	4.85	6.9	7.50246	8.1	7.9	7.45	5.25
1.916501312	10.625	10.625	7.675	5.625	3.025	5	5.50205	6	5.85	5.525	3.925
2.166501312	6.8	6.8	4.4	3.3	1.2	3.1	3.50164	3.9	3.8	3.6	2.6
2.416501312	4.6	4.6	2.7	2.15	0.35	2.1	2.40123	2.7	2.65	2.5	1.85
2.666501312	2.4	2.4	1	1	-0.5	1.1	1.30082	1.5	1.5	1.4	1.1
2.916501312	0.9	0.9	0.5	0.4	-1	0.5	0.60041	0.7	0.7	0.6	0.5
3.166501312	0	0	0	0	0	0	0	0	0	0	0
3.416501312	-0.5	-0.5	-0.6	-0.2	-1.6	-0.3	-0.4004	-0.5	-0.4	-0.4	-0.3
3.666501312	-0.5	-0.5	-0.4	-0.2	-1.7	-0.4	-0.6008	-0.8	-0.7	-0.7	-0.6
3.916501312	-0.2	-0.2	0.1	-0.05	-1.65	-0.45	-0.701	-0.95	-0.85	-0.85	-0.8
4.166501312	0.1	0.1	0.6	0.1	-1.6	-0.5	-0.8012	-1.1	-1	-1	-1
4.416501312	0.725	0.725	1.225	0.425	-1.45	-0.35	0	-1.025	-0.95	-0.95	-1.05
4.666501312	1.35	1.35	1.85	0.75	-1.3	-0.2	0	-0.95	-0.9	-0.9	-1.1
4.916501312	1.975	1.975	2.475	1.075	-1.15	-0.05	0	-0.875	-0.85	-0.85	-1.15
5.166501312	2.6	2.6	3.1	1.4	-1	0.1	0	-0.8	-0.8	-0.8	-1.2

The data in Table A1 contains the data to reproduce the calibration, smiles, surfaces and cubes done in previous sections. All values are in per cent unless other stated. The implied volatility for a specific swaption should be read by adding or subtracting a term to the ATM volatility. For example, the implied volatility for a 1M4Y swaption at strike K = 0.213859081 is (33.4 + 37.0984) = 70.4984% while the implied volatility for a 20Y20Y ATM swaption (i.e. K =3.166501312) is (16.1+0) = 16.1%.

Appendix 3 – Selected Code

This appendix contains some of the MATLAB code used for calibrating the SABR model. For a full account of MATLAB code, contact either author. We begin with the functions for two different methods of calibrations that based on the functions written by Rouah. The first function *EstimateAllParameters* corresponds to Method 2, where all the SABR parameters are estimated simultaneously.

```
function y = EstimateAllParameters(params,MktStrike,MktVol,F,T,b)

% -------------------------------------------------------------------------
% Returns the following SABR parameters:
% a = alpha
% r = rho
% v = vol-of-vol
% Required inputs:
% MktStrike = Vector of Strikes
% MktVol    = Vector of corresponding volatilities
% F = spot price
% T = maturity
% b = beta parameter
% -------------------------------------------------------------------------
a = params(1);
r = params(2);
v = params(3);

N = length(MktVol);

% Define the model volatility and the squared error terms
for i=1:N
    ModelVol(i) = SABRvol(a,b,r,v,F,MktStrike(i),T);
    error(i) = (ModelVol(i) - MktVol(i))^2;
end;

% Return the SSE
y = sum(error);

% Impose the constraint that -1 <= rho <= +1 and that v>0
if abs(r)>1 | v<0
    y = 1e100;
end
```

