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Risk Measures with Normal Distributed Black Options Pricing Model

Supervisors: Anatoliy Malyarenko, Jan Röman

Author: Wenqing Huang

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Abstract

The aim of the paper is to study the normal Black model against the classical Black model in a trading point of view and from a risk perspective. In the section 2, the derivation of the model is subject to the assumption that implementation of a dynamic hedging strategy will eliminate the risk of holding long or short positions in such options. In addition, the derivation of the formulas has been proved mathematically by the notable no-arbitrage argument. The idea of the theory is that the fair value of any derivative security is computed as the expectation of the payoff under an equivalent martingale measure. In the third section, the Greeks have been derived by differentiation. Also brief explanation regarding how one can approximate log-normal Black with normal version has been explored in the section 4. Eventually, with the aid of Excel VBA, there is an empirical test for swaptions on an At-The-Money volatility surface, given as Black (log-normal) volatilities, which is translated into a normal volatility surface. Then calculate and plot how delta and vega differs between the models.

Key Words: Black model, Normal Black model, Interest rate derivatives, Forward swap rate, Swaptions, Volatilities, Greeks, Yield curve, Risk disparities

Contents

1. Introduction

1.1. Negative interest rates

When it comes to the question regarding how low interest rates can reach, both the economists and traders around the globe believe that interest rates could go negative. Although that sounds unbelievable, there are precedents for negative interest rates.

To name just a few macroeconomic cases, in the late 1990s zero interest rate came into being briefly, as Japanese savers were so unsecured about the stock market which had collapsed, they would rather deposit their money in the bank even at a negative interest rate. In 2009, Sweden's Riksbank, the first central bank to adopt negative interest rates, actually lowered its deposit rate to a -0.25% in the midst of the financial crisis. On a regular basis, negative real interest rates, when nominal interest rates are below inflation rate, become a monetary policy tool of the governments to tax on money as a mean of driving money out of the banks and into investments to benefit the economy, as well as dealing with variables like inflation and unemployment.

In the financial market, another example of a negative interest rate, a rate below zero whereby the lender pays interest to the borrower, is a private placement with institutional investor of debt-plus-warrants, a security called Squarz from Berkshire Hathaway Corporation governed by the Wall Street legend Warren Buffett. Goldman Sachs engineered the structure in mid-2002, and called it the "first ever negative-coupon security." With the aid of high stock market volatility and low market interest rates, Berkshire would pay around 3% annually on the bonds being issued. Instead the investor would receive a warrant allowing the purchase of Berkshire stock as well, and to keep the warrant alive, investors would have to pay a higher rate, perhaps 3.75%, meanwhile Berkshire made the interest payments. To sum up, the net effect amounted to a negative interest rate.

In recent decades, the interest rate derivative instruments, which are products whose payoffs depend in some way on the level of interest rate, are increasingly arousing interest from either such institutional investors as banks or individual investors with customized cashflow demands or as a speculative tool to profit from their specific views on the movement of the market interest rates, say directional movements or volatility movements.

1.2. Notations and assumptions

For the whole paper, denote the current future price *f*, with $-\infty \leq f \leq \infty$. For instruments like swaptions, *f* stands for the forward rates which are the rates of interest implied by current zero rates for periods of time in the future. We do not take in account the case of an option on a normally distributed spot price, as this is an obvious special case of an option on a forward price. Let *C* and *P* be the value of European call and put options respectively. The strike price and time to maturity are denoted by *K* and $T-t$ respectively. The annual risk-free interest rate is denoted by *r* and the annual volatility rate (or the annual standard deviation of the price of the asset) is denoted by σ. Finally, I will use $N(X)$ which denotes the standard normal cumulative distribution function, $N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{z^2}{2}}dz$; $N'(x)$ which denotes the standard normal probability density function, $N'(x)=\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$.

At the same time, the following explicit assumptions are made:

- There is no arbitrage opportunity (i.e., there is no way to make a riskless profit).
- It is allowed to borrow and lend cash at a known constant risk-free interest rate.
- It is likely to buy and sell any amount, even fractional, of the underlying (this includes short selling).
- The above transactions do not incur any fees or costs (i.e., frictionless market).
- The underlying does not pay a dividend.

- The options can not be exercised earlier than the maturity time, i.e. European type options.
- The current future price follows the following normal distibution in a risk-neutral world:

$df = \mu dt + \sigma dW_t$, where μ and σ are constant.

1.3. A short review of the literature on the Black option pricing model

In 1976, Fischer Black, one of the fathers of the Black-Scholes model coined in 1973, demonstrated how the Black-Scholes model could be modified in order to value European call or put options on futures contracts. For options on forward or futures, derived directly from Black-Scholes model with:

$$f = e^{r(T-t)} S \quad [1] \qquad [1.3.1]$$

then the Black formulas[2] are

$$P_{Call} = e^{-r(T-t)}[fN(d_1) - KN(d_2)] \qquad [1.3.2]$$

$$P_{Put} = e^{-r(T-t)}[KN(-d_1) - fN(-d_1)] \qquad [1.3.3]$$

where $d_2 = \frac{ln(\frac{f}{K}) - \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$

Why are futures options in the equity market and the Black pricing model popular? In terms of futures options, Hull (2008) summarized that there are a number of reasons: 1).Compared with spot options, which are exercised as soon as the sale or purchase of the asset at the agreed-on price take place, futures options are exercised to give the holder a right to enter into a futures contract at a certain futures price by a given date. In most case, a future contract is more liquid and easier to trade than the underlying asset.

2).Besides, a future price is informed readily from trading on the futures exchange, while the spot price of the underlying may not be known immediately. Take Treasury bonds as an example, the market for Treasury bond futures is far more active than the

[1] *S* is the spot price of the underlying asset.

[2] For their Greeks, see Appendix.

one for any Treasury bond. The current market price of a bond can not be available but contacting dealers, whereas a bond futures price is obtained shortly from trading on the Chicago Board of Trade.

3).One of beauties of a future option for most capitalists is that exercising it does not, more often than not, result in delivery of the underlying asset, since the underlying futures contract is closed out prior to delivery commonly. They are eventually settled in cash.

4).Another advantage for futures options is that they are traded in the same exchange and facilitates speculation hedging and arbitrage and tends to require lower transaction costs than spot options to make the markets more efficient.

Traders prefer to Black pricing model to price not only European options on physical commodities, forward or futures, but also interest rates derivative instruments, including bond options, interest rate caps and floors and swaptions primarily. They are widely used to either speculate on the future course of interest rates or to hedge the interest payments or receipts on an underlying position. Besides, they allow an investor to benefit from changes in interest rates while limiting any downside losses. For instance, Black's model can be used to imply a term structure of forward rates from actively traded index option.

1.4. Problem formulation

Therefore, it has become a hot issue in the study of pricing interest rate derivatives like interest rate cap & floors, options of Forward Rate Agreement or European swaptions. The Black model, alternatively referred as the Black-76 model, is available as the standard model for valuing these over-the-counter interest rate options. This model is classified into a class of models known as log-normal forward models under the assumption that the underlying asset, i.e. the interest rate, is lognormal distributed. Unfortunately, in the current interest rate market situation with very low or even negative interest rates, different from that of the equity market, the parameter of strike price (and/or the forward rate ("the price")can take zero or negative values which makes a log-normal model hardly take effect and give accurate market prices.

To handle this puzzle, a normal distributed Black model is required. As a matter of fact, this case was initially considered by Bachelier's model illustrated as below [1.4.1] of arithmetic Brownian motion in 1900, it came to be was regarded as an instructive dead end though. The main reason is that it took time value of money (i.e. lack of the discount factor) out of consideration. Nonetheless, I suggest here that it is premature to conclude that an option pricing model with a normal underlying is of no use. In addition to the work of Bachelier (1990), I would like to mention papers with introduction related to the research questions by Hagan and Woodward (1998), Iwasawa (2001), Henrard (2005), Blake, Dawson and Dowd (2007), Grunspan (2011), together with Benhamou and Nodelman (2013).

$$C(S,T) = SN(\frac{S-K}{\sigma\sqrt{T-t}}) - KN(\frac{S-K}{\sigma\sqrt{T-t}}) + \sigma\sqrt{T-t}N'(\frac{K-S}{\sigma\sqrt{T-t}}) \quad [1.4.1]$$

Accordingly, this brings us to the problems of the paper: when the interest rate is modeled with a normal distributed stochastic process, what does the model look like? Do the two models give the identical price with the right volatility? Will the risk, i.e. the Greeks (delta, gamma, vega, theta and rho), differ when we are shifting the yield curve? Then the interest rate volatility is normal while in Black the volatility is log-normal. Both models are supposed to give the same prices and there exists formula to convert from normal to log-normal or vice verse. Therefore, in a trading point of view and from a risk perspective, a study of a normal and a log-normal Black model is of great interest.

For this paper, the author describes the current state of the interest rate derivatives market in the context of normal distributed Black options pricing model against classical Black model. The theory depends upon various areas of applied mathematics with specialization in financial engineering, including stochastic calculus, the implementation of a dynamic hedging strategy, the notable no-arbitrage argument, the application of Feynman-Kač, the equivalent martingale measure and particularly risk measures. Also singular perturbation theory explains how one can approximate normal volatility with log-normal one. The paper unifies couples of results scattered throughout the mathematical and financial literature and papers, as well as it tests empirically new outcomes from this highly promising area by the author, with the assistance of Excel VBA.

2. Derivation of equation and formulas

Start by constructing a certain portfolio, called the delta hedged portfolio, consisting of being long delta shares of future contract and short one derivative in question. Say, call it Π. Let us also denote the value of derivative by g. Then, the value of the delta-hedged portfolio is given by:

$$\Pi = g - \frac{\partial g}{\partial f} f \qquad [2.1]$$

So applying Ito's lemma using the SDE given above into the changes of the portfolio value, one gets:

$$d\Pi = dg - \frac{\partial g}{\partial f}df = (\frac{\partial g}{\partial t} + \mu\frac{\partial g}{\partial f} + \frac{1}{2}\sigma^2\frac{\partial^2 g}{\partial f^2})dt + \sigma\frac{\partial g}{\partial f}dW - \frac{\partial g}{\partial f}(\mu dt + \sigma dW)$$
$$= (\frac{\partial g}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 g}{\partial f^2})dt \qquad [2.2]$$

Notice that the dW term has vanished. Thus uncertainty has been eliminated and the portfolio is effectively riskless. The rate of return on this portfolio must be equal to the rate of return on any other riskless instrument; otherwise, there would be opportunities for arbitrage. We want the above riskless portfolio to be a martingale under the discounted expectation. This is to say that the above quantity equals the gain from the risk free interest rate for the portfolio value. So, over the time period $[t, t+\Delta t]$ we have:

$$d\Pi = r\Pi dt \qquad [2.3]$$

Since it cost nothing to enter into a futures contract at the beginning, one has: Π = g. Thus, we arrive at the following partial differential equation:

$$\frac{\partial g}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 g}{\partial f^2} = rg \qquad [2.4]$$

To test the effectiveness of this strategy, simulate the returns to dealers with short positions in payer and receiver swaptions respectively who perform daily rehedging over the lifetime of the swaptions. Monte Carlo simulation is available to model the evolution of the underlying forward swap price, assuming a normal distribution. It is assumed that a dealer starts with zero cash and borrowing or depositing at the riskless interest rate in response to the cashflows gained by the dynamic hedging strategy.

As Merton (1973) and Blake, Dawson and Dowd (2007) indicate, since the portfolio requires zero investment, it must be that to avoid "arbitrage" profits, the expected and realized return on the portfolio with this strategy is zero. Merton's model was predicated on rehedging in continuous time, which would bring about expected and realized returns being identical. In practice, traders tend to use discrete time rehedging alternatively. One outcome of this is that, over a large number of simulations, the expected return will be zero, although on any individual simulation, the realized return may differ from zero.

In a risk neutral world, the process followed by the variable *V* known as a Wiener process is giving as $df = \sigma dV_t$ with the trivial solution, from integration over the interval [*t, T*]:

$$f_T = f_t + \sigma(V_T - V_t) = f_t + \sigma\sqrt{T-t}z \qquad [2.5]$$

As far as we can see , since z has a standardized normal distribution with mean 0 and variance 1, then f_T is a Gaussian process; $N[f_t, \sigma^2(T-t)]$, i.e. with mean f_t and variance $\sigma^2(T-t)$. According to the above parabolic partial differential equation [2.4], the terminal payoff is $g(t,f) = f(f)$.By the application of Feynman-Kač to compute the expectations of random process equivalent to the integral of a solution to a diffusion equation, we obtain the following solution:

$$g(t, f_T) = e^{-r(T-t)} E^Q[\Phi(T)] = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} \Phi(T) e^{-\frac{(f_T - f_t)^2}{2\sigma^2(T-t)}} df_T \quad [2.6]$$

where $\Phi(T) = \begin{cases} (f_T - K)^+ & \text{for a Call}_3 \\ (K - f_T)^+ & \text{for a Put} \end{cases}$

Intuitively, couple [2.5] with [2.6] and the solution can also be expressed as:

$$g(t, f_T) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} \Phi(T) e^{\frac{(\sigma\sqrt{T-t}z)^2}{2\sigma^2(T-t)}} \sigma\sqrt{T-t}dz$$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(T) e^{-\frac{z^2}{2}} dz \quad [2.7]$$

Therefore, the formula for the above can be simplified by simply expanding the expression inside the integral. The detail will be shown for more general audience.

For the call, we have:

$$\Pi_C(t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f_T - K)^+ e^{-\frac{z^2}{2}} dz$$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f_t + \sigma\sqrt{T-t}z - K)^+ e^{-\frac{z^2}{2}} dz$$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{z_0}^{\infty} (f_t + \sigma\sqrt{T-t}z - K) e^{-\frac{z^2}{2}} dz$$

$$= A - B \quad [2.8]$$

Set $f_t = F$ and with $z_0 = \frac{F-K}{\sigma\sqrt{T-t}}$ we get

$A = \frac{e^{-r(T-t)}}{\sqrt{2\pi}}(F-K) \int_{z_0}^{\infty} e^{-\frac{z^2}{2}} dz = e^{-r(T-t)}(F-K)N[z_0]$

$B = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{z_0}^{\infty} \sigma\sqrt{T-t}ze^{-\frac{z^2}{2}} dz = e^{-r(T-t)} \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} e^{-\frac{z_0^2}{2}}$

[3] $(f_T - K)^+$ refers to $max[(f_T - K), 0]$; Similarly, $(K - f_T)^+$ refers to $max[(K - f_T), 0]$.

Then, the fair values of call C and corresponding put P are given as:

$$C = e^{-r(T-t)}[(F-K)N(d_1)+\frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}] = e^{-r(T-t)}[(F-K)N(d_1)+\sigma\sqrt{T-t}N'(d_1)] \quad [2.9]$$

$$P = e^{-r(T-t)}[(K-F)N(-d_1)+\frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}] = e^{-r(T-t)}[(K-F)N(-d_1)+\sigma\sqrt{T-t}N'(d_1)] \quad [2.10]$$

where $d_1 = \frac{(F-K)}{\sigma\sqrt{T-t}}$.

