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Division of Applied Mathematics

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Bridging the Gap: An Analysis into the relationship between Duration and Key Rate Duration

by

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Abstract

As financial instruments continue to increase in their complexity, the ways in which risk is measured needs to adapt accordingly. In this paper the aim is to communicate the importance of using Key Rate Duration as a risk measure as opposed to the more popular Duration when measuring risk of fixed income instruments as well as the relationship between the two.

The paper will give insights into the deficiencies of the Duration measure with particular emphasis on when interest shifts on the term structure are done in a non-parallel manner. It will also show how when shifts are done in a non-parallel manner Key Rate Duration is the better measure.

Finally this paper will show, using market data relating to Swedish interest rates, that individual nodes have varying risk buckets and these can be substantial thereby creating potential risk compliance issues.

Keywords: Duration, Key Rate Duration, Term Structure, Interpolation, Bootstrapping

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Chapter 1

Introduction

In the financial services industry it is of paramount importance to ensure that risk relating their products, be calculated both accurately and provide relevant information to for decision making.

The research to be undertaken is to communicate the differing methodologies between the Risk Department and the Trading Floor for calculating risk relating to interest rate movements on fixed income instruments and `"Bridging the Gap"` between the two.

1.1 Background

To better understand the purpose of the research the differing methodologies need to be explained using a Forward Rate Agreement (FRA) for illustration. The first will be the view of

1.1.1 Trading floor perspective

The Trading Floor calculates risk by shifting quotes of the fixed instrument as shown in Figure 1.1,

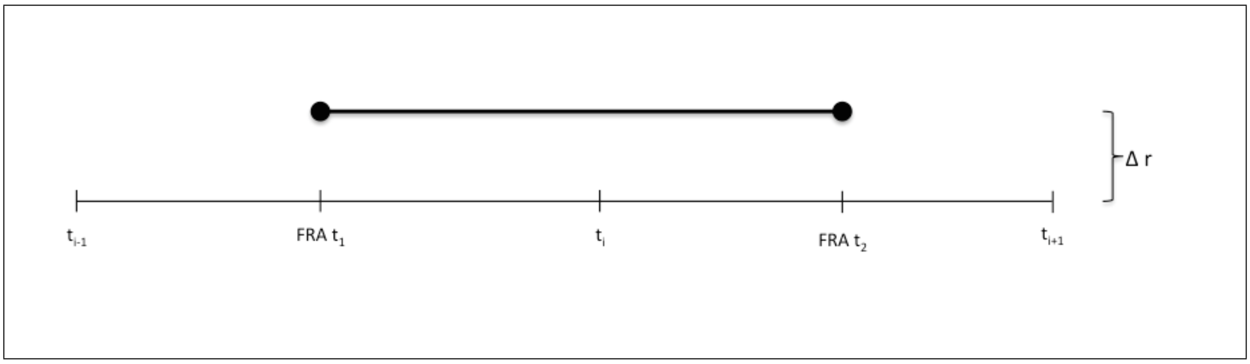

Figure 1.1: The floor trading perspective

and this is done as a way of hedging their trades [19]. The value at risk would then be calculated as

$$\text{Risk} = \text{notional amount} \times \text{time} \times \text{change in interest rate.}$$

The calculation of interest rate risk by the trading floor is very similar to *Duration*. This is due to both calculations being based on shifting the interest only of the specific instrument.

1.1.2 Risk management perspective

On the other side of the coin, the Risk Department shifts the zero coupon curve (primarily done to aggregate the risk in multiple currencies) in specific nodes as shown in Figures 1.2–1.4. The shifting done at these curve-nodes is similar to *Key Rate Duration*.

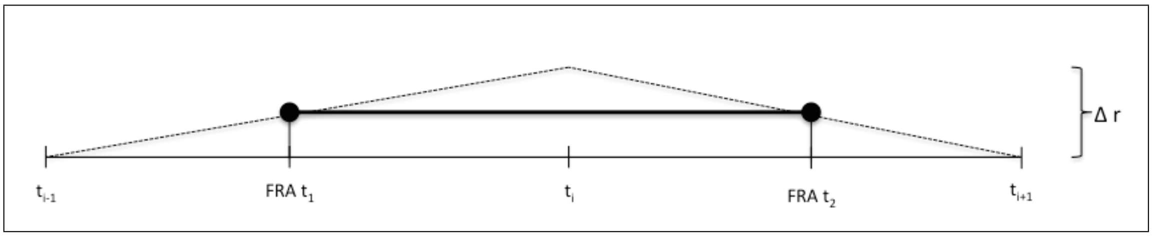

Figure 1.2: Risk manager view with shift at node t_i

The first shift on the rates is at t_i but the instrument is shifted to an interest rate change determined by interpolation methods [8]. Figure 1.2 would show interest shift risk associated with that node.