The second and the third function *EstimateRhoAndVol* and *findAlpha* are used for Method 1. As mentioned above, alpha is estimated through ATM volatility while rho and volvol is found by minimizing the SSE.

```
function y = EstimateRhoAndVol(params,MktStrike,MktVol,ATMVol,F,T,b)

% ---------------------------------------------------------------------------
% Returns the following SABR parameters:
% r = rho
% v = vol-of-vol
% Uses ATM volatility to estimate alpha
% Required inputs:
% MktStrike = Vector of Strikes
% MktVol    = Vector of corresponding volatilities
% ATMVol = ATM volatility
% F = spot price
% T = maturity
% b = beta parameter
% ---------------------------------------------------------------------------
r = params(1);
v = params(2);
a = findAlpha(F,F,T,ATMVol,b,r,v);
N = length(MktVol);

% Define the model volatility and the squared error terms
for i=1:N
    ModelVol(i) = SABRvol(a,b,r,v,F,MktStrike(i),T);
    error(i) = (ModelVol(i) - MktVol(i))^2;
end;

% Return the SSE
y = sum(error);

% Impose the constraint that -1 <= rho <= +1 and that v>0
% via a penalty on the objective function
if abs(r)>1 | v<0
    y = 1e100;
end
```

```
function y = findAlpha(F,K,T,ATMvol,b,r,v)
% By Fabrice Douglas Rouah

% ---------------------------------------------------------------------------
% Finds the SABR "alpha" parameter by solving the cubic equation described
% in West (2005) "Calibration of the SABR Model in Illiquid Markets"
% The function can return multiple roots.  In that case, the program
% eliminates negative roots, and select the smallest root among the
% positive roots that remain.
% Required inputs:
% F = spot price
% K = strike price
% T = maturity
% ATMvol = ATM market volatility
% b = beta parameter
% r = rho parameter
% v = vol of vol parameter
% ---------------------------------------------------------------------------
```

```
% Find the coefficients of the cubic equation for alpha
C0 = -ATMvol*F^(1-b);
C1 = (1 + (2-3*r^2)*v^2*T/24);
C2 = r*b*v*T/4/F^(1-b);
C3 = (1-b)^2*T/24/F^(2-2*b);

% Return the roots of the cubic equation (multiple roots)
AlphaVector = roots([C3 C2 C1 C0]);

% Find and return the smallest positive root
index = find(AlphaVector>0);
Alpha = AlphaVector(index);
y = min(Alpha);
```

This following function, *SABRvol* will return the SABR volatility for each strike and maturity in both methods.

```
function y = SABRvol(a,b,r,v,F,K,T);
% ---------------------------------------------
% Returns the SABR volatility.
% Required inputs:
% a = alpha parameter
% b = beta parameter
% r = rho parameter
% v = vol of vol parameter
% F = spot price
% K = strike price
% T = maturity
% ---------------------------------------------
% By Fabrice Douglas Rouah

% Separate into ATM case (simplest case) and
% Non-ATM case (most general case)

if abs(F-K) <= 0.001  % ATM vol

    Term1 = a/F^(1-b);
    Term2 = ((1-b)^2/24*a^2/F^(2-2*b) + r*b*a*v/4/F^(1-b) + (2-
3*r^2)*v^2/24);
    y = Term1*(1 + Term2*T);

else    % Non-ATM vol

    FK = F*K;
    z = v/a*(FK)^((1-b)/2)*log(F/K);
    x = log((sqrt(1 - 2*r*z + z^2) + z - r)/(1-r));
    Term1 = a / FK^((1-b)/2) / (1 + (1-b)^2/24*log(F/K)^2 + (1-
b)^4/1920*log(F/K)^4);
    if abs(x-z) < 1e-10
        Term2 = 1;
    else
        Term2 = z / x;
    end
    Term3 = 1 + ((1-b)^2/24*a^2/FK^(1-b) + r*b*v*a/4/FK^((1-b)/2) + (2-
3*r^2)/24*v^2)*T;
    y = Term1*Term2*Term3;
```

```
end
```

For all code above, we refer to the eminent work of Fabrice Douglas Rouah with free MATLAB functions available at www.volopta.com.

Below is a very short section from our code for calibrating one example of a 1M5Y swaption, given the above functions.

```
%% Calibrating the smile for a 1M5Y swaption

%Define the starting values and options for fminsearch
start = [0.3,0.3];
options = optimset('MaxFunEvals', 1e5, 'TolFun', 1e-8, 'TolX', 1e-10);

% Parameter estimation method 1.  Set Beta = 0.5.
% Estimate rho and v, and at each iteration step,
% Find alpha as the cubic root using the findAlpha function
Beta = 0.5;
[param, feval] =
fminsearch(@(par)EstimateRhoAndVol(par,K5Y,Vol5Y(:,1),Vol5ATM(1),Swapcurve(
5),Expiry(1)/12,Beta),start,options);
r1 = param(1);
v1 = param(2);
a1 =
findAlpha(Swapcurve(5),Swapcurve(5),Expiry(1)/12,Vol5ATM(1),Beta,r1,v1);
for j=1:length(K5Y);
    SABRVol1M5Y_method1(j) =
SABRvol(a1,Beta,r1,v1,Swapcurve(5),K5Y(j),Expiry(1)/12);
end
```