Consider European swaptions, the holder of the option has the right to enter a swap which commences at a specified time t_0, the first payment being one time period later, at t_1, and lasts until time t_n. Then are two possibilities exist:

(a) A payer swaption, which gives the holder the right but not the obligation to receive a floating rate, and pay a fixed rate L_K (a call on the floating rate).

(b) A receiver swaption, which gives the holder the right but not the obligation to receive a fixed rate L_K, and pay a floating rate (a put on the floating rate).

The value of the swaption per unit of nominal derived from the normal Black formula is expressed as

$$V_a = S_n^N(t)[a(F - L_K)N(ad_1) + \sigma\sqrt{T-t}N'(d_1)]; \qquad d_1 = \frac{(F-L_K)}{\sigma\sqrt{T-t}} \quad [2.11]$$

where $a = 1$ stands for a payer swaption, $a = -1$ for a receiver swaption. Here $S_n^N(t)$ is called the accrual factor, the value of a basis point (PV01, DV01), the level or the annuity[4].

3. Risk measures

A financial institution is always faced with the problem of managing its risk when selling an option to a client in the over-the-counter markets. The institution can neutralize its exposure by buying the identical option as it has sold on the exchange,

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[4] The annuity factor is $\frac{1-\frac{1}{(1+\frac{F}{m})^{(T-t)m}}}{F}e^{-rT}$, where m represents the compounding frequency per year in swap rate. It is as same as the derived one of Black-76 model. Concerning the derivation of this factor, see Appendix 2.

as long as the option happens to be the same as one that is traded on an exchange. However, when the option has been customized to the demands of a client and does not correspond to the standardized products traded by exchanges, then hedging the exposure is much trickier.

To solve this, alternative approaches are commonly referred to as the ‘Greeks’. The Greeks are vital tools in risk management. Each Greek letter measures the risk in a different dimension in an option position and the purpose of a trader is to manage the Greeks so that all risks are acceptable enough. Limits are defined for each Greek letter. For example, the delta limit is often expressed as the equivalent maximum position in the underlying asset. Besides, the vega limit is usually expressed as a maximum dollar exposure per 1% change in the volatility. And special permission is necessary if a trader intends to exceed a limit at the end of a trading day. Moreover, the first-order Greeks (delta, gamma, vega, theta and rho) are computed by simple differentiation of the above formulas, as exhibited below one by one in this section.

3.1. Delta

Most importantly, the delta $(\Delta)$of an option is the rate of change of its price with respect to the price of the underlying asset. According to the put–call parity, a long call plus a short put (a call minus a put) replicates a forward, which has delta equal to 1. That is, for a European call and put option for the same strike price and time to maturity of underlying, and without dividend yield, the sum of the absolute values of the delta of each option will be 1.00.

Since the delta of underlying asset is always 1.0, the trader could delta-hedge his entire position in the underlying by buying or shorting the number of shares indicated by the total delta. For example, if the delta of a portfolio of options in Z (expressed as shares of the underlying) is +1.75, the trader would be able to delta-hedge the portfolio by selling short 1.75 shares of the underlying. This portfolio will then keep its total value whichever direction the price of Z moves. The delta of an option varies over time, thus the position in the underlying asset has to be rebalanced by traders at least on a daily basis.

$$\Delta_{(call)} = \frac{\partial C}{\partial F} = e^{-r(T-t)}[(F-K)\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\frac{1}{\sigma\sqrt{T-t}}] + e^{-r(T-t)}N(d_1) +$$

$$e^{-r(T-t)}\frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{\frac{-d_1^2}{2}}[-\frac{(F-K)}{\sigma\sqrt{T-t}}][\frac{1}{\sigma\sqrt{T-t}}] = e^{-r(T-t)}N(d_1) \qquad [3.1]$$

$$\Delta_{(put)} = \frac{\partial P}{\partial F} = e^{-r(T-t)}[(K-F)\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\frac{-1}{\sigma\sqrt{T-t}}] - e^{-r(T-t)}N(-d_1) +$$

$$e^{-r(T-t)}\frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{\frac{-d_1^2}{2}}[-\frac{(F-K)}{\sigma\sqrt{T-t}}][\frac{1}{\sigma\sqrt{T-t}}] = -e^{-r(T-t)}N(-d_1) \qquad [3.2]$$

In the equity market, delta is trivial since an underlying instrument can vary in price. Nevertheless, in the context of interest rate theory, the changes occur as a change in the interest rate curve. Not a single value (point), but the whole curve. This implies that delta risk is the risk associated with a shift in the zero curve. Thus, delta can be defined in several alternatives:

i) By shifting the swap rate (F), i.e., the fixed rate in an underlying swap. It is the standard in some trading systems to take the analytical derivative of the swaption price with respect to the forward swap rate, ignoring the annuity term which also depends on the swap rate via the discount factors.

ii) By shifting the yield curve (the zero coupon curve) with one basis-point (1bp = 0.01 %). This is sometimes termed as DV01 or PV01[5]. This approach assumes implicitly a parallel evolution of the interest rate curve.

iii) By shifting the quoted rate, before bootstrapping the market quotes to a zero curve. Traders argue that the zero curve can changes only if the quote for the instruments used to calculate the zero curve changes. In this delta definition, one computes as many as deltas as there are market instruments. Each delta corresponds to the isolated influence of the change of the market quote of one

[5] PV01 is the variation in "Present Value" of a 1 basis-point shift of the rate; DV01 refers to the same ratio expressed in "Dollar". They are analogous to the delta in derivative pricing and almost the same.

market instrument. Thus it makes sense to outline the exposures arising from the changes in the quote.

iv) By calculating the change in the value of the swaption with respect to the change of the underlying swap value when making a shift in the curve as ii) or iii).

v) By shifting of certain section or time buckets of the interest rate curve. The risk profile is aggregated, as bucket delta measures the impact of shifting the rates of a given bucket by one basis point while the other buckets stay unchanged. The fair forward swap rate is dependent upon the bootstrap and interpolation method associated with the construction of the yield curve. When applying continuous compounding of the interest rate mathematically to express the forward rate, the yield on all maturities can change by the same number of basis points and a parallel shift in the yield curve occurs. Unfortunately, the world is far from that simple because the zero rates can tilt up or down over a long term and the risk is placed unevenly in time buckets. As a result, a change in the yield curve with different maturities in which the changes in yields do not occur evenly in the financial market. This method is quite useful to give a condensed overview of the risk when the traders intend to hedge partially risks.

3.2. Gamma

Once an option position has been made delta neutral, the next step is to focus on its gamma (Γ).It is the rate of change of its delta with respect to the price of the underlying asset. It is a measure of the curvature of the relationship between the option price and the asset price. The impact of this curvature on the performance of delta hedging can be decreased by making an option position gamma neutral. To be more exactly, if is the gamma of the position being hedged, this decrease is often completed by taking a position in a traded option that has a gamma of $-\Gamma$. It is evident from the Put-Call parity that by differentiating the put and call formula twice with respect to the underlier establishes the equality of gamma of put and call for option models.

$$\Gamma_{(Call)} = \Gamma_{(Put)} = \frac{\partial^2 C}{\partial F^2} = e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\frac{1}{\sigma\sqrt{T-t}}$$

$$= \frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} = \frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}N'(d_1) \quad [3.3]$$

3.3. Vega

Practically, volatilities of the underlying asset are stochastic throughout the time, while delta and gamma hedging are under the assumption that the volatility remains constant. The vega(ν) of an option or an option portfolio measures the rate of change of its value with respect to volatility. Vega can be an important Greek to monitor for an option trader, especially in volatile markets. Making the position vega neutral can help trader hedge an option position against volatility changes. Again, from the Put-Call parity, by differentiating the put and call formula twice with respect to σ comes to the equality of put and call for option models.

$$\nu_{(Call)} = \nu_{(Put)} = \frac{\partial C}{\partial \sigma} = e^{-r(T-t)}(F-K)\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}[-\frac{(F-K)}{\sigma^2\sqrt{T-t}}]+$$

$$e^{-r(T-t)}\frac{\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} + e^{-r(T-t)}\frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}[-\frac{(F-K)}{\sigma\sqrt{T-t}}][-\frac{(F-K)}{\sigma^2\sqrt{T-t}}]$$

$$= e^{-r(T-t)}\frac{\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} = e^{-r(T-t)}\sqrt{T-t}N'(d_1) \quad [3.4]$$

Put-Call Parity: $P = C - e^{-r(T-t)}(F-K) \Rightarrow \frac{\partial P}{\partial \sigma} = \frac{\partial C}{\partial \sigma} - \frac{\partial}{\partial \sigma}e^{-r(T-t)}(F-K) = \frac{\partial C}{\partial \sigma}$

Since vega is conventionally presented by practitioners in terms of a one percentage point change in volatility, then vega can also be $\frac{e^{-r(T-t)}\sqrt{T-t}N'(d_1)}{100}$. To create both gamma and vega neutrality, two traded derivatives dependent on the underlying asset must be used. Unlike delta, typically it is far from feasible to maintain gamma and vega neutrality regularly. If they get too large, trading is curtailed or corrective action is taken.

3.4. Theta and Rho

Another measure of the risk of an option position is theta(Θ) which measures the rate of change of the value of the position with respect to the passage of time, with the rest holding constant. As demonstrated below, the chain rule and the product rule as well as the sum rule are applied implicitly. By convention, practitioners quote theta as the change in an option's value as one day passes, as exhibited [3.5B]&[3.6B]. On top of those, to measure the rate of change of the value of the position with respect to the interest rate, rho (ρ) is available. The value of an option is generally less sensitive to changes in the risk free interest rate than to changes in other parameters. For this reason, rho is the least used of the first-order Greeks.

$$\Theta_{(Call)} = \frac{\partial C}{\partial(T-t)}$$

$$= -re^{-r(T-t)}[(F-K)N(d_1) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}] + e^{-r(T-t)}[(F-K)\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}(-\frac{1}{2}\frac{F-K}{\sigma(T-t)^{\frac{3}{2}}})]$$

$$+e^{-r(T-t)}[\frac{\sigma}{2\sqrt{2\pi(T-t)}}e^{-\frac{d_1^2}{2}}] + e^{-r(T-t)}[\frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}(-\frac{(F-K)}{\sigma\sqrt{T-t}})(-\frac{1}{2}\frac{F-K}{\sigma(T-t)^{\frac{3}{2}}})]$$

$$= -re^{-r(T-t)}[(F-K)N(d_1) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}] + e^{-r(T-t)}(\frac{\sigma}{2\sqrt{2\pi(T-t)}}e^{-\frac{d_1^2}{2}})$$

$$= -rC + \frac{e^{-r(T-t)}\sigma N'(d_1)}{2\sqrt{T-t}} \qquad [3.5A]$$

$$= \frac{2\sqrt{T-t}rC - e^{-r(T-t)}\sigma N'(d_1)}{730\sqrt{T-t}} \qquad [3.5B]$$

$$\Theta_{(Put)} = \frac{\partial P}{\partial(T-t)}$$

$$=\text{-re}^{-r(T-t)}[(K-F)N(-d_1) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}] - e^{-r(T-t)}[(K-F)\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}(-\frac{1}{2}\frac{F-K}{\sigma(T-t)^{\frac{3}{2}}})]$$

$$+\text{e}^{-r(T-t)}[\frac{\sigma}{2\sqrt{2\pi(T-t)}}e^{-\frac{d_1^2}{2}}] + e^{-r(T-t)}[\frac{\sigma\sqrt{(T-t)}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}(-\frac{(F-K)}{\sigma\sqrt{T-t}})(-\frac{1}{2}\frac{F-K}{\sigma(T-t)^{\frac{3}{2}}})]$$

$$=\text{-re}^{-r(T-t)}[(K-F)N(-d_1) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}] + e^{-r(T-t)}(\frac{\sigma}{2\sqrt{2\pi(T-t)}}e^{-\frac{d_1^2}{2}})$$

$$= -rP + \frac{e^{-r(T-t)}\sigma N'(d_1)}{2\sqrt{T-t}} \qquad [3.6A]$$

$$= \frac{2\sqrt{T-t}rP - e^{-r(T-t)}\sigma N'(d_1)}{730\sqrt{T-t}} \quad [3.6B]$$

$$\rho_{(Call)} = \frac{\partial C}{\partial r} = -(T-t)C \quad [3.7]$$

$$\rho_{(Put)} = \frac{\partial P}{\partial r} = -(T-t)P \quad [3.8]$$

It has been proved previously that the price of a single derivative dependent on a future contract must satisfy the differential equation [2.4]. It follows that the value of Π of a portfolio of such derivatives also satisfies the differential equation

$$\frac{\partial \Pi}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \Pi}{\partial f^2} = r\Pi. \quad [3.9]$$

Since

$$\theta = \frac{\partial \Pi}{\partial t}, \Gamma = \frac{\partial^2 \Pi}{\partial f^2},$$

then it follows that

$$\theta + \frac{1}{2}\sigma^2 \Gamma = r\Pi. \quad [3.10]$$

4. Equivalence between the normal and the lognormal implied volatility

In the real market, it is standard practice to quote the swaptions in term of log-normal volatility (Black volatility) which is inserted into the Black model to find the price. Meanwhile, normalized volatility is also the market convention though, primarily because normalized volatility deals with basis point changes in rates rather than, as in lognormal volatility, with percentage changes in rates. Therefore one needs to calculate the implied normal volatility.

In this derived model, the interest rate is modeled with a normal distributed stochastic process. Then the interest rate volatility is normal while the volatility is log-normal in Black-76 model. Both models are supposed to give the same prices and there exist formula to convert from normal to log-normal or log-normal to normal. Such formula can be derived with the help of perturbation theory which is applicable if the problem at hand can be formulated by adding a "small" term to the mathematical description of the exactly solvable problem.

4.1 Singular perturbation expansion

Consider a European call with expiration date t_{ex}, settlement date t_s, and strike K. As before, let $F(t)$ be the stochastic process for the forward price as seen at date t with "adjustment". We are assuming that

$$dF = \alpha(t)A(F)dW \qquad [4.1]$$

under the forward measure. Under this measure, the value of the option at date t is $V(t, F(t))$, where the function $V(t, f)$ is given by the expected value

$$V(t, f) = D(t, t_s)E\{[F(t_{ex}) - K]^+ | F(t) = f\} \qquad [4.2]$$

Here $D(t, t_s)$ is the discount factor to the settlement date t_s at date t.