Next is to shift the node at t_{i-1} in this case only one part of the instrument is affected the shift.

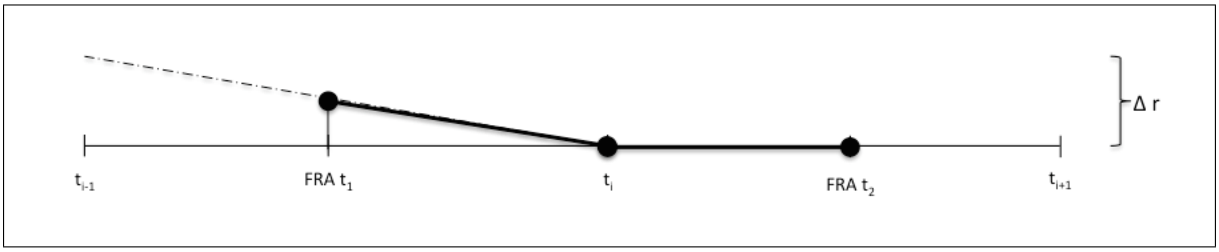

Figure 1.3: Risk manager view with shift at node t_{i-1}

Finally a shift is done at the node t_{i+1}

When all the affected nodes have been shifted the result is as in Figure 5.

The relationship between these two calculations is

$$\text{Trading floor risk} = \sum_{i=1}^{m} \text{Risk department risk, calculated at each node } t_i,$$

and is similar to the relationship of Key Rate Duration(KRD) and Duration [12].

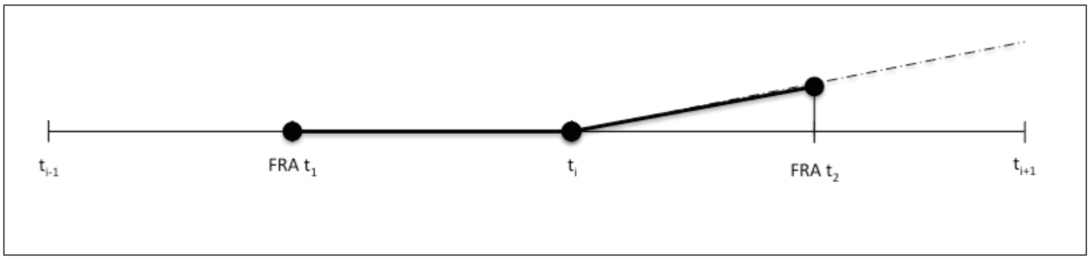

Figure 1.4: Risk manager view with shift at node t_{i+1}

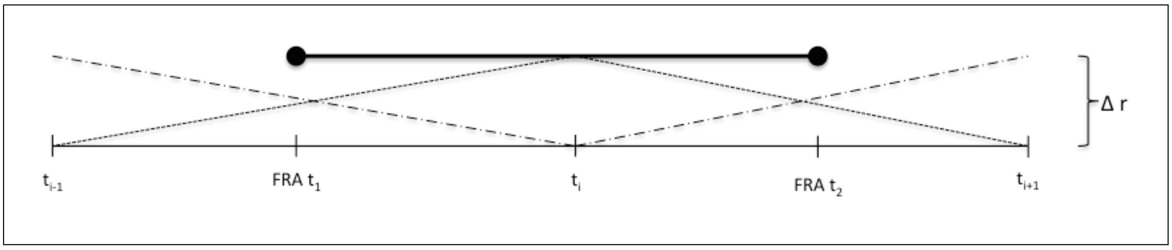

Figure 1.5: Risk manager view

$$\text{Duration} = \sum_{i=1}^{m} \text{KRD at } t_i.$$

where m is the number of key rates to be used.

These two different methods have implications regarding whether the trades are within the prescribed risk limits. This would affect compliance on one hand as well as lead to blind spots regarding unstated risk from the different nodes.

As this is an applied science course the aim is not to deliver a theoretical output but a usable product, hence the aim is to deliver a better general bootstrap for determining the risk.

1.2 Literature Review

1.2.1 Duration

Originally defined in 1938 by Professor Macaulay, Duration [14] is the mean length of time that would pass before the present value would be returned by a stream of known fixed payments. Duration is given by [10]

$$D = \sum_{i=1}^{n} t_i \left(\frac{c_i e^{-y t_i}}{B} \right). \tag{1.1}$$

where

- t_i is the time when cash flow occurs,
- c_i is the cashflow,

- B is the bond price and
- y is the yield to maturity.

Though initially used as a replacement for maturity it was Hicks who first demonstrated the risky-proxying property of Duration [14]. This led to it's wide usage as a risk-proxy but it would be Hopewell and Kaufman [13] that would bring to light inconsistencies regarding the use of Duration as a risk measure. They noted that the relationship between the maturity and interest rate sensitivity of bonds does not hold for bonds selling at a discount.