Appendix 4 – Ordinary Least Squares

To minimize the sum squared of errors (SSE) in our calibration, the technique implied is Ordinary Least Squares (OLS) regression estimation. Given a single variable regression model below

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

Where Y_i and X_i is the observed market data, $\widehat{Y_i}$ is estimated data, OLS will chose β_0 and β_1 that gives the minimum squared ϵ_i all over sample data points. Where

$$\epsilon_i = Y_i - \widehat{Y_i}$$

$$\widehat{Y_i} = \widehat{\beta_0} + \widehat{\beta_1} X_i$$

So, in detail, OLS minimize

$$\sum_i e_i^2 = \sum_i (Y_i - \widehat{\beta_0} - \widehat{\beta_1} X_i)^2$$

For only one independent variable, $\widehat{\beta_1}$ and $\widehat{\beta_0}$ are calculated as follow

$$\widehat{\beta_1} = \frac{\sum_{i=1}^{N}[(X_i - \bar{X})(Y_i - \bar{Y})]}{\sum_{i=1}^{N}(X_i - \bar{X})^2}$$

$$\widehat{\beta_0} = \bar{Y} - \widehat{\beta_1 \bar{X}}$$

However, in our model, there are two or three independent variables in the regression model (ρ, ν and α). In which, the Equation (3.4.1) has to be rewritten where ρ, ν and α are $\widehat{\beta_1}$, $\widehat{\beta_2}$ and $\widehat{\beta_3}$ respectively. Now, the method is called multivariate regression model but it has the same underlying principle as the single variable model. The goal now is to minimize the SSE in the same mathematical approach but with one variable at a time while keeping others constant. The formula for different β is very cumbersome and vast in size. We refer to A. H. Studenmund (2011) for the content of this section and more information.

7 SUMMARY OF REFLECTION OF OBJECTIVES IN THE THESIS

Objective 1: ***demonstrate knowledge and understanding in the major field of study, including knowledge of the field's scientific basis, knowledge of applicable methods in the field, specialisation in some part of the field and orientation in current research questions.***

- Survey of literature with comments related to the current research questions: ***Section 1.3 Review of Current Literature***.
- Survey and comparison of alternative methods related to the subject of the project: **Section 2.7 Local Volatility Models.**
- Deeper presentation of specific methods supposed to be used in the project: ***Section 2 Theoretical Framework.***

In Section 1.3, we presented the findings from papers that have examined the SABR model. We found that in a majority of papers that the SABR has successfully captured the smile effect of volatility for different types of option. However, there is controversy about the SABR hedging ability. Many papers also studied and suggested methods of calibration and how the parameters effect the smile produced by the model.

Section 2.7 addresses one of the alternative models that can be used to capture the observed volatility smiles on the markets. We choose this model in particular since this is the most known one before the SABR and the one that Hagan et al. (2002) studied. It was somewhat challenging to capture the model, however we still claim that Section 2.7 sufficiently shows the general idea so that the reader can grasp why the SABR model was developed. Here we have learned another usage of the Binomial tree that was very interesting.

In Section 2, we presented a broad framework regarding pricing of swaps and swaptions. There we carefully explained the rates and other inputs that are used in valuation and pricing model. Many other papers take this for granted, just stating the pricing formula. However, since our aim is to explain the SABR model for swaptions, we found that the explanation and valuation of these derivatives were crucial. The Black-Scholes and the Black-76 frameworks are important in order to understand how these options can be priced under certain assumptions. This section has also developed our understanding about pricing certain derivatives and has thus developed our skills in applied (financial) mathematics where we had to express how to price them in reality. In addition, we also gained better understanding about Martingale and Arbitrage Pricing Theory.

Objective 2: ***demonstrate the ability to search, collect, evaluate and critically interpret relevant information in a problem formulation and to critically discuss phenomena, problem formulations and situations.***

- Analysis of data, their quality, volume, shortage, etc. (if any): ***Section 1.5 Data Selection, Assortment and Limitations***

- Description of the model and comparisons with alternative models: ***Section 3 The SABR Model*, Section 3.5.2 An updated Set of Greeks, Section 3.6 Refinement of the SABR model.**

In Section 1.5, we address the quality and source of the data and discuss upon this matter. We also stress the point that the result in the calibration of the SABR model depends very much of the quality of the data. This data is generally not available to the public and was handed to use by our supervisor Jan Röman who works in one of Sweden's major banks.