By using singular perturbation methods to solve the scaled problem, we analyze Black's model to determine the volatility σ_B which would yield the same value of the option. As Hagan and Woodward (1998) proved previously, the value of the call option is

$$V(t, f) = D(t, t_s)G(\tau^*, f - K), \qquad [4.3]$$

where $\tau^* = A^2(K)\tau[1_{\nu 1}(f - K) + \frac{4\nu_2 + \nu_1^2}{12}(f - K)^2 + \frac{2\nu_2 - \nu_1^2}{12}A^2(K)\tau + ...];$

$\nu_1 = \frac{A'(K)}{A(K)}; \nu_2 = \frac{A''(K)}{A(K)}; \epsilon \equiv A(K) \ll 1.$

Lastly, the equivalent Black volatility implied by this price is computed as below:

$$\sigma_B = a\frac{A(f_{av})}{f_{av}}\{1 + (\gamma_2 - 2\gamma_1^2 + \frac{2}{f_{av}^2})\frac{(f-K)^2}{24} + (2\gamma_2 - \gamma_1^2 + \frac{1}{f_{av}^2})\frac{a^2A^2(f_{av})(t_{ex} - t)}{24} + ...\}$$

Where $f_{av} = \frac{1}{2}(f + K), \gamma_1 = \frac{A'(f_{av})}{A(f_{av})}, \gamma_2 = \frac{A''(f_{av})}{A(f_{av})}, and\ \alpha = \frac{1}{t_{ex} - t\int_t^{t_{ex}}}\alpha^2(t')dt'.$ [4.4]

To yield the more precise equivalent volatility formula, arbitrarily high order can be performed via $O(\epsilon^4)$. A similar analysis shows that the implied volatility for a European put option is given by the same formula.

4.2 Conversion between log normal and normal volatility

Black's model is $dF = \sigma_B F dW, F(0) = f$ where f is present forward swap (or caplet) rate and σ_B is the implied log normal volatility, while the normal model is $dF = \sigma_N F dW, F(0) = f$, where $\sigma_N$is the normal volatility. To translate from normal to lognormal vol., the formula proved earlier by Hagan(1998) and Viorel and Dan (2011) is

$$\sigma_N = \sigma_B \frac{f-K}{logf/K} * \frac{1}{1+\frac{1}{24}(1-\frac{1}{120}log^2 f/K)\sigma_B^2\tau + \frac{1}{5760}\sigma_B^4\tau^2} \quad [4.5]$$

as $|\frac{f-K}{K}| \geq 0.001.$

Alternatively, considering some terms are too small to affect its precision, the formula could be simplified as below:

$$\sigma_N = \sigma_B \sqrt{fK} \frac{1+\frac{1}{24}log^2 f/K}{1+\frac{1}{24}\sigma_B^2\tau + \frac{1}{5760}\sigma_B^4\tau^2} \quad [4.6]$$

when $|\frac{f-K}{K}| < 0.001.$With these formulas and the normal volatility, then make use of a global Newton method to find log-normal one approximately.

5. Empirical test on disparities between Black model and normal Black model

In this section, to take a close graphical look at how the risks for a swaption differ between the log-normal Black model and normal one, some empirical tests are performed by the Excel VBA.[6] Starting from the setup, the values of parameters are defined specifically. Only Swedish calendar is valid to bootstrap, as well as 360 days per year actual days per month. The beginning date is 2013/4/19. The rates quoted in the market are par yields from which the zero coupon rate is derived. To bootstrap a zero-coupon curve, there are liquid instruments on the Swedish market, i.e. an over-night rate (O/N), a tomorrow-next rate (T/N), deposit rates for one week, one, two and three month maturities, some OMX STIBOR Forward Rate Agreements

[6] This VBA application was programmed originally by Jan Röman. See the demo of input-output interfaces in the Appendix.

(FRA) and finally swaps from 4 year up to 30 year. Swap rate is calculated as pure interpolation of the bootstrapped yield curve. For the years when there is shortage of swap rates, use the linear extrapolation to find the zero rates and then in the same way to calculate the discount factors as well as forward rate in turn.

Since the log normal volatility is the normal way to quote swaptions in the market, an at-the-money normal volatility surface can be converted from log-normal volatility surface as figure 1, applying the formula [4.5] or [4.6]. However, as brief experiment has showed in figure 2, two models could always reach same option price for a certain receiver swaption of 4 years to expiration and around 1.72% forward rate, under the circumstance of a range of strike rates and log-normal volatilities from 2% to 40%

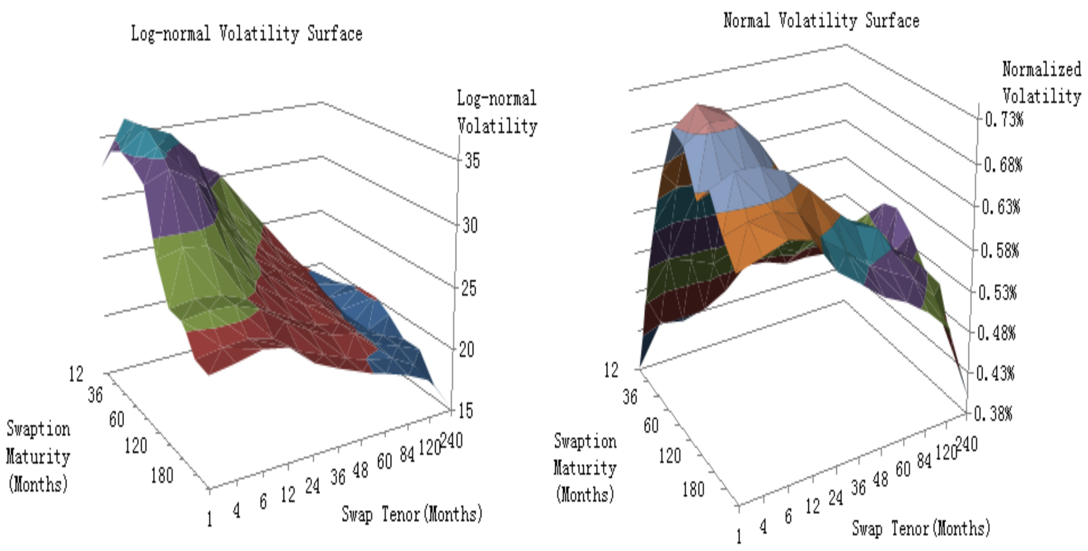

Figure 1. Conversion between log normal and normal volatility surface

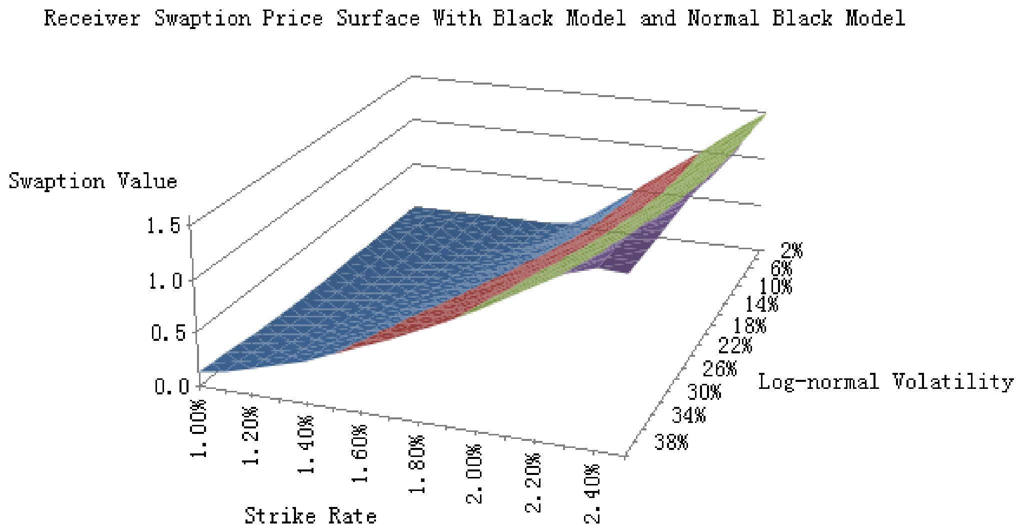

Figure 2. Approximation experiments on receiver Swaption price surface with Black model and normal Black model

In terms of delta and vega, their disparities between models are also of significance. On one hand, when the forward rate is identical to strike rate and the log-normal volatility increases, its vega decreases and its delta rises, while the curves of their normal counterparties hovers, exhibited as figure 3. On the other hand, if the forward rate differs from the strike rate, the difference between log-normal delta and normal one accelerates with respect to the growing volatility; whereas the gap between log-normal vega and normal vega tend to not only go up majorly but move down to zero somewhere by all means, as demonstrated in figure 4&5.

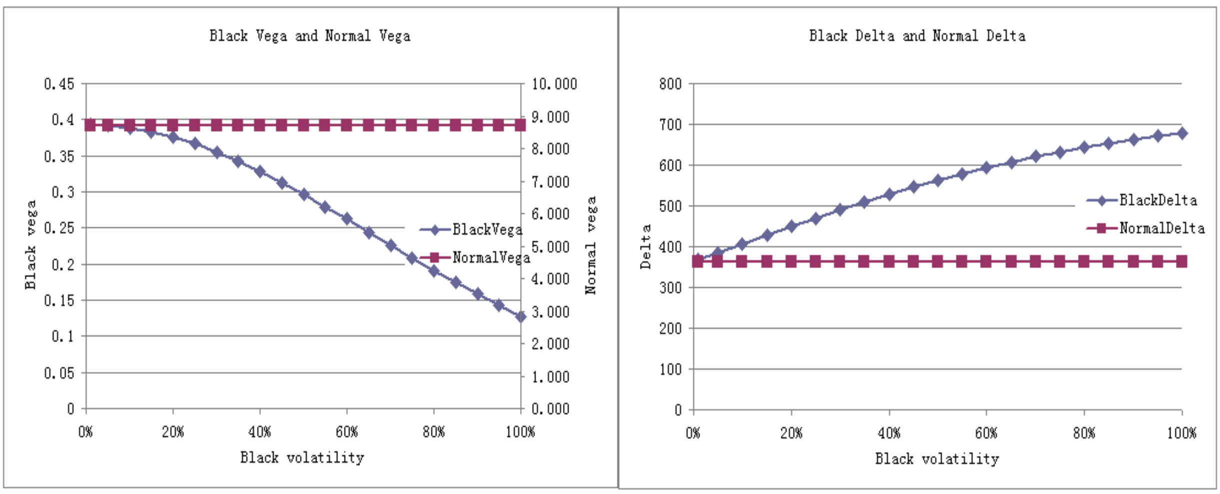

Figure 3. Disparity between the Black Vega(Delta) and normal Vega(Delta) with respect to Black volatility when forward rate is as same as strike rate (At the Money)

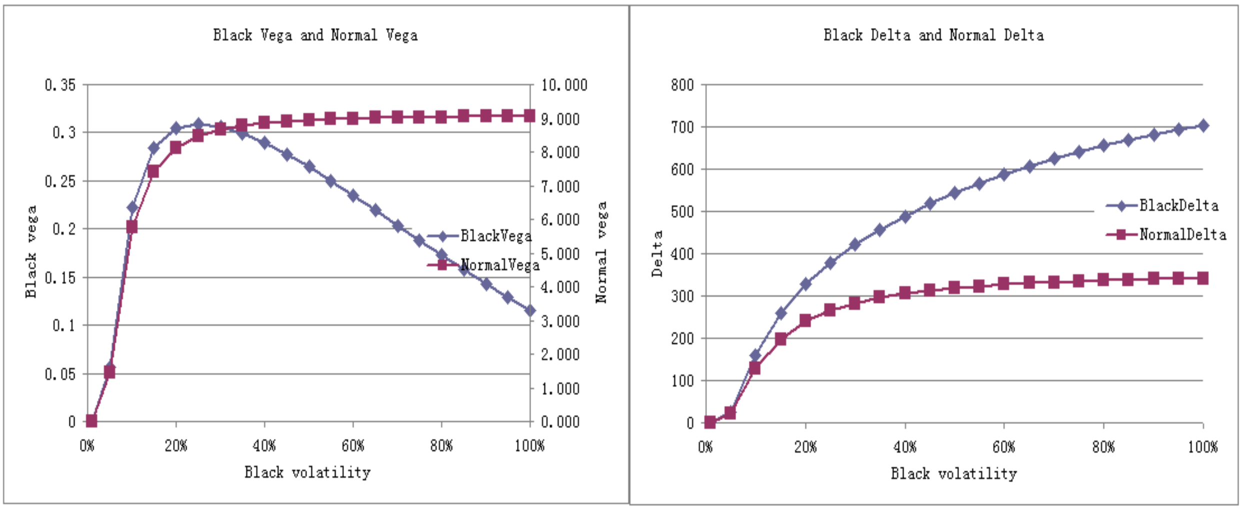

Figure 4. Disparity between the Black Vega(Delta) and normal Vega(Delta) with respect to Black volatility when forward rate is lower than strike rate (Out of the Money)

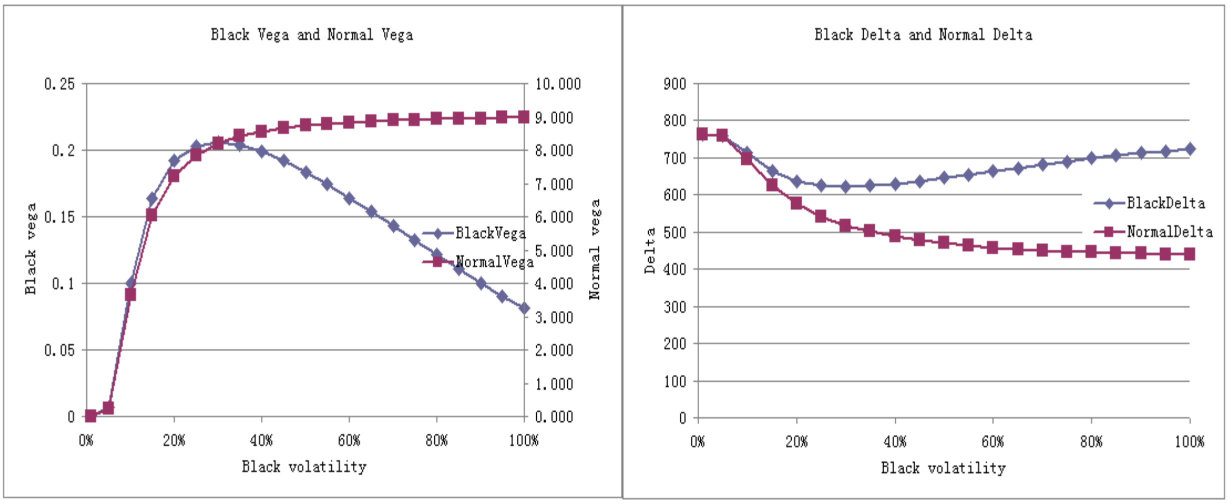

Figure 5. Disparity between the Black Vega(Delta) and normal Vega(Delta) with respect to Black volatility when forward rate is higher than strike rate (In the Money)

The indicated results of experiments coincide with the results of proposed solution of Henrard (2005) who discussed that all six models[7] with sufficient degrees of freedom lead to the same price for standard options. But, he studied that the difference of risks inferred will be substantial owing to the different implicit hypotheses within the models. For instance, the delta of nomal-like models for a receiver swaption are lower

[7] Classical Black model of geometric Brownian motion of the forward swap rate; its normal version of arithmetic Brownian motion of the forward swap rate; Hull-White model of arithmetic Brownian motion of the continuously compounded rates and stochastic volatility models of Stochastic Alpha Beta Rho, one with the elasticity parameter β equal to 0 (normal) or with 1 (log-normal) and one with no correlation between rates and volatility (ρ = 0).

than that of log-normal-like Black models. The difference in the delta can reach 10% or more of the underlying even if models are calibrated to the identical prices. Therefore, it is convincible enough to ensure that the above investigated analysis holds strongly.