This led to the following key limitations of duration as a risk-proxy:

- when a 100 basis point change occurs, Duration as measure of risk is useless,
- when shifts on the yield curve are non-parallel, Duration becomes a poor risk measure,
- due to it's use of yield to maturity as opposed to forward interest rates Duration requires a flat term structure.

Despite the shortcomings of Duration it is widely used, for the purposes of this study the non-parallel shift deficiency will be the main defiency to be researched.

1.2.2 Key Rate Duration

Given the weakness of Duration in dealing with non-parallel shifts Key Rate Duration model was [11] proposed as an alternative. It's premise is the shifts in the term structure as a discrete vector representing the changes in the key spot rates of various maturities. Interest changes at other maturities are derived from these values via linear interpolation, it is important to note that only linear interpolation can be used as other interpolation methods require more data points.

1.3 Problem Formulation

Currently Traders use Duration as their main form of risk proxy when dealing with Fixed Income instruments and related portfolios. Though this measure is attractive to them as they can use it for the purposes of hedging it falls short at identifying risk at key rates. As these rates can at times move independent of other rates it leaves them exposed to unforeseen risk.

The centre of interest for this thesis document is the utilisation of Key Rate Duration as a method for calculating of risk in Fixed Income instruments. The type of instruments to be used in this study are Deposits, Forward Rate Agreements and Interest Rate Swaps. The study will utilise bootstrapping techniques to obtain the term structure curve and then linear interpolation to determine the risk amounts in the individual key rates.

The results of this research show that when calculating interest movement dependent risk in Fixed Income instruments Key Rate Duration provides a solution with better insights. It gives the Trader(viewed as primary user of study) both the total risk during the entire period of the instrument and the risk due to interest rate movements that occur at only one rate. This will result in the Trader gaining an understanding of how different key rates carrying varying risk and hence require differing strategies.

In Chapter 2 the study will detail the instruments to be used for building the term structure curve that will also be analysed to show how key rate shifts operate. In Chapter 3 the focus will be on the mathematics of how to build a term structure and detail the bootstrapping techniques used and the importance of curve smoothness in obtaining accurate results.

Chapter 4 gives insight into Duration as a risk proxy and its utilisation as a tool for hedging. The will also be a discussion on how it has certain deficiencies and how these compromise the results that one obtains. Chapter 5 deals with how Key Rate Duration is utilised to give good insights on the amount of risk at key rates.

Chapter 6 involves the building of a term structure curves using bootstrapping techniques discussed in Chapter 3. The next is to calculate risk based on Duration and then Key Rate Duration. From these relationships between the two will shown and also that amount of risk in certain key rates can at times significantly higher than that which is calculated from Duration.

Chapter 2

Fixed Income Instruments

This chapter deals with the main types of *Fixed Income Instruments* and how they are valued. These are also the key instruments used in the construction of an *interest rate curve* and also be used to show the risk implications of Duration and Key Rate Duration .

2.1 Bonds

A bond is the most commonly known fixed income security. The general theory is that an *Issuer* of the bond sells it to an *Investor* who is lending money to the issuer in return for interest payments and the repayment of the money borrowed [7].

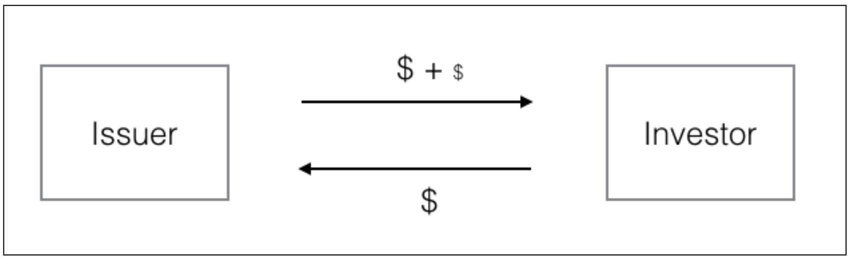

Figure 2.1: Diagram depicting bond relationship

2.1.1 Valuation of Bonds

The theoretical price of a bond is the PV of the interest payments and the principal and calculated as follows

$$PV = \sum_{i=1}^{n} C \cdot p(0,T_i) + M \cdot p(0,T_n).$$

where

- C is the coupon payment,

- $p(0,T_i)$ is the discounting factor at the period T_i,
- M is the principal amount and
- n is number of payment periods.

The different types of bonds are

- Callable bonds,
- Putable bonds,
- Convertible bonds,
- Zero-coupon bonds and
- Foreign currency bonds.