In Section 3, the SABR Model is presented. We aimed to give a good understanding about the SABR model so that the plausible reader does not have to turn to secondary sources for additional explanation. We also decided to give a deeper presentation on how to calibrate the model since this will affect the final result of the fit to the market data. This section has developed our knowledge about stochastic processes, calibration, optimization techniques and the interpretation of Greeks.

Section 3.5.2 and Section 3.6 have in turn pointed out alternative method to calculate the implied volatility besides the original and another set of Greeks for better hedging risks. However, we decided to present the main formulas in Section 3.5.2 shortly and skip the detail derivation. We believe the derivation is complicated and the main formulas are sufficient to implement.

Objective 3: ***demonstrate the ability to independently identify, formulate and solve problems and to perform tasks within specified time frames.***

- Formulation of the problem studied in the project and the goals of the project: ***Section 1.4 The Aim of the Thesis.***

- Evaluation of possible solution in the time framework and presentation of solution (algorithms, results of experiments, description of programs, presentation of input-output interfaces, etc.): ***Section 4 Calibration and Results.***

- Program codes: ***Appendix 3 Selected Code.***

The topic was initiated by our supervisor, Jan Röman. From his idea, we have formulated the aim of the thesis, found in Section 1.4. As mentioned earlier, we wanted to give a broad description of the model for readers that are new to the topic like ourselves can understand

and follow the paper. This led us to develop Section 2 where we guided readers through the process that leading up the creation of the SABR model.

In Section 4, we presented the results we found. We argued that the most interesting cases one could imagine have been covered since we compare various methods of calibration, different beta and the influences of a refinement technique.

In appendices, only a small segment of MATLAB code is included. This should be the focal idea on how calibration should be done. Our actual code consists of almost 1000 lines of MATLAB code where we first imported the data then categorized it. We later performed all the operations required to calibrate the SABR model. In this part, we had to be careful so that the result is correct and we argue that we succeed in doing so. After writing this application, we have gained much skill in coding with MATLAB.

Objective 4: ***demonstrate the ability to present orally and in writing and discuss information, problems and solutions in dialogue with different groups.***

- Print of the oral presentation of the project: ***N/A***
- Improved English and the thesis structure (abstract, table of contents, sections, conclusion, references): ***The thesis structure is completed with all requirements and English has been checked.***
- The place of results in the area; the list of main results and achievements; potential use of results; possible future continuation of the project: ***Section 4.4 Additional Remarks, Section 5 Conclusion.***

The presentation of the project was done as a seminar with a smaller audience where our supervisors, examiner and a few colleagues were present. We had prepared a 20-slide long PowerPoint presentation for approximately 20 minutes. During this presentation, we tried to capture the main points about the SABR model so that a listener with only limited knowledge could follow. We maintain that we succeeded to present a rather complicated topic during the time we were given.

We decided to follow of structure of the paper that lead up to the SABR model in Section 3. We think that this structure of the report, with a literature review before presenting the actual SABR model should make this paper comprehensible for any plausible reader.

Our paper adds to the ones examining the aptness of the SABR model. Several suggestions about further conceivable studies are presented after results had been discussed and compared with previous studies.

Objective 6: demonstrate ability in the major field of study make judgements with respect to scientific, societal and ethical aspects.

- Popular presentation of project and its results: ***Section 1.1 Background, Section 1.2 Problem Statement.***
- Remarks concerned the use of Internet and correctness of citations: ***In this paper we assure the correctness of citations and use of Internet sources.***
- Ethically correct description of contribution of co-authors if any: **N/A.**
- Acknowledgement: ***Page V.***

Röman's (and others) contributions are acknowledged on page V, before the popular presentation of the project, Section 1.1 Background, is presented. Here we tried to give the reader a broad picture of the problem with stochastic volatility and how the history had lead us up to where we are today. Since derivative trading has an enormous turn-over on the world's exchanges as well as OTC, this paper can appeal to financial engineers, investors and traders.

We use very few Internet resources for this paper. We have for example used figures that represent volatility smiles that we found on the Internet. Where it is done, we have stated the Internet source.

Neither of the co-authors is personally responsible for an entire section. Instead, we have helped each other to write the sections in this report so that both of the authors have a good overview about the SABR model.

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