6.Conclusion

Ever since it became clear that a geometric Brownian motion process provides a more plausible model of asset prices than an arithmetic Brownian motion process, it has been taken for granted that there was no point in developing an option pricing model for a normally distributed underlying. Nonetheless, it has been argued that there are potential in which we might need such a model when the forward rate is zero and/or when the strike rate is equal zero or negative, and a contemporary example is when the interest rate is negative.

In the second section, the derivation of the model is subject to the assumption that implementation of a dynamic hedging strategy will eliminate the risk of holding long or short positions in such options. Additionally, the derivation of the formulas has been proved mathematically by the famous no-arbitrage argument. The idea of the theory is that the fair value of any derivative security is computed as the expectation of the payoff under an equivalent martingale measure. In the section 3, the Greeks have been derived analytically by differentiation. Also brief explanation regarding how one can approximate log-normal Black with normal version has been explored in the fourth section. Eventually, there is an empirical test for swaptions on an at-the-money volatility surface, given as Black (log-normal) volatilities, which is converted to a normal volatility surface. Then calculate and plot how delta and vega differs between the models.

Admittedly, both models have limitations in routine pricing and do not provide a description of how interest rates evolve through time, concerning pricing interest rate derivatives such as American style swap options. Interest rate derivatives are tougher to value than equity and foreign exchange derivative due to the complicated behavior of an individual interest rate and the varying volatilities of different points on the yield curve. As a matter of fact, other notable models, like Hull-White model, can be calibrated to the market as well. I believe that either model can be valuable in my future study.

To sum up, the crucial risk sensitivities for such fixed income derivatives as swaptions are delta (PV01, DV01) and Vega among Geeks, as a result of their risk limits set by the trading desk. Thanks to the conversion formula between log-normal volatility and normal volatility, it has been found in this paper that both models results in same option value despite of the disparities in their risks. Accordingly, an option-pricing model based on a normal underlying is not some flawed relative of Black, as it is usually considered to be, but is instead the key to correctly pricing this type of derivatives– and hence, from a risk perspective as well as in a trading point of view, a very helpful tool in the rapidly emerging interest rate derivatives market.

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Remarks concerned the use of Internet, correctness of citations:

I certify that I have checked the correctness of citations in this thesis, and I did not copy material from any Internet Web page, but used the Internet preprint. The data in the Excel are taken by Jan Röman from one of trading system at Swedbank, Murex , Mx3.

Appendix

Table 1: Risk measures for the Black model

Greeks	**Call**	**Put**
Delta	$e^{-r(T-t)}N(d_1)$	$e^{-r(T-t)}(N(d_1)-1)$
Gamma	$e^{-r(T-t)}\frac{N'(d_1)}{f\sigma\sqrt{T-t}}$	$e^{-r(T-t)}\frac{N'(d_1)}{f\sigma\sqrt{T-t}}$
Vega	$fe^{-r(T-t)}N'(d_1)\sqrt{T-t}$	$fe^{-r(T-t)}N'(d_1)\sqrt{T-t}$
Theta	$-\frac{fe^{-r(T-t)}N'(d_1)\sigma}{2\sqrt{T-t}}+rfe^{-r(T-t)}N(d_1)-rKe^{-r(T-t)}N(d_2)$	$-\frac{fe^{-r(T-t)}N'(d_1)\sigma}{2\sqrt{T-t}}-rfe^{-r(T-t)}N(-d_1)+rKe^{-r(T-t)}N(-d_2)$
Rho	$Ke^{-r(T-t)}(T-t)N(d_2)$	$-Ke^{-r(T-t)}(T-t)N(-d_2)$

2. Black formulas expressed for Caps/Floors and Swaptions & annuity factor

The Black formula for the time-*t* value of a caplet and a floorlet are expressed as:

$$C(t) = \frac{N\tau}{1+F\tau}e^{-r(T-t)}[FN(d_1 - KN(d_2)]$$

$$F(t) = \frac{N\tau}{1+F\tau}e^{-r(T-t)}[KN(-d_2 - FN(-d_1)]$$

Where τ is the tenor, N the face value and *F* the implied forward rate between time *t* and at the caplets/floorlets maturity, *T*.

From Black model, a payer swaption and a receiver swaption are expressed as :

$$PS = \frac{1-\frac{1}{(1+\frac{F}{m})^{(T-t)m}}}{F}e^{-rT}[FN(d_1) - KN(d_2)]$$

$$RS = \frac{1-\frac{1}{(1+\frac{F}{m})^{(T-t)m}}}{F}e^{-rT}[FN(-d_2) - KN(-d_1)]$$

$$d_1 = \frac{ln(\frac{F}{K})+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$$

where

T-t = Tenor of swap in years (time between swaption maturity and swap maturity).
F = Forward rate of the underlying swap.
K = Strike rate of the swaption.
r = Risk-free interest rate.
T = Time to swaption expiration in years.
σ = Volatility of the forward-starting swap rate.
m = Compounding's per year in swap rate.

Derivation of annuity factor of swaptions[8]

To derive the factor for a swaption, start by studying a forward starting swap. That is a swap that starts at a future time where we exchange floating for fixed cash flows. A $T_n \times (T_N - Tn)$ swap means a swap that start at time T_n and have maturity at time T_N.

Define the reset days for any swap as: T_0, T_1, T_N and denote a_i as $T_i - T_{i-1}$ The holder of a forward starting $T_n \times (T_N - Tn)$ payer swap with tenor $T_N - Tn$ receives fixed payments at times $T_{n+1}, T_{n+2}, ..., T_N$and pay at the same times floating payments.

For each period $[T_i, T_{i+1}]$, the LIBOR rate $L_{i+1}(T_i)$is set at time T_i and the floating leg $a_{i+1}L_{i+1}(T_i)$ is received at T_{i+1}. For the same period the fixed leg $a_{i+1}F$ is

[8] Röman, J., (2012), *Lecture Notes in Analytical Finance II*, Mälardelen University, p370-373.

paid at T_{i+1} where *F* is the (fixed) swap rate.

The non-arbitrage value at $t < T_n$of the floating payment made at T_i is given by $p(t, T_{i+1})$. The total value of the floating legs at time *t* for $t \leq T_n$equals

$\Sigma_{i=n}^{N-1} a_i f(t, T_i) p(t, T_{i+1}) = \Sigma_{i=n}^{N-1} a_{i+1} \frac{1}{a_{i+1}} \frac{p(t,T_i) - p(t,T_{i+1})}{p(t,T_i)} p(t, T_i)$

$= \Sigma_{i=n}^{N-1} [p(t, T_i) - p(t, T_{i+1})] = p(t, t_n) - p(t, t_N)$

where the forward rate is given by:

$p(0, t_i) = p(0, t_{i-1}) \frac{1}{1 + a_i f(t_{i-1}, ti)}$

$f(t_{i-1}, t_i) = \frac{1}{a_i} \frac{p(0,t_i) - p(0,t_{i-1})}{p(0,t_i) =}$

The value at the starting day is the same as the face value = 1. In a swap, there is not any final payment of the face value. This gives the swap value at the starting day *t* = 0, as 1 – *p*(0, *T*). Between to resets therefore must the swap value must be as: $p(t, t_0) - p(t, T)$ where is t_0 the time for the next reset day. This explains the formula above.

The total value at time *t* for the fixed side equals

$\Sigma_{i=n}^{N-1} F a_{i+1} p(t, T_{i+1}) = F \Sigma_{i=n+1}^{N} a_i p_i(t)$

where *F* is called the swap rate. This is a par rate since it makes the price of the swap to be equal zero when entering the swap contract. So the total value of the payer swap is given by

$PS_n^N(t, F) = p_n(t) - p_N(t) - F \Sigma_{i=n+1}^{N} a_i p_i(t)$

Thus define the forward swap rate (at par) $R_n^N(t)$ of the $T_n \times (T_N - T_n)$swap as the value of *F* for which the total value above is zero. I.e.,

$R_n^N(t) = \frac{p_n(t) - p_N(t)}{\Sigma_{i=n+1}^{N} a_i p_i(t)}$

In addition, define for each pair *n, k* with *n* < *k*, the process

$S_n^k(t) = \Sigma_{i=n+1}^{N} a_i p_i(t)$

as the accrual factor or the value of a basis point (also called the *level, DV01, PV0l, annuity* or *numerical duration* of the swap).

Then express the swap value as:

$R_n^N(t) = \frac{p_n(t) - p_N(t)}{S_n^N(t)}$

In the market there are no quoted prices for different swaps. Instead there are market quotes for the par swap rates. Calculate the arbitrage free price for a payer swap with

the strike rate *K* as
$PS_n^N(t, R_n^N(t), K) = (R_n^N(t) - K)S_n^N(t)$

A payer swaption is then a contract given by:

$P_n^N(t, R_n^N(t), K) = max(R_n^N(t) - K, 0)S_n^N(T_n)$

This contract gives the holder the right to enter a swap contract at time T_N with swaption strike (fixed rate) *K*.

Under the numeraire process S_n^N a payer swaption is then a call option on $R_n^N$with strike\ price *K*. The value of this contract is given by the Black-76 formula: $P_n^N(t) = S_n^N(t)\{R_n^N(t)N(d_1) - KN(d_2)\}$

The Black formula can be written as:

$P_n^N(t) = \Sigma_{i=n+1}^N a_i p_i(t)\{R_n^N(t)N(d_1) - KN(d_2)\}$

$= \frac{p_n(t) - p_N(t)}{R_n^N(t)}\{R_n^N(t)N(d_1) - KN(d_2)\}$

If denote the Forward swap-rate between $T_n$and T_N as *F*, att_n it is:

$p_n(t) - p_N(t) \equiv p(t, t_n) - p(t, t_N) = p(t, t_n)\{1 - p(t_n, t_N)\}$

$= p(t, t_n)\{1 - \frac{1}{(1+f(t_n, t_N)^{t_N - t_n})}\}] \equiv p(t, t_n)\{1 - \frac{1}{(1+F)^{t_N - t_n})}\}$

Now let $T = t_n$ be the maturity of the swaption, *F* the Forward swap-rate $R_n^N(t)$ above) and introducing *m* reset days per year (the frequency).

3. Presentation of input-output interfaces

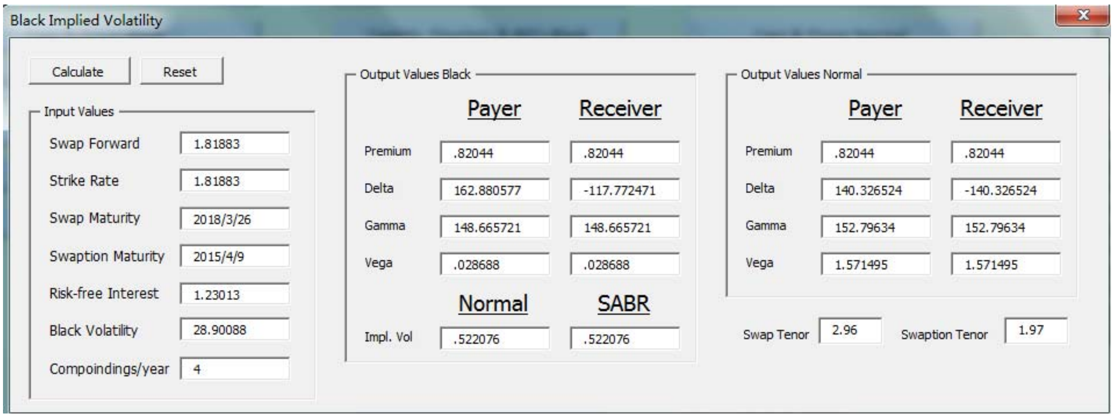

Figure 6. Demo of classical Black and normal Black swaption calculator given Black implied volatility