2.2 Forward Rate Agreements

A *Forward Rate Agreement*, is an over-the-counter contract that a reference rate (normally LIBOR rate) will be exchanged for a specified interest rate during a specified future period of time. This instrument is used to both hedge interest rate exposure and speculate on interest rates. To better understand an FRA and it's valuation familiarisation with the following terms, needed [19]

- Notional (N)- the principal underlying the contract,
- Trade date - the date on which the contract is dealt,
- Spot date - the date in with the contract must be delivered,
- Settlement date (T_1)- the date on which the contract period begins,
- Fixing date - the date on which the reference rate is determined,
- Maturity date (T_2)- the date on which the contract expires,
- FRA rate (r_{FRA})- the rate at which the FRA is traded,
- Reference rate (r_{Ref})- the rate which is agreed upon to be used to reference FRA contract on and
- Settlement sum - the difference between FRA rate and reference rate as a percentage of the notional and is paid on the settlement date.

Figure 2.2 shows these dates as they would be on a timeline [19].

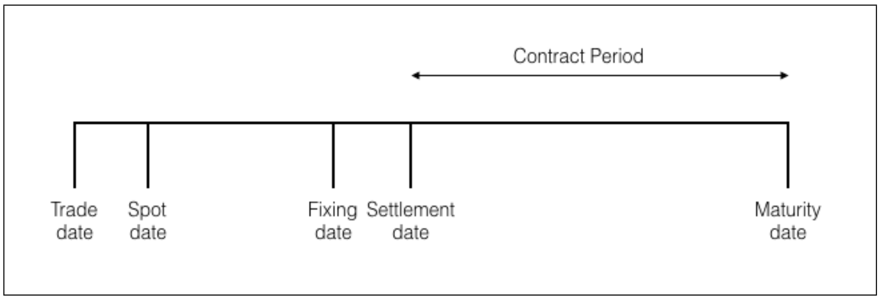

Figure 2.2: Key time periods in FRA contracts

2.2.1 Valuation of FRAs

The valuation of the FRA is then derived as which rate you receive your payout in and discounted using the riskfree rate at T_2 which will be r_2 [10]:

1. $r_{FRA} = r_{Ref}$ where the value is 0,
2. if payout is r_{FRA} then $V_{FRA} = N(r_{FRA} - r_{Ref})(T_2 - T_1)e^{-r_2 T_2}$ and
3. if payout is r_{Ref} then $V_{FRA} = N(r_{Ref} - r_{FRA})(T_2 - T_1)e^{-r_2 T_2}$

2.3 Interest Rate Swaps

A *swap* is an over-the-counter agreement between two counterparts to exchange periodic interest rate payments where it's dollar amount is based on some pre-determined principal. The agreement gives certain dates when cash flows are to be paid and the way in which they are to be calculated [15]. The best type of swap to illustrate the concept with is a plain vanilla swap as it operates of the basic definition of a swap. Swaps are extremely liquid due to the huge size of the swaps market.

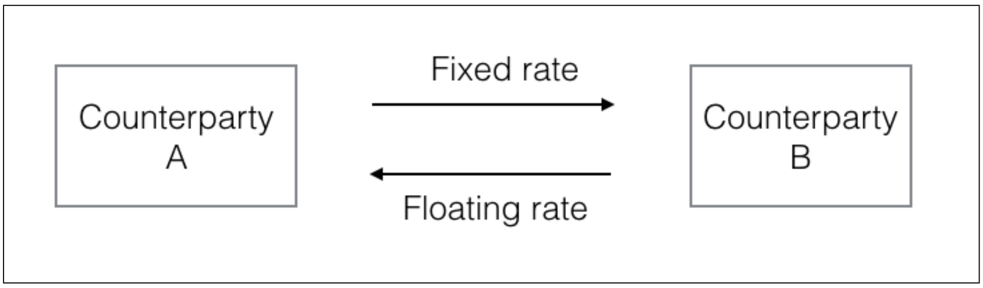

Figure 2.3: Diagram depicting plain vanilla swap relationship

2.3.1 Valuation of Swaps

In interest rate swaps no principle payments are made, the most basic illustration of the valuation is [7]

- from view of a floating-rate payer as a long position in a fixed rate bond and a short position on a floating rate bond

$$V_{swap} = B_{fixed} - B_{floating},$$

- from view of a fixed rate payer as a long position in a floating rate bond and a short position on a fixed rate bond.

$$V_{swap} = B_{floating} - B_{fixed}.$$

To then get a value for the swap it, necessary to determine the fixed leg and floating leg.

Floating leg

To understand the valuation of the *floating leg* the following terms need to be explained

- discount factors $P(T_{i-1}, T_i)$ - this allows for valuing of money received in T_i at T_{i-1} where $T_{i-1} \leq T_i$ and 0 is present with the following relationship

$$P(T_{i-1}, T_i) = \left(\frac{P(0, T_i)}{P(0, T_{i-1})} \right), \tag{2.1}$$

- forward rates $f(T_{i-1}, T_i)$ - are rates of interest implied by current zero for periods of time in the future. It's relationship with discount factors is shown below with δ_i being the day count fraction

$$f(T_{i-1}, T_i) = \frac{1}{\delta_i} \left(\frac{1}{P(T_{i-1}, T_i)} - 1 \right),$$

As floating-rate payments are set at the beginning of the payment period and paid at the end of the period. As these payments are on the interest only for that period they are similar to zero coupon bonds. The value of the swap can therefore be seen as a sum of zero coupon bonds with payments calculated using forward rates, leading to the following relationship

$$PV_{floating} = \sum_{i=1}^{n_{floating}} \delta_i f(T_{i-1}, T_i) P(0, T_i),$$

The first thing do to simplify

$$PV_{floating} = \sum_{i=1}^{n_{floating}} \frac{\delta_i}{\delta_i} \left(\frac{1}{P(T_{i-1}, T_i)} - 1) P(0, T_i \right),$$

which leads to

$$PV = \sum_{i=1}^{n_{floating}} P(0, T_{i-1}) - P(0, T_i) = P(0, T_0) - P(0, T_{n_{floating}}),$$

and the resulting final relationship is

$$PV_{floating} = 1 - P(0, T_{n_{floating}}).$$

Fixed leg

The pricing of the fixed leg is simpler as it is the discounting of the swap rate $= C$ as shown

$$PV_{fixed} = \sum_{i=1}^{n_{fixed}} \delta_i C P(0, T_i).$$

It is important to note that the frequencies of the fixed and floating payments need not be similar hence n_{fixed} and $n_{floating}$[19].

2.4 Chapter Summary

In this chapter the discussion is on the key fixed asset instruments and their valuation. The first was bonds which are the simplest and most commonly known instrument and also the factors that affect their pricing. The next were Forward Rate Agreements and Swaps and their valuation. These instruments are of importance when constructing a yield curve as will be discussed in the next chapter.

Chapter 3

Curve Construction

One of the key purposes of a term structure is to allow the consistent valuing of instruments under a singular valuation framework. The framework to be used is the continuous zero curve as it is standard format for all option pricing formulae [7]. The accuracy of the model implementation are highly dependent on material found in this chapter.

3.1 Term structure estimation methods

There exist two term structure estimation methodologies [8]

1. Theoretical term structure methods typically based on an explicit structure for a variable known as the short rate of interest, whose value depends on a set of parameters that might be determined using statistical analysis of market variables with Vasicek [20] being a notable example.

2. Empirical methods, unlike the theoretical methods, are independent of any model or theory of the term structure. The empirical methods utilise observed interest rate data and then uses these data points to try and find a close representation of the term structure at any point in time.

For the purposes of this research the focus will be on the empirical methods only.

3.2 Mathematics of Yield Curves

The term structure of interest rates is defined as the relationship between the yield to maturity on a zero coupon bond and the bond's maturity. From the outset, continuously-compounded rates will be used as most derivatives are modelled in continuous time [1].

To develop the zero curves certain assumptions need to be made

- Market trades continuously over the time horizon,
- Market is frictionless,

- Market is competitive every trader,
- Market is efficient,
- Market is complete,
- No arbitrage and
- All trades are rational.

These assumptions though important in the development of the zero curve do not occur in reality.

The first step is to give the zero yield in terms of the bond price. This relationship is given by

$$P(t,T) = \exp(-(T-t)r(t,T)), \tag{3.1}$$

where $t \leq T \leq t_\infty$ and

- t is current time,
- T is maturity time,
- $P(t,T)$ price of the bond and
- $r(t,T)$ yield at time t of a bond that matures at time T.

This forms the basis of the mathematics of a zero yield curve.

The forward price of a bond is provided that $t < T_1 < T_2$ is given by

$$P(t,T_1,T_2) = \left(\frac{P(t,T_2)}{P(t,T_1)}\right), \tag{3.2}$$

at the same time an implied forward rate, as seen at time t for the period T_1 to T_2 symbolised as $f(t,T_1,T_2)$ is defined by

$$P(t,T_1,T_2) = \exp(-(T_2-T_1)f(t,T_1,T_2)),$$

Using Equation (3.1) and substituting it into Equation (3.2) the following is obtained

$$exp(-(T_2-t)r(t,T_2)) = \exp(-(T_1-t)r(t,T_1)) \times \exp(-(T_2-T_1)f(t,T_1,T_2)), \tag{3.3}$$

Rearranging Equation (3.3) yields a result that shows how the forward rate is determined

$$f(t,T_1,T_2) = \left(\frac{(T_2-t)r(t,T_2)-(T_1-t)r(t,T_1)}{(T_2-T_1)}\right), \tag{3.4}$$