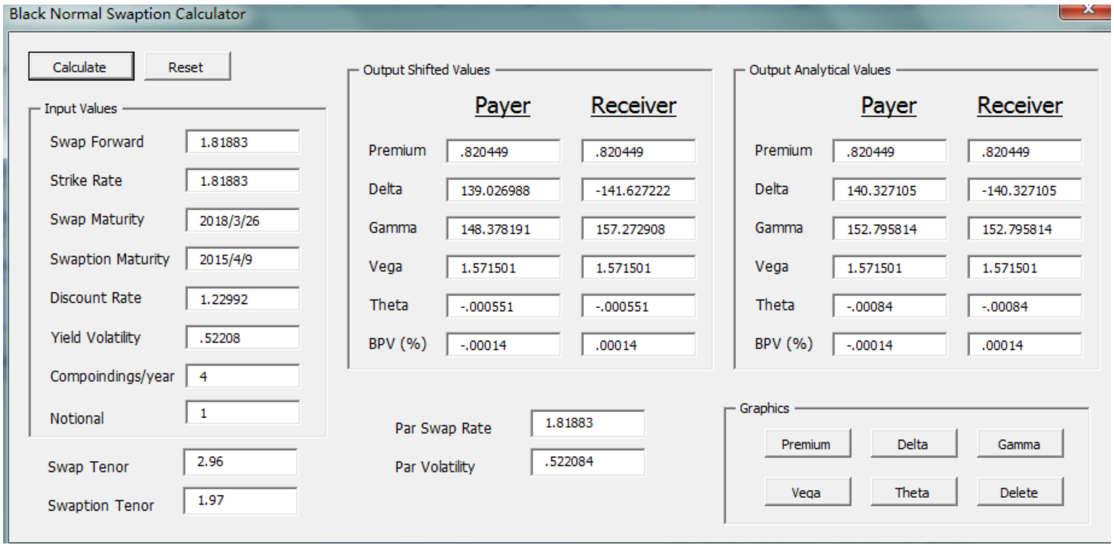

Figure 7. Demo of Black normal swaption calculator

4. Extract of VBA program codes

```
' Base function for the normal Black model (C = Call, P = Put options)
'For all functions below, the following is used.
' The SwapRate, StrikeRate, r and vol is given in %. I.e., 3.4 % as 0.034
' The SwapTenor and SwaptionMaturity are given in years.
' F is the frequency, i.e., the mumber of cash-flows per year
' N (the face value, notional) is not used (use this outside this function)
' ========================================================
Function Annuity(SwapRate As Double, Tenor As Double, F As Double) As Double
Annuity = (1 - 1 / Pow(1 + SwapRate / F, F * Tenor)) / SwapRate
End Function
' ------------------------------------------------------------------------
Function BlackNormalC(SwapRate As Double, StrikeRate As Double, maturity As_
Double, r As Double, vol As Double) As Double Dim d1 As Double, d2 As Double,_
nd1 As Double

d1 = (SwapRate - StrikeRate) / (vol * Sqr(maturity))
d2 = vol * Sqr(maturity / (2 * 3.141592654)) * Exp(-d1 * d1 / 2)
nd1 = CND(d1)
BlackNormalC = Exp(-r * maturity) * ((SwapRate - StrikeRate) * nd1 + d2)
End Function
' ------------------------------------------------------------------------
Function BlackNormalP(SwapRate As Double, StrikeRate As Double, maturity As_
Double, r As Double, vol As Double) As Double Dim d1 As Double, d2 As Double,_
nd1 As Double

d1 = (SwapRate - StrikeRate) / (vol * Sqr(maturity))
d2 = vol * Sqr(maturity / (2 * 3.141592654)) * Exp(-d1 * d1 / 2)
```

```
nd1 = CND(-d1)
BlackNormalP = Exp(-r * maturity) * ((StrikeRate - SwapRate) * nd1 + d2)
End Function
' ----------------------------------------------------------------------
Function BlackNormalDeltaC(SwapRate As Double, StrikeRate As Double, maturity_
As Double, r As Double, vol As Double) As Double Dim d1 As Double

d1 = (SwapRate - StrikeRate) / (vol * Sqr(maturity))
BlackNormalDeltaC = Exp(-r * maturity) * CND(d1)
End Function
' ----------------------------------------------------------------------
Function BlackNormalDeltaP(SwapRate As Double, StrikeRate As Double, _
maturity As Double, r As Double, vol As Double) As Double Dim d1 As Double

d1 = (SwapRate - StrikeRate) / (vol * Sqr(maturity))
BlackNormalDeltaP = -Exp(-r * maturity) * CND(-d1)
End Function
' ----------------------------------------------------------------------
Function BlackNormalDeltaCS(SwapRate As Double, StrikeRate As Double, _
SwapTenor As Double, SwaptionMaturity As Double, _r As Double, F As Double,_
vol As Double) As Double Dim d1 As Double

d1 = BlackNormalSwaptionPayer(SwapRate + 0.000001, StrikeRate, SwapTenor, _
SwaptionMaturity, r, F, vol)
d1 = d1 - BlackNormalSwaptionPayer(SwapRate - 0.000001, StrikeRate, _
SwapTenor,    SwaptionMaturity, r, F, vol)
BlackNormalDeltaCS = d1 / 0.000002
End Function
' ----------------------------------------------------------------------
Function BlackNormalDeltaPS(SwapRate As Double, StrikeRate As Double, _
SwapTenor As Double, SwaptionMaturity As Double, r As Double, F As Double, _
vol As Double) As DoubleDim d1 As Double

d1 = BlackNormalSwaptionReceiver(SwapRate + 0.000001, StrikeRate, SwapTenor,_
SwaptionMaturity, r, F, vol)
d1 = d1 - BlackNormalSwaptionReceiver(SwapRate - 0.000001, StrikeRate, _
SwapTenor, SwaptionMaturity, r, F, vol)
BlackNormalDeltaPS = d1 / 0.000002
End Function
' ----------------------------------------------------------------------
Function BlackNormalVeg(SwapRate As Double, StrikeRate As Double, _
maturity As Double, r As Double, vol As Double) As Double Dim d1 As Double

d1 = (SwapRate - StrikeRate) / (vol * Sqr(maturity))
```

```
BlackNormalVeg = Exp(-r * maturity) * nd(d1) * Sqr(maturity)
End Function
' ------------------------------------------------------------------------
Function BlackNormalVegS(SwapRate As Double, StrikeRate As Double, _
SwapTenor As Double, SwaptionMaturity As Double, r As Double, F As Double, _
vol As Double) As Double     Dim d1 As Double

d1 = BlackNormalSwaptionPayer(SwapRate, StrikeRate, SwapTenor, _
SwaptionMaturity, r, F, vol + 0.0001)
d1 = d1 - BlackNormalSwaptionPayer(SwapRate, StrikeRate, SwapTenor, _
SwaptionMaturity, r, F, vol - 0.0001)
   BlackNormalVegS = d1 / (0.0002)
End Function

' ==========================================================
```

'Conversion between log normal and normal volatility

```
' The formulas here is found in Hagans article. The same can be found in
' Lecture Notes in Analytical Finance II by Jan.
' For all functions below, the following is used.

' The SwapRate, StrikeRate, rates and vol is given in %. I.e., 3.4 % as 0.034
' The SwapTenor and SwaptionMaturity are given in years.
' F is the frequency, i.e., the mumber od cash-flows per year
' N (the face value, notional) is not used (use this outside this function)
' ==========================================================
Function BlackVol2Norm(SwapRate As Double, StrikeRate As Double, _
SwaptionMaturity As Double, v As Double) As Double
   If (Abs((SwapRate - StrikeRate) / StrikeRate) < 0.001) Then
      BlackVol2Norm = v * Sqr(SwapRate * StrikeRate) * (1 + (1 / 24) * _
                        Log(SwapRate / StrikeRate) * Log(SwapRate / StrikeRate))
      BlackVol2Norm = BlackVol2Norm / (1 + Pow(v, 2) * SwaptionMaturity / _
                        24 + Pow(v, 4) * Pow(SwaptionMaturity, 2) / 5760)
   Else
      BlackVol2Norm = v * (SwapRate - StrikeRate) / Log(SwapRate / StrikeRate)
      BlackVol2Norm = BlackVol2Norm / (1 + (1 - Log(SwapRate / StrikeRate) * _
                        Log(SwapRate / StrikeRate) / 120) * Pow(v, 2) * _
SwaptionMaturity / 24 + Pow(v, 4) * Pow(SwaptionMaturity, 2) / 5760)
   End If
End Function
' ------------------------------------------------------------------------
Function NormVol2Black(SwapRate As Double, StrikeRate As Double, _
SwaptionMaturity As Double, vol As Double) As Double

NormVol2Black = NewtonRaphson2(SwapRate, StrikeRate, SwaptionMaturity, vol, _
```

```
2 * vol / (SwapRate + StrikeRate))
End Function
' ---------------------------------------------------------------------------
Function NewtonRaphson2(SwapRate As Double, StrikeRate As Double, _
SwaptionMaturity As Double, cm As Double, Optional initial As Double) As Double
    Dim dVol As Double
    Dim EPSILON As Double
    Dim maxIter As Double
    Dim vol_1 As Double
    Dim vol_2 As Double
    Dim vol_3 As Double
    Dim old_err As Double
    Dim i As Double
    Dim dX As Double
    Dim Value_1 As Double
    Dim Value_2 As Double

    dVol = 0.00001
    EPSILON = 0.00001
    maxIter = 100
    vol_1 = initial
    i = 1
    old_err = 9E+99
    Do
        Value_1 = BlackVol2Norm(SwapRate, StrikeRate, SwaptionMaturity, vol_1)
        vol_2 = vol_1 - dVol
        Value_2 = BlackVol2Norm(SwapRate, StrikeRate, SwaptionMaturity, vol_2)
        dX = (Value_2 - Value_1) / dVol
        If Abs(old_err) < EPSILON Or i > maxIter Or dX = 0 Then Exit Do
        old_err = -(cm - Value_1) / dX
        vol_1 = vol_1 - (cm - Value_1) / dX
        Debug.Print vol_1
        i = i + 1
    Loop
    Debug.Print vol_1
    NewtonRaphson2 = vol_1
End Function
' ---------------------------------------------------------------------------
```

5. Summary of reflection of objectives in the thesis

Objective 1.

- Survey of literature with comments related to the current research questions: **Second paragraph of section 1.4.**

- Survey and comparison of alternative methods related to the subject of the project: **Section 3.1**
- Deeper presentation of specific methods supposed to be used in the project: **Section 2, 3, 4, 5.**

Objective 2.

- Description of the model and comparisons with alternative models**: Section1.3**
- Analysis of data, their quality, volume, shortage, etc. (if any)**: First paragraph of section 5**

Objective 3.

- Formulation of the problem studied in the project and the goals of the project: **Section 1.4**
- Evaluation of possible solution in the time framework and presentation of solution (algorithms, results of experiments, description of programs, presentation of input-output interfaces, etc)**: Section 5, Appendix 3. The algorithms are based on the formulas of section 3,4.**
- Program codes**: Appendix 4**

Objective 4.

- Print of the oral presentation of the project**. PowerPoint**
- Improved English and the thesis structure (abstract, table of contents, sections, conclusion, references). **Everywhere**
- The place of results in the area; the list of main results and achievements; potential use of results; possible future continuation of the project: **Section 6.**

Objective 6.

- Popular presentation of project and its results: **Last paragraph of Section 1.4.**
- Remarks concerned the use of Internet and correctness of citations: **In the end of section 7.**
- Acknowledgement: **Page 2**

Bachelor Degree Project in Financial Mathematics

Risk Measures with Normal Distributed Black Options Pricing Model

Wenqing Huang

Supervised by **Anatoliy Malyarenko & Jan Röman**

MÄLARDALENS HÖGSKOLA
ESKILSTUNA VÄSTERÅS

Contents

Black Option Pricing Model

- Fisher Black, 1976

- A generalization of Black-Scholes model
- The underlying is log-normal distributed

$$f = e^{r(T-t)}S$$

$$P_{Call} = e^{-r(T-t)}[fN(d_1) - KN(d_2)]$$

$$P_{Put} = e^{-r(T-t)}[KN(-d_1) - fN(-d_1)]$$

$$d_2 = \frac{ln(\frac{f}{K}) - \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

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- Equity market
- Interest rate derivatives market
    - Bond options
    - Caps and floors
    - Swaptions
    - FRA
    - …

- Future option VS Spot one
- More liquid , price known readliy, settle in cash, lower cost

Negative Interest Rate

- Macroeconomics

  Japan, in 1990s

  Sweden’s Riksbank,-0.25% in 2009

Real Interest Rate=Nominal IR – Inflation Rate

- Finance

  Debt-plus-warrants called Squarz by Goldman Sachs

Normal Distributed Option Pricing Model

- Bachelier's Model, in 1900
- Arithmetic Brownian Motion

$$C(S,T) = SN\left(\frac{S-K}{\sigma\sqrt{T-t}}\right) - KN\left(\frac{S-K}{\sigma\sqrt{T-t}}\right) + \sigma\sqrt{T-t}N'\left(\frac{K-S}{\sigma\sqrt{T-t}}\right)$$

Normal Distributed Black Options Pricing Model

- Partial differential equation
- Formulas of call and put options

Assumptions

- No arbitrage opportunity
- Borrow and lend cash at a known constant risk-free interest rate

- Buy and sell any amount, even fractional, of the underlying.
- Frictionless market: no transactions costs& no dividend.

- European type options.
- The current future price follows the following normal process in a risk-neutral world

$$df = \mu dt + \sigma dW_t$$

- Delta hedged portfolio

$$\Pi = g - \frac{\partial g}{\partial f} f$$

- Ito's lemma

$$d\Pi = dg - \frac{\partial g}{\partial f}df = (\frac{\partial g}{\partial t} + \mu\frac{\partial g}{\partial f} + \frac{1}{2}\sigma^2\frac{\partial^2 g}{\partial f^2})dt + \sigma\frac{\partial g}{\partial f}dW - \frac{\partial g}{\partial f}(\mu dt + \sigma dW)$$
$$= (\frac{\partial g}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 g}{\partial f^2})dt$$

- Gain from the risk free interest rate for the portfolio value. So, over the time period $[t, t + \Delta t]$

$$d\Pi = r\Pi dt$$

- Cost nothing to enter into a futures contract at the beginning, one has: $\Pi = g$.

$$\frac{\partial g}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 g}{\partial f^2} = rg \qquad \theta + \frac{1}{2}\sigma^2\Gamma = r\Pi$$

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To test the effectiveness of this strategy:

"It's risk-free... we've got Goliath."

- Tool: Monte Carlo simulation
- Assume: normal distribution for Underlying forward swap price
- Zero cash & borrowing /depositing at the riskless interest rate
- Short positions in payer and receiver swaptions
- Perform daily rehedging over the lifetime of the swaptions
- Result: the expected return will be zero, over a large number of simulations

$$df = \sigma dV_t$$

$$f_T = f_t + \sigma(V_T - V_t) = f_t + \sigma\sqrt{T-t}\,z$$

Gaussian process $N[f_t, \sigma^2(T-t)]$ $\quad z \to N[0,1]$

Application of Feynman-Kač $\quad g(t,f) = f(f)$

$$g(t,f_T) = e^{-r(T-t)}E^Q[\Phi(T)] = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{\infty}\Phi(T)e^{-\frac{(f_T-f_t)^2}{2\sigma^2(T-t)}}df_T$$

$$C = e^{-r(T-t)}\left[(F-K)N(d_1)+\frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\right] = e^{-r(T-t)}((F-K)N(d_1)+\sigma\sqrt{T-t}N'(d_1))$$

$$P = e^{-r(T-t)}\left[(K-F)N(-d_1)+\frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\right] = e^{-r(T-t)}[(K-F)N(-d_1)+\sigma\sqrt{T-t}N'(d_1)]$$

$$d_1 = \frac{(F-K)}{\sigma\sqrt{T-t}}$$

- European swaptions:
- A payer swaption (a call on the floating rate).
- A receiver swaption (a put on the floating rate).

- The value of the swaption per unit of nominal is

$$V_a = L\left[a(F - L_K)N(ad_1) + \sigma\sqrt{T-t}\,N'(d_1)\right] \quad d_1 = \frac{(F-L_K)}{\sigma\sqrt{T-t}}$$

$$\frac{1-\frac{1}{(1+\frac{F}{m})^{(T-t)m}}}{F}e^{-rT}$$

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- Delta

$$\Delta_{(call)} = \frac{\partial C}{\partial F} = \Delta_{(put)} = \frac{\partial P}{\partial F} = -e^{-r(T-t)}N(-d_1)$$

- Gamma

$$\Gamma_{(Call)} = \Gamma_{(Put)} = \frac{\partial^2 C}{\partial F^2} = \frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}N'(d_1)$$

- Vega

$$\nu_{(Call)} = \nu_{(Put)} = \frac{\partial C}{\partial \sigma} = e^{-r(T-t)}\sqrt{T-t}N'(d_1)$$

- Theta

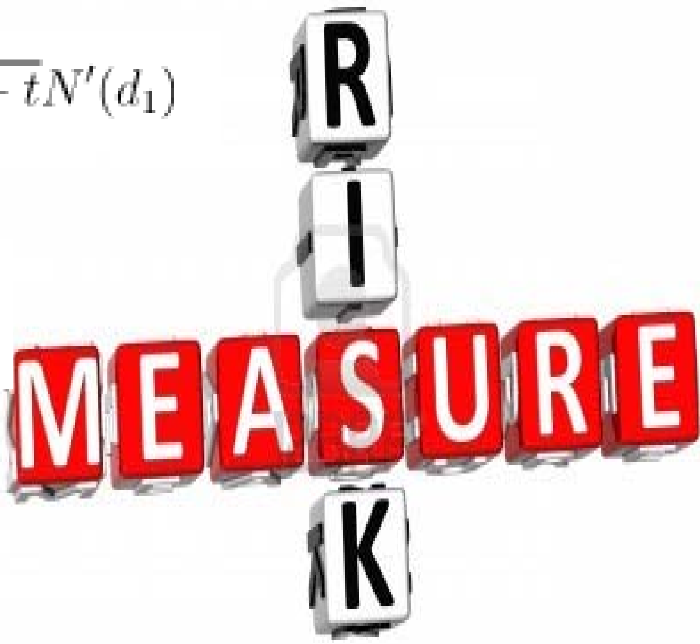
$$\Theta_{(Call)} = \frac{\partial C}{\partial(T-t)} = \frac{2\sqrt{T-t}rC - e^{-r(T-t)}\sigma N'(d_1)}{730\sqrt{T-t}}$$

$$\Theta_{(Put)} = \frac{\partial P}{\partial(T-t)}, = \frac{2\sqrt{T-t}rP - e^{-r(T-t)}\sigma N'(d_1)}{730\sqrt{T-t}}$$

- Rho

$$\rho_{(Call)} = \frac{\partial C}{\partial r} = -(T-t)C$$

$$\rho_{(Put)} = \frac{\partial P}{\partial r} = -(T-t)P$$



Alternatives to define Delta

- i) By shifting the swap rate (F), i.e., the fixed rate in an underlying swap.
- ii) By shifting the yield curve (the zero coupon curve) with one basis-point (1bp = 0.01 %).
- iii) By shifting the quoted rate (before bootstrapping the quotes to a zero curve).
- iv) By calculating the change in the value of the swaption with respect to the change of the underlying swap value when making a shift in the curve as ii) or iii).
- v) By shifting of certain section or buckets of the interest rate curve.

Equivalence between the normal and the lognormal implied volatility

- Singular perturbation expansion

- Conversion between log normal and normal volatility

$$\sigma_N = \sigma_B \sqrt{fK} \frac{1 + \frac{1}{24} log^2 f/K}{1 + \frac{1}{24}\sigma_B^2 \tau + \frac{1}{5760}\sigma_B^4 \tau^2}$$

$$|\frac{f-K}{K}| < 0.001.$$

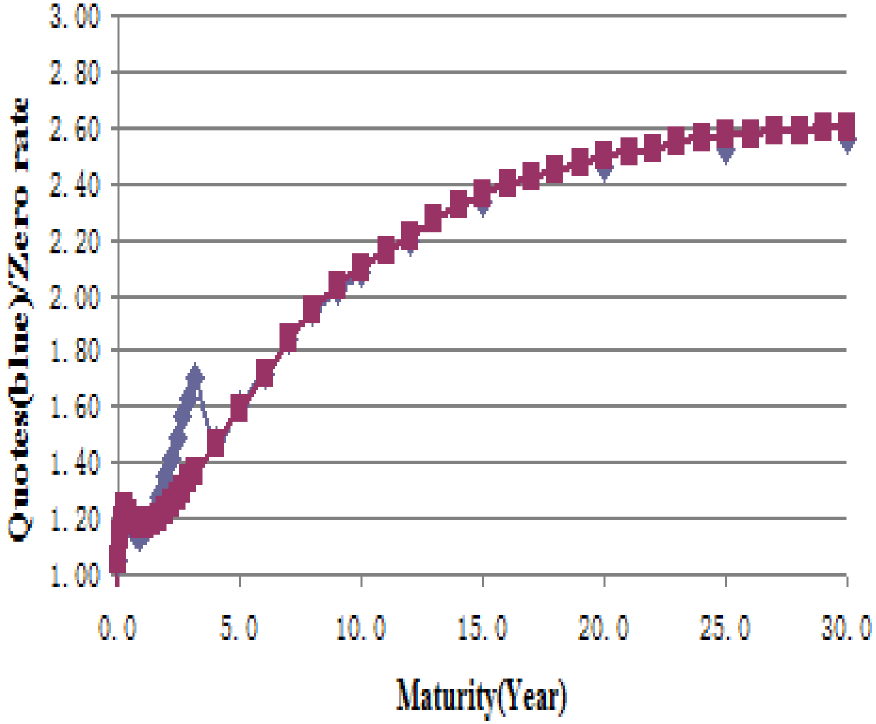
Parameter	Value		Choices
Calendar	SWE		Only SWE
Days per Year	360		Act, 365 or 360
Days per Month	Act		Act or 30
Interpolation	LIN		LIN or POL
Discounting	Continuous		Spot, Forward or Continuous
Spot Days	2		
ForwardCalc	Continuous		Normal, Continuous, MoneyMarket or FDiscount
Debug	FALSE		True or False
Today	2013/4/19		Date (like 2012-12-15) or False for TODAY
Precision	6		Number of decimals in output result
Swap Rate Model	2		Model for calculating SwapRate (1-4) see info

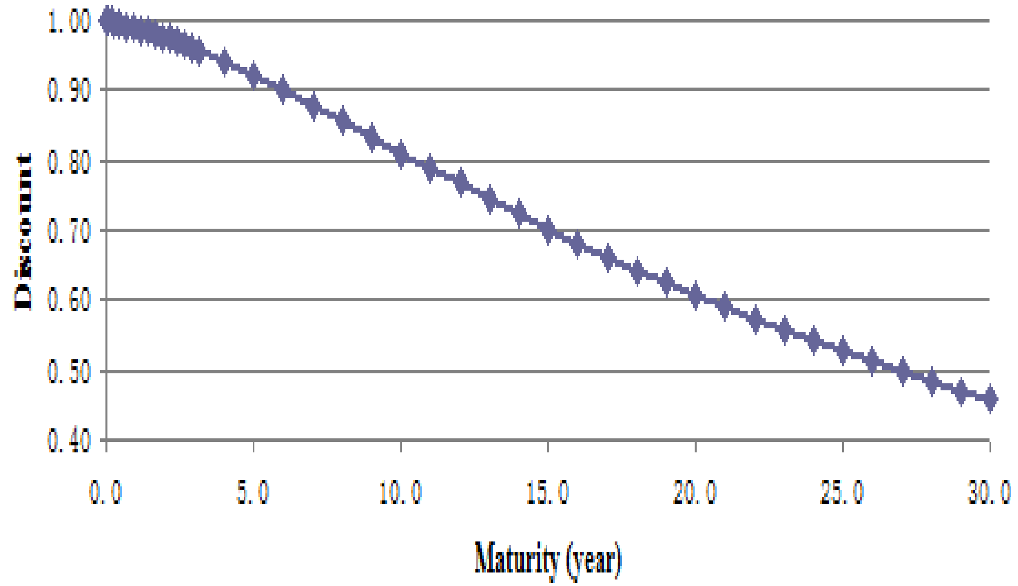
Info
Only SEK can be Bootstrapped in the current version, as in Sheet Bootstrap.
Only linear interpolation and cubic spline interpolation exist in this version.
Swap Rate can be calculated as: 1.) the fixed rate of a Bond to match a Floating Rate Note. 2.) pure interpolation of the bootstrapped yield curve. 3.) the definition of the Swap Rate using Discount Factors. 4.) using the parity relation: Cap - Floor = Swap (= 0).

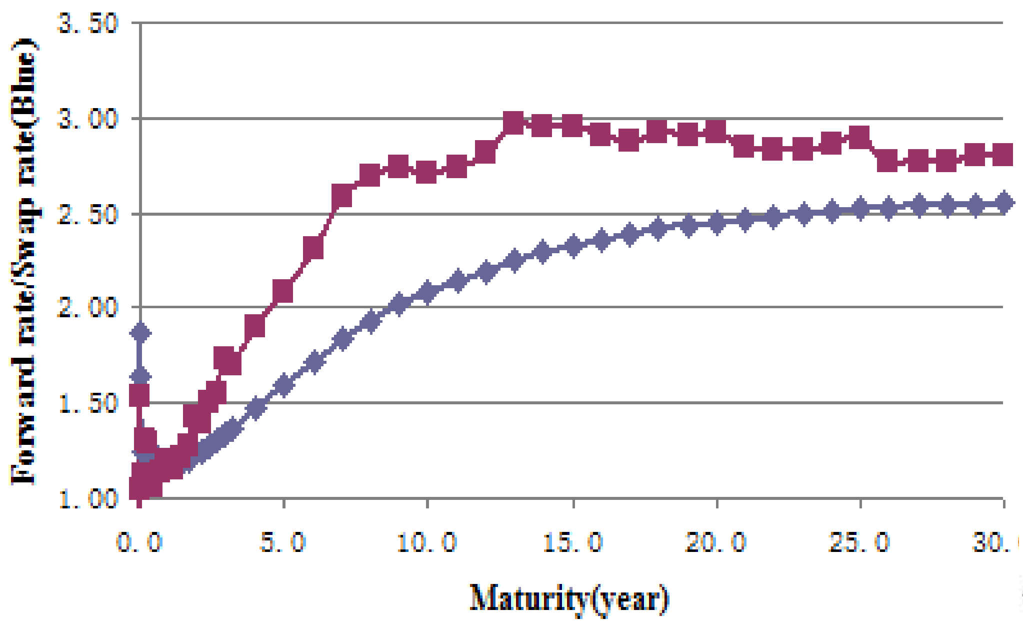
Deposit Cash Maturity	Bid	Ask
O/N	0.8000	1.0000
T/N	1.0520	1.0520
1W	1.0980	1.0980
1M	1.1540	1.1540
2M	1.1940	1.1940
3M	1.2400	1.2400

Short Fut OMX STIBOR FRA 3M Maturity	Bid	Ask
jun-13	1.1975	1.2175
sep-13	1.1363	1.1563
dec-13	1.1300	1.1500
mar-14	1.1500	1.1700
jun-14	1.1938	1.2238
sep-14	1.2633	1.2933
dec-14	1.3350	1.3650
mar-15	1.4050	1.4350
jun-15	1.4738	1.5138
sep-15	1.5463	1.5863
dec-15	1.6140	1.6540
mar-16	1.6850	1.7250

Swap SEK STIBOR A 3M Maturity	Bid	Ask
4Y	1.4525	1.5025
5Y	1.5725	1.6225
6Y	1.6950	1.7450
7Y	1.8125	1.8625
8Y	1.9125	1.9625
9Y	1.9975	2.0475
10Y	2.0625	2.1125
12Y	2.1675	2.2275
15Y	2.3025	2.3625
20Y	2.4150	2.4950
25Y	2.4800	2.5600
30Y	2.5050	2.6050







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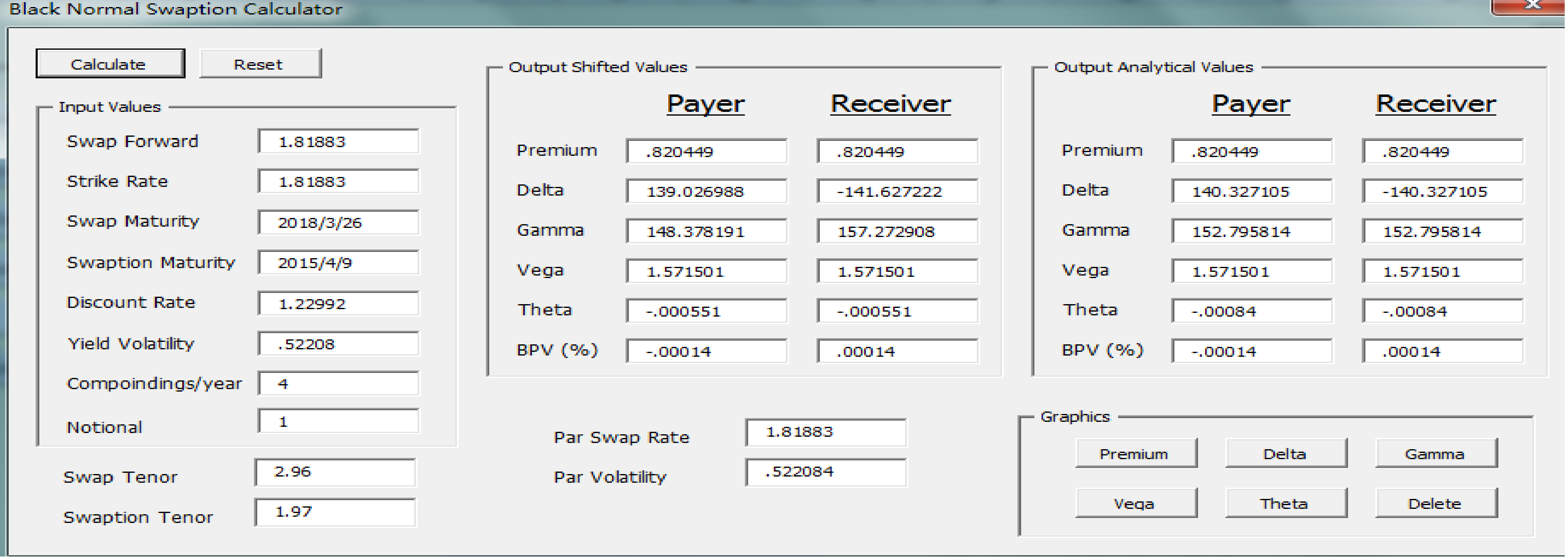
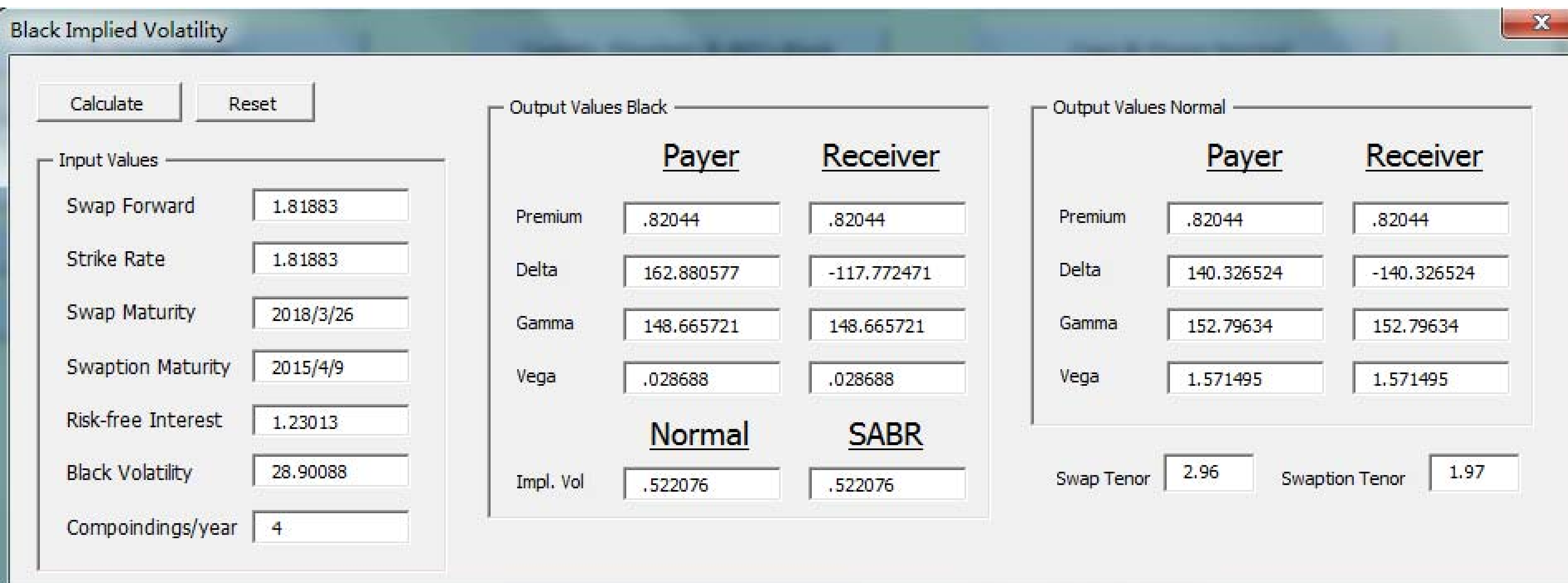