In Equation (3.4), the period forward rate is defined. However, the instantaneous forward rate is of much greater importance in the theory of the term structure. The instantaneous forward rate for time T, as seen at time t, is denoted by $f(t,T)$ and is the continuously compounded rate defined by

$$f(t,T) = \lim_{h\to 0} f(t,T,T+h) = r(t,T) + (T-t)r_T(t,T), \tag{3.5}$$

where

$$r_T(t,T) = \frac{\partial}{\partial T} r(t,T),$$

A re-arrangement of Equation (3.1) gives the following

$$-\ln(P(t,T)) = (T-t)r(t,T), \tag{3.6}$$

Differentiating Equation (3.6) with respect to T gives

$$\frac{-P_T(t,T)}{P(t,T)} = y(t,T) + (T-t)y_T(t,T), \tag{3.7}$$

Finally, by direct comparison of Equation (3.4) and Equation (3.7),

$$f(t,T) = \left(\frac{-P_T(t,T)}{P(t,T)}\right),$$

Equation (3.4) implies $f(t,t) = y(t,t)$ and noting that $P(t,t) = 1$, Equation (3.7) can be used to obtain

$$r(t,t) = \lim_{T\to t} \frac{-P_T(t,T)}{P(t,T)} = -\lim_{T\to t} P_T(t,T).$$

Though defined in terms of the zero yields, the zero curve can be defined in terms of the instantaneous forward rates [1]. In this case, the zero curve is defined by the instantaneous forward rates $f(t)$ for $t \le T \le t_\infty$. The zero yield in terms of the instantaneous forward rate is obtained by integrating Equation (3.5) and yields

$$r(t,T)T = r(t,t)t + \int_t^T f(t,u)du.$$

This gives an overview of the essential mathematical theory of zero curves. In the following sections, the relevance of this theory to the interpolation of zero curves is shown and a look into smoothest forward-rate interpolation.

3.3 Interpolation

To construct zero curves from market data, it's assumed that the n data values are

$$\{(T_1,y_1),(T_2,y_2),...,(T_n,y_n)\},$$

where $0 \le T_1 < T_2 < ... < T_n < T_\infty$ are the times to maturity of $n \ge 1$ zero coupon bonds.

Mathematical theory of zero curves discussed earlier assumes that the value of $r(t,T)$ for $0 \leq T < t_\infty$ is known. In reality, the current zero curve is not defined by this infinite set of values implied by the complete market assumption but, rather, by a set of discrete data values each value comprising a time to maturity and a zero rate [2].

To use the mathematics of zero curves derived earlier, the discrete set of values should be extended to an infinite set. To achieve this define the current zero curve by a combination of the set of discrete data values and a method for interpolating those values.

Given the need for interpolation, suppose a time variable t is given such that $t_{i-1} < t < t_i$. The desire is to attain the estimated value of $r(t)$ given that the rates for $r(t_{i-1})$ and $r(t_i)$ are known, this is where interpolation comes in.

There are various interpolation methods with varying levels of accuracy and difficulty, it is important to note the following when comparing them [9]:

- The positivity and continuity of the forward rates to avoid arbitrage,
- The localisation of interpolation, which relates to if an input is changed does the interpolation function only change in close proximity or the change is more widespread,
- The stability of forward rates; this is observed by changing an input in the interpolation function and noting the maximum change on the forward curve,
- The delta risk assigned to hedging instruments that have maturites close to the given tenors or is it more widespread and
- The effect the number of instruments has on it; if it is small does it give an exact empirical curve and on large set does the algorithm give a small degree of error.

3.3.1 Linear Interpolation

The most common form of interpolation is linear and the methods formulated are thereby also implicitly linear. There are numerous ways that linear interpolation can be applied these are

Linear on rates

To determine $r(t)$ given the following conditions $t_{i-1} < t < t_i$ the interpolation formula is

$$r(t) = \left(\frac{t - t_{i-1}}{t_i - t_{i-1}}\right) r_i + \left(\frac{t_i - t}{t_i - t_{i-1}}\right) r_{i-1},$$

Using the relationship that $f(t) = \frac{d}{dt}(r(t))t$ the following is obtained for forward rates

$$f(t) = \frac{d}{dt}\left(\left(\frac{t^2 - tt_{i-1}}{t_i - t_{i-1}}\right) r_i + \left(\frac{tt_i - t^2}{t_i - t_{i-1}}\right) r_{i-1}\right),$$

differentiation with respect to t yields

$$f(t) = \left(\frac{2t - t_{i-1}}{t_i - t_{i-1}}\right) r_i + \left(\frac{t_i - 2t}{t_i - t_{i-1}}\right) r_{i-1}.$$

Linear on discount factors

As similar approach to that done on the rates can be taken on the discount factors

$$D(t) = \left(\frac{t - t_{i-1}}{t_i - t_{i-1}}\right) D_i + \left(\frac{t_i - t}{t_i - t_{i-1}}\right) D_{i-1},$$