Black Normal Swaption Calculator
Calculate
Reset
Input Values
Swap Forward 1.81883
Strike Rate 1.81883
Swap Maturity 2018/3/26
Swaption Maturity 2015/4/9
Discount Rate 1.22992
Yield Volatility .52208
Compoindings/year 4
Notional 1
Swap Tenor 2.96
Swaption Tenor 1.97
Output Shifted Values
Payer Receiver
Premium .820449 .820449
Delta 139.026988 -141.627222
Gamma 148.378191 157.272908
Vega 1.571501 1.571501
Theta -.000551 -.000551
BPV (%) -.00014 .00014
Par Swap Rate 1.81883
Par Volatility .522084
Output Analytical Values
Payer Receiver
Premium .820449 .820449
Delta 140.327105 -140.327105
Gamma 152.795814 152.795814
Vega 1.571501 1.571501
Theta -.00084 -.00084
BPV (%) -.00014 .00014
Graphics
Premium Delta Gamma
Vega Theta Delete


Black Implied Volatility
Calculate
Reset
Input Values
Swap Forward 1.81883
Strike Rate 1.81883
Swap Maturity 2018/3/26
Swaption Maturity 2015/4/9
Risk-free Interest 1.23013
Black Volatility 28.90088
Compoindings/year 4
Output Values Black
Payer Receiver
Premium .82044 .82044
Delta 162.880577 -117.772471
Gamma 148.665721 148.665721
Vega .028688 .028688
Normal SABR
Impl. Vol .522076 .522076
Output Values Normal
Payer Receiver
Premium .82044 .82044
Delta 140.326524 -140.326524
Gamma 152.79634 152.79634
Vega 1.571495 1.571495
Swap Tenor 2.96
Swaption Tenor 1.97

Swap

Vol SEK Black

Swaption

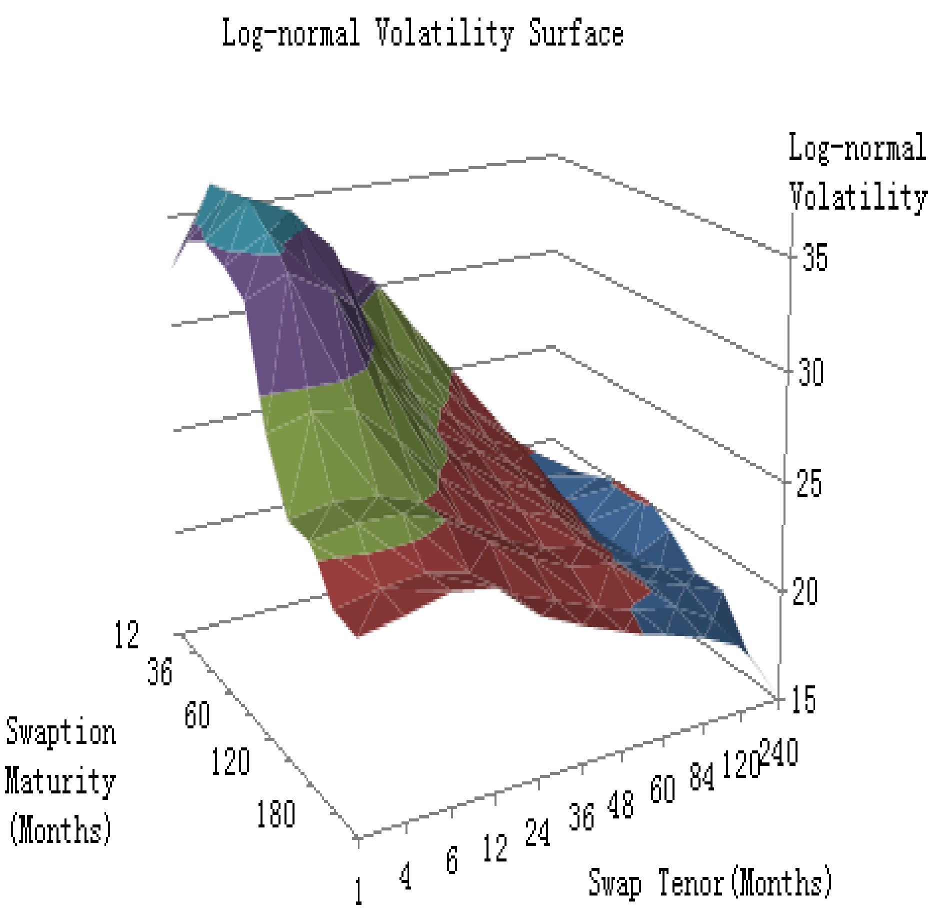
Months	1	4	6	12	24	36	48	60	84	120	240
12	32.6	36.3	35.1	33.8	31.5	30.4	27.8	25.3	22.5	20.3	18.2
24	34.4	36.4	35.1	32.5	30.5	27.5	25.9	23.7	22.2	19.4	19.2
36	35	36.6	35.8	33.7	28.7	26	24.5	23	21.8	18.9	19.7
48	34.5	35.7	33.5	31.6	27.5	25	23.7	22.4	21.1	18.6	20.2
60	33.6	34.5	33.3	29.6	26.5	24.4	22.9	22.1	20.3	18.4	20.2
84	28.5	29.1	28.8	27	24.9	23.4	22.3	21.6	19.9	18.6	19.4
120	25.6	25.9	25.8	25	23.6	22.4	21.4	21	19.7	19	18.2
136	25.6	25.9	25.8	25	23.6	22.4	21.4	21	19.7	19	18.2
180	23.6	24	24.1	23.6	22.2	21.4	20.5	19.9	19.4	18.4	16
240	23.4	23.8	24.2	23.8	22.1	21.2	20.3	19.7	19	18	15.1

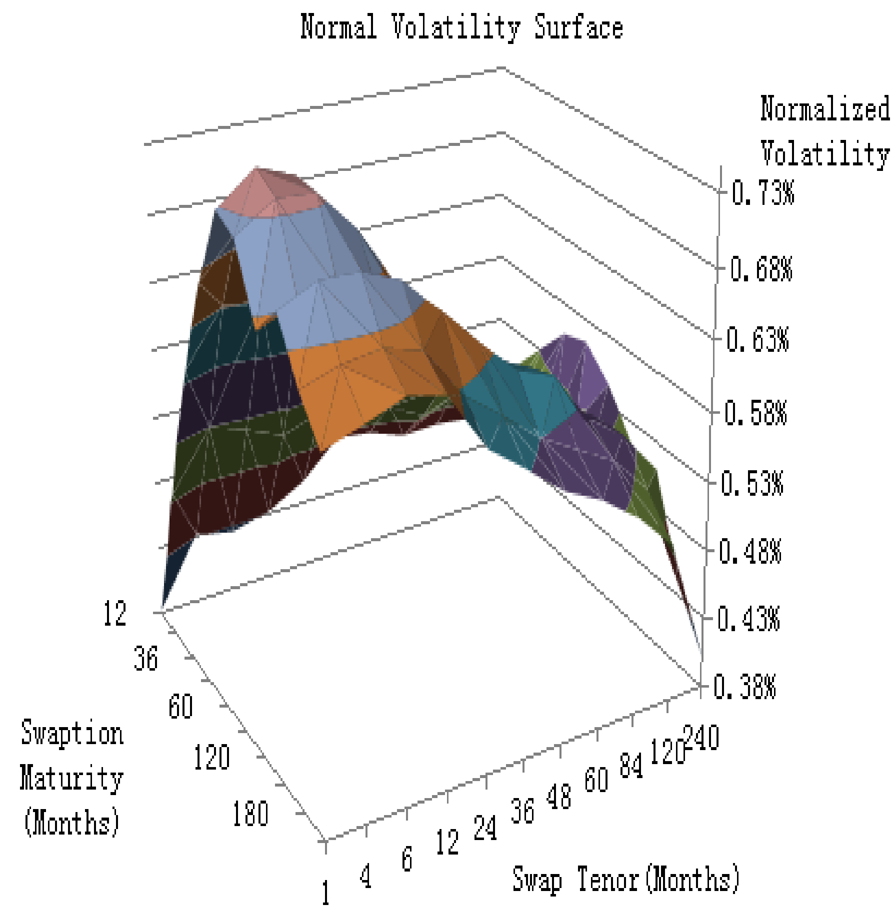
Forwd SEK

Months	1	4	6	12	24	36	48	60	84	120	240
12	1.1744	1.1973	1.2163	1.2858	1.4285	1.5655	1.6943	1.8217	2.0491	2.2542	2.5787
24	1.4330	1.4717	1.4960	1.5713	1.7053	1.8305	1.9557	2.0770	2.2555	2.4084	2.6568
36	1.7212	1.7550	1.7765	1.8394	1.9602	2.0839	2.2034	2.2973	2.4193	2.5463	2.7206
48	1.9690	1.9994	2.0198	2.0810	2.2061	2.3248	2.4118	2.4756	2.5494	2.6572	2.7712
60	2.2181	2.2490	2.2696	2.3313	2.4466	2.5221	2.5743	2.6030	2.6561	2.7440	2.8099
84	2.5913	2.6059	2.6220	2.6729	2.7019	2.7072	2.7180	2.7399	2.7997	2.8352	2.8437
120	2.7016	2.7116	2.7210	2.7501	2.7889	2.8426	2.8690	2.8850	2.8901	2.8933	2.8555
136	2.8133	2.8273	2.8366	2.8679	2.9100	2.9232	2.9270	2.9228	2.9166	2.9047	2.8452
180	2.8964	2.8904	2.8945	2.9100	2.9026	2.9029	2.9027	2.9015	2.8863	2.8757	2.7851
240	2.8469	2.8404	2.8420	2.8492	2.8485	2.8483	2.8490	2.8499	2.8314	2.8177	2.7105

Vol SEK Normal

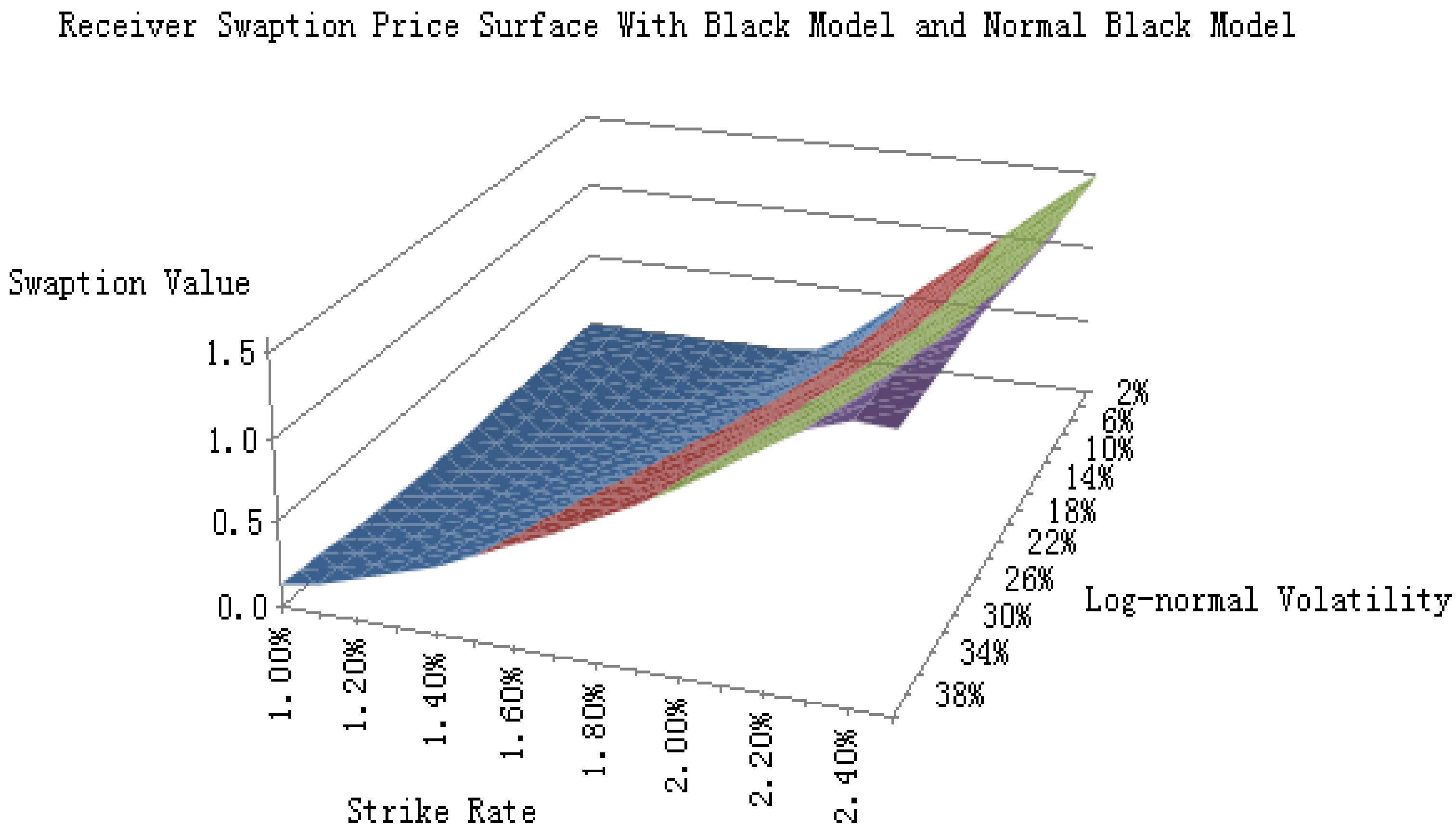
Months	1	4	6	12	24	36	48	60	84	120	240
12	0.3812%	0.4322%	0.4248%	0.4325%	0.4481%	0.4741%	0.4695%	0.4597%	0.4601%	0.4568%	0.4687%
24	0.4881%	0.5299%	0.5197%	0.5062%	0.5161%	0.5002%	0.5037%	0.4899%	0.4987%	0.4658%	0.5085%
36	0.5933%	0.6317%	0.6260%	0.6112%	0.5568%	0.5373%	0.5358%	0.5249%	0.5243%	0.4791%	0.5334%
48	0.6661%	0.6989%	0.6642%	0.6468%	0.5991%	0.5752%	0.5663%	0.5499%	0.5340%	0.4914%	0.5560%
60	0.7281%	0.7571%	0.7387%	0.6777%	0.6390%	0.6078%	0.5831%	0.5695%	0.5346%	0.5014%	0.5628%
84	0.7214%	0.7400%	0.7373%	0.7066%	0.6608%	0.6235%	0.5974%	0.5839%	0.5508%	0.5221%	0.5457%
120	0.6732%	0.6832%	0.6830%	0.6700%	0.6432%	0.6237%	0.6024%	0.5949%	0.5603%	0.5416%	0.5126%
136	0.6985%	0.7097%	0.7095%	0.6964%	0.6691%	0.6396%	0.6131%	0.6012%	0.5642%	0.5426%	0.5098%
180	0.6605%	0.6695%	0.6730%	0.6636%	0.6251%	0.6039%	0.5798%	0.5634%	0.5470%	0.5181%	0.4386%
240	0.6370%	0.6454%	0.6556%	0.6474%	0.6048%	0.5820%	0.5591%	0.5438%	0.5222%	0.4938%	0.4016%


Log-normal Volatility Surface
Log-normal Volatility
35
30
25
20
15
Swaption Maturity (Months)
12
36
60
120
180
Swap Tenor(Months)
1
4
6
12
24
36
48
60
84
120
240


Normal Volatility Surface
Normalized Volatility
0.73%
0.68%
0.63%
0.58%
0.53%
0.48%
0.43%
0.38%
Swaption Maturity (Months)
12
36
60
120
180
Swap Tenor(Months)
1
4
6
12
24
36
48
60
84
120
240

T =	2
TT =	4
fRate	1.7162%
Strike =	[1, 2.5]
BlackVol =	[2, 40]
f =	4
N =	1
r =	1.2355%

Strike	2
BVol =	2
Nvol =	0.5257%

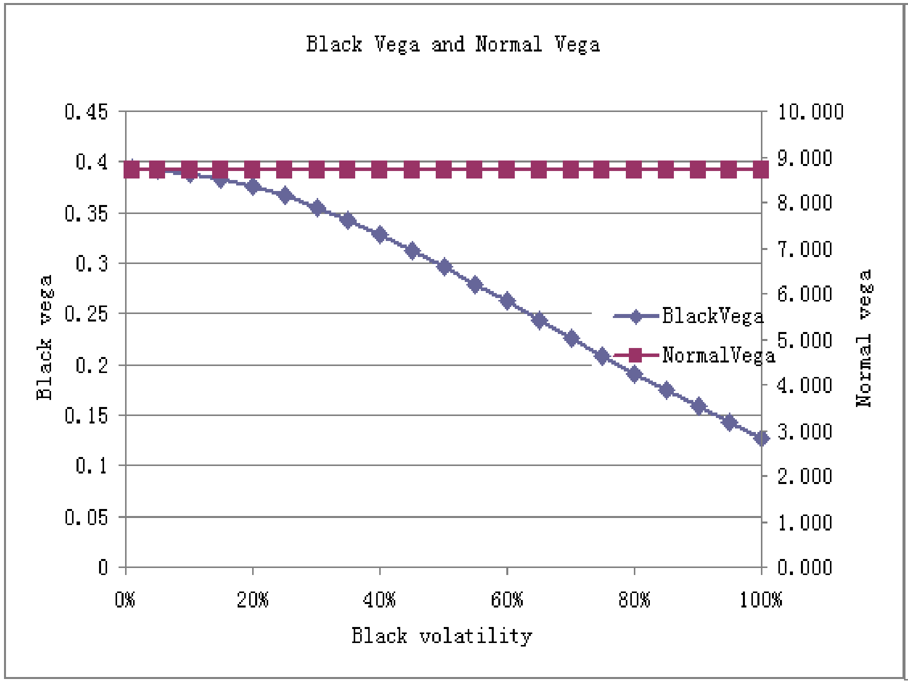


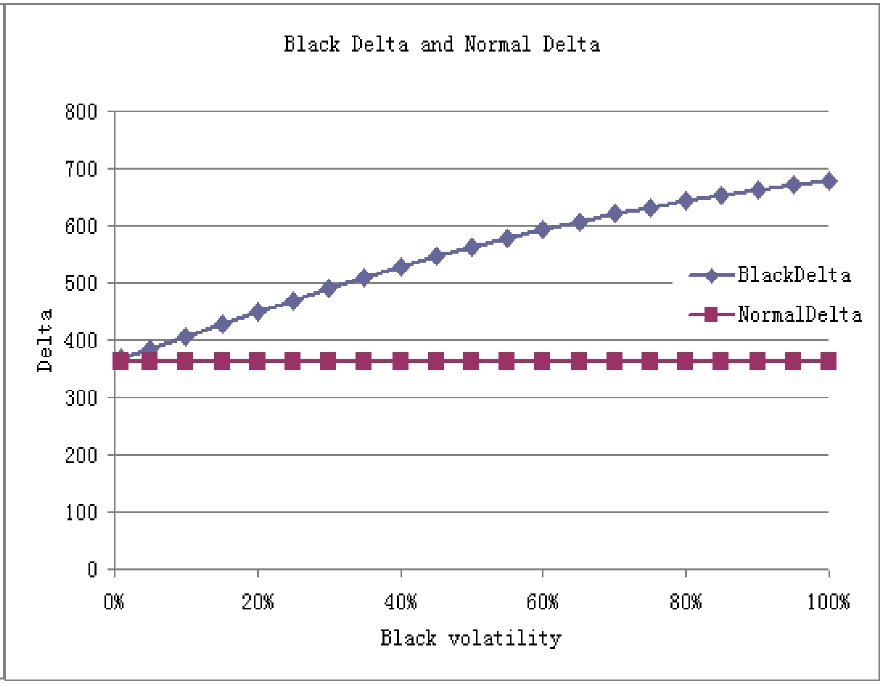
Swap Tenor	10
Swaption Maturi	9
ForwRate	3.0000%
StrikeRate	4.0000%
RiskFree Rate	1.3423%
Frequency	4

Black Vol	NormalVol	Black Vega	Normal vega	Black Delta	NormalDelta
1%	0.0348%	3.84179E-21	0.00000	0	0
5%	0.1736%	0.056593417	1.44688	24.93543279	20.94867471
10%	0.3463%	0.222918168	5.74745	159.7135184	128.1299831
15%	0.5170%	0.283773774	7.42001	258.9682697	198.0935118
20%	0.6849%	0.304262902	8.11383	327.2379682	239.0625155
25%	0.8491%	0.309191283	8.45648	378.9770309	265.0621454
30%	1.0087%	0.306586801	8.64853	421.1637715	282.7756918
35%	1.1630%	0.299613141	8.76632	457.2185209	295.5036269
40%	1.3113%	0.289774051	8.84352	488.9482942	305.0203492
45%	1.4531%	0.277916329	8.89672	517.3756681	312.3544801
50%	1.5879%	0.264594009	8.93484	543.1144796	318.1402009
55%	1.7152%	0.250216362	8.96303	566.5536912	322.7885796
60%	1.8348%	0.235112984	8.98440	587.9536891	326.5772418
65%	1.9464%	0.219563831	9.00093	607.4995949	329.7002044
70%	2.0499%	0.203813346	9.01393	625.3320667	332.2970112
75%	2.1453%	0.188077011	9.02430	641.565635	334.470516
80%	2.2325%	0.172544267	9.03265	656.2997948	336.2981447
85%	2.3118%	0.157379747	9.03944	669.6257137	337.8392585
90%	2.3831%	0.142723844	9.04498	681.6302005	339.1401031
95%	2.4467%	0.12869315	9.04953	692.3979219	340.2372196
100%	2.5029%	0.115381047	9.05325	702.0124854	341.1598435

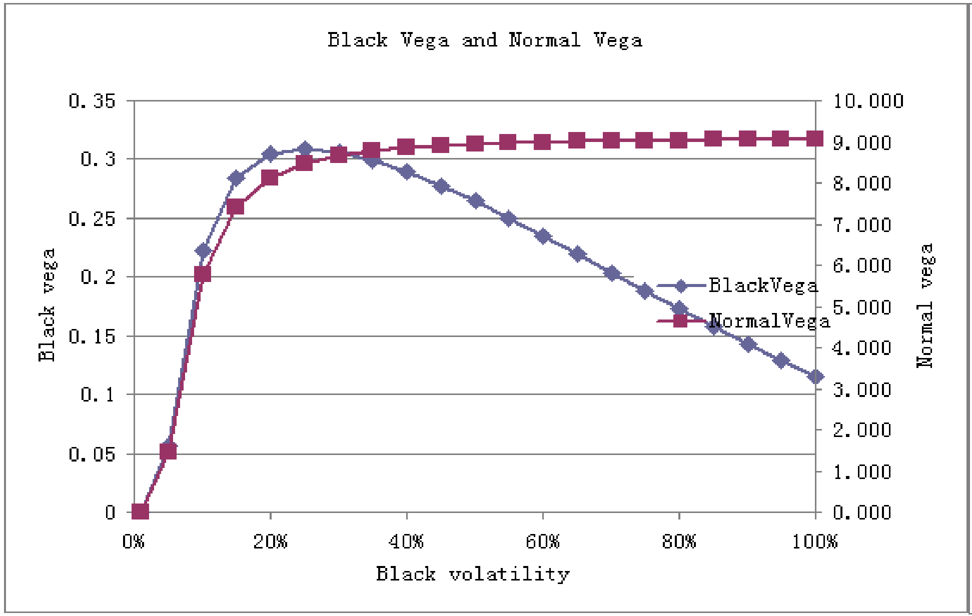
Disparity between the Black Vega(Delta) and normal Vega(Delta) with respect to Black volatility

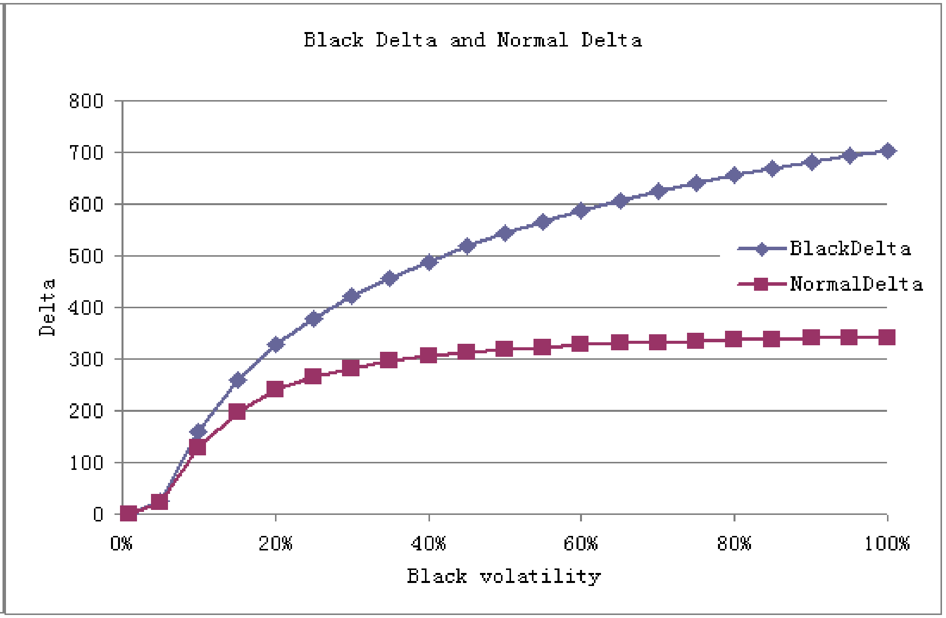
At the Money



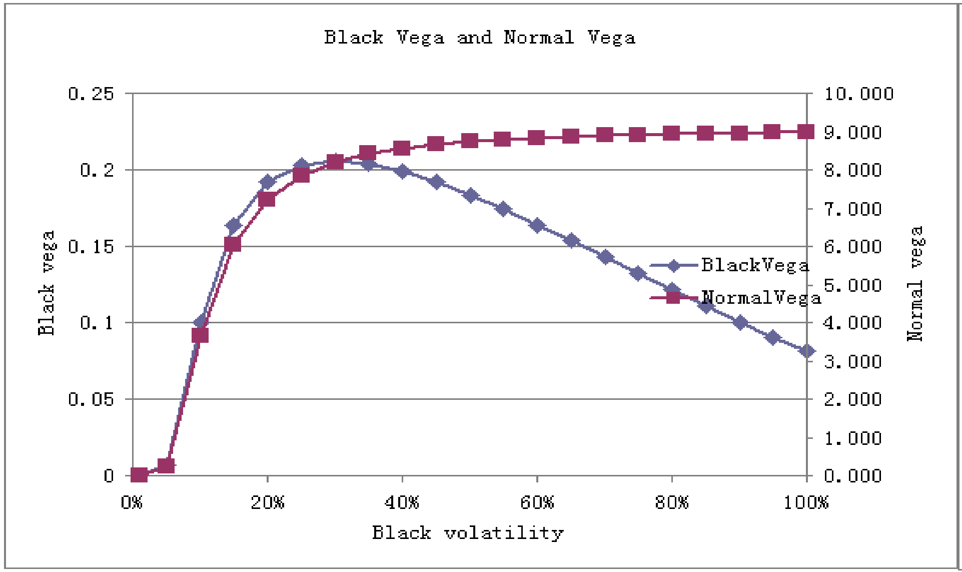


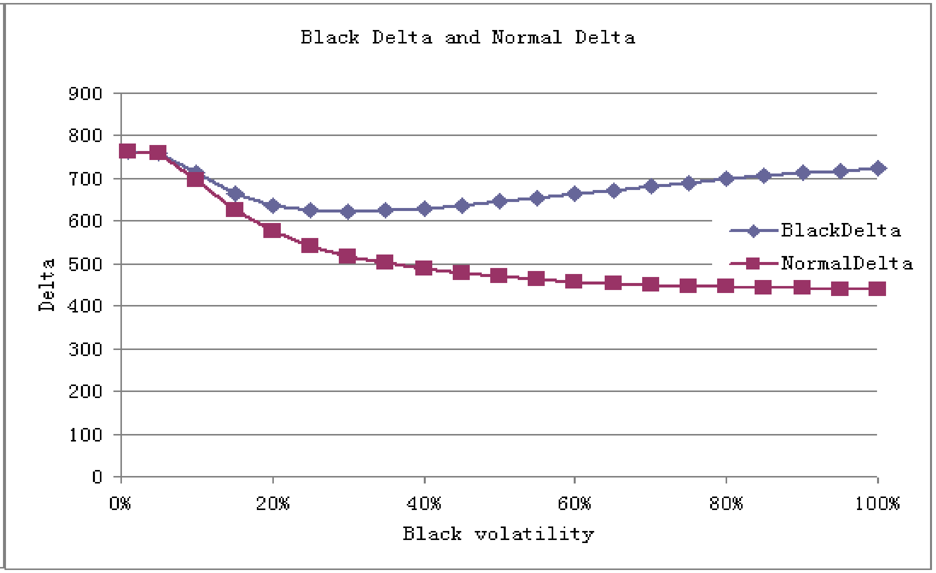
Out of the Money





In the Money





Conclusion