Given the relationship $D(t) = \exp(-r(t)t)$ the following calculation can be done

$$r(t) = -\frac{1}{t} \ln\left(\left(\frac{t - t_{i-1}}{t_i - t_{i-1}}\right) D_i + \left(\frac{t_i - t}{t_i - t_{i-1}}\right) D_{i-1}\right),$$

This allows for the use of the relationship $f(t) = \frac{d}{dt}(r(t))t$ to obtain forward rates by firstly

$$f(t) = \left(\frac{\frac{-1}{t_i - t_{i-1}} D_i - \frac{1}{t_i - t_{i-1}} D_{i-1}}{\frac{t - t_{i-1}}{t_i - t_{i-1}} D_i + \frac{t_i - t}{t_i - t_{i-1}} D_{i-1}}\right),$$

then

$$f(t) = \left(\frac{D_{i-1} - D_i}{(t - t_{i-1})D_i + (t_i - t_{i-1})D_{i-1}}\right).$$

as can observed the forward rate is not continuous.

Linear on the log of rates

In this instance the interpolation formula for the conditions $t_{i-1} < t < t_i$ is

$$\ln(r(t)) = \left(\frac{t - t_{i-1}}{t_i - t_{i-1}}\right) \ln(r_i) + \left(\frac{t_i - t}{t_i - t_{i-1}}\right) \ln(r_{i-1}),$$

which gives the following rate formula

$$r(t) = \left(r_i^{\frac{t - t_{i-1}}{t_i - t_{i-1}}} r_{i-1}^{\frac{t_i - t}{t_i - t_{i-1}}}\right),$$

The key failing of this interpolation method especially in the current interest rate environment is that it does not allow for negative interest rates.

To determine the forward rate first define

$$\ln(r(t)t) = \left(\frac{t - t_{i-1}}{t_i - t_{i-1}}\right) \ln(r_i) + \left(\frac{t_i - t}{t_i - t_{i-1}}\right) \ln(r_{i-1}) + \ln(t),$$

differentiation leads to

$$\frac{1}{r(t)t} f(t) = \left(\frac{1}{t_i - t_{i-1}} \ln\left(\frac{r_i}{r_{i-1}}\right) + \frac{1}{t}\right),$$

finally giving the forward rates as

$$f(t) = r(t)\left(\frac{t}{t_i - t_{i-1}}\ln\left(\frac{r_i}{r_{i-1}}\right) + 1\right).$$

As can be observed if $\frac{t}{t_i - t_{i-1}}\ln\left(\frac{r_i}{r_{i-1}}\right) < -1$ the forward rates can become negative leading to arbitrage opportunities.

Raw Interpolation Linear on log of discount factors

The following method is very stable and is usually a starting point for interpolation investigations. It is used to ensure no errors occur in fancier methods by comparing the raw method with the more sophisticated method[9].

The first step in this method is to define the forward discount factor which is given by,

$$Z(t) = \exp(-\int_0^t f(s)ds), \tag{3.8}$$

The discrete forward rate for the interval $[t_{i-1}, t_i]$ is equal to the average of the instantaneous forward rate over any the given intervals,

$$\frac{r_i t_i - r_{i-1}t_{i-1}}{t_i - t_{i-1}} = \frac{1}{t_i - t_{i-1}}\int_{t_{i-1}}^{t_i} f(s)ds,$$

this will leads to

$$r(t)t = r_i t_i + \int_{t_i}^{t_{i-1}} f(s)ds,\ t \in [t_{i-1}, t_i],$$

which is an important interpolation formula.

Given the relationship in equation this formula can be re-written as

$$r(t)t = r_i t_i + (t_i - t_{i-1})\left(\frac{r_i t_i - r_{i-1}t_{i-1}}{t_i - t_{i-1}}\right),$$

by making $t_i - t_{i-1}$ a common denominator the following relationship is obtained

$$r(t)t = \left(\frac{t - t_{i-1}}{t_i - t_{i-1}}\right) r_i t_i + \left(\frac{t_i - t_{i-1}}{t_i - t_{i-1}}\right) r_{i-1}t_{i-1}.$$

This method guarantees that all instantaneous forwards are positive which is key to avoiding arbitrage, because every instantaneous forward is equal to the discrete forward for the *parent* interval. As can be seen $r(t)t$ is a log on the discount factors.

Linear interpolation methods have continuity difficulties associated with them. Thus, they should not be used for anything other than basic yield curve construction.

Polynomial Interpolation

Polynomial interpolation is the process of approximating complicated curves (yield curve being one). It smears out sharp edges to deliver a continuous curve. If we have $n+1$ discrete data points $(t_i, r(t_i))$, to construct an interpolation polynomial to fit the data to a polynomial of degree n through all the points [19]

$$p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_2 t^2 + a_1 t + a_0.$$

this can be simplified as

$$p(t) = \sum_{i=0}^{n} a_i t^i,$$

To solve for multiple points $p(t_i) = r(t_i)$ a system of linear equations can be utilised giving

$$\begin{bmatrix} t_0^n & t_0^{n-1} & t_0^{n-2} & \dots & t_0 & 1 \\ t_1^n & t_1^{n-1} & t_1^{n-2} & \dots & t_1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ t_n^n & t_n^{n-1} & t_n^{n-2} & \dots & t_n & 1 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_0 \end{bmatrix} = \begin{bmatrix} r(t_0) \\ r(t_1) \\ \vdots \\ r(t_n) \end{bmatrix}.$$

Alternatively using Lagrange polynomials the following general equation can be obtained

$$p(t) = \sum_{i=0}^{n} \left(\prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{t - t_j}{t_i - t_j} \right) r(t_i).$$

Cubic Spline Interpolation

A polynomial spline is a function which is piecewise in each interval a polynomial, with the coefficients arranged to ensure at least that the spline coincides with the input data (and so is continuous). It is preferred to polynomial interpolation as it gives similar results without having any instability.

In a cubic spline the aim is to find the coefficients (a_i, b_i, c_i, d_i) for $1 \le i \le n-1$. Given these coefficients, the function value at any term t will be [6]

$$r(t) = a_i + b_i(t - t_i) + c_i(t - t_i)^2 + d_i(t - t_i)^3,$$

It is important to note that the equation is three times differentiable as shown.

$$\begin{aligned} r'(t) &= b_i + 2c_i(t - t_i) + 3d_i(t - t_i)^2, \\ r''(t) &= 2c_i + 6d_i(t - t_i), \\ r'''(t) &= 6d_i. \end{aligned}$$

The common constraints when using splines are:

1. the interpolating function needs to meet the given data points,

2. the entire interpolating function is continuous and

3. the entire function is differentiable, this is important as when the forward function $f(t) = \frac{d}{dt}(r(t))t$ is continuous the following is obtained:

$$f(t) = \frac{d}{dt}(a_i + b_i(t-t_i) + c_i(t-t_i)^2 + d_i(t-t_i)^3)t,$$

differentiation with respect to t yields

$$f(t) = a_i + b_i(2t-t_i) + c_i(t-t_i)(3t-t_i) + d_i(t-t_i)^2(4t-t_i).$$

This method of using splines is very good at producing smooth curves both forward and zero and this shall be explored in the following section.

3.4 Smoothing of curve

The smooth interpolation of interest rates is of keen interest for risk managers. Though smooth interpolation gives an aesthetically appealing curve, there is little published research on numerical benefits.

There exist an intuitive belief that smooth interpolation gives more accurate results [2].

What was proposed by them is the following, they defined the smoothest forward rate curve on an interval $(0,T)$ as one that minimises the function

$$Z = \int_t^x (f''(u))^2 du, \tag{3.9}$$

which is a common mathematical expression used in defining smoothness.

Using this definition it can be shown that a cubic spline fitted to the discount function produces the smoothest possible discount function according to this definition of smoothness.

The maximum smoothness criterion is useless unless it is combined with observable points on the yield curve. If the observable points take the form of m given zero-coupon bond prices Equation (3.9) will be minimised subject to Equation (3.8) acting as a constraint.

In this case let $f(t) = f(t; a_1, a_2, ..., a_n)$ be the forward rate curve as a known function of parameters $a_1, a_2, ..., a_n$.

Introducing the Lagrange multipliers λ_i for $i = 1, ..., m$, the minimisation problem can be written as

$$\min(Z + \sum_{i=1}^{m} \lambda_i x[P(t_i) - P_i]),$$

The solution then becomes

$$\frac{\partial}{\partial a_i}\left[\int_t^x (f''(u))^2 du + \sum_{i=1}^{m} \lambda_i x \left[\exp(-\int_{t_i}^x f(u)du) + P_i\right]\right].$$

To then determine the maximum smoothness term structure within all possible functional forms a theorem from Vasicek [20] is utilised.

Theorem 1. *The term structure $f(t)$, $0 \leq t \leq T$ of forward rates that satisfies the maximum smoothness criterion while fitting observed points is a fourth-order spline with the cubic term absent and given by,*

$$f(t) = c_i t^4 + b_i t + a_i, \text{ for } t_{i-1} < t \leq t_i, \text{ for } i = 1,2,..,m+1,$$

where $0 = t_0 < t_i < ... < t_m < t_{m+1} = T$. The coefficients a_i, b_i, c_i for $i = 1,2,...,m+1$ such that the following equations are satisfied:

$$\begin{aligned} c_i t_i^4 + b_i t_i + a_i &= c_{i+1} t_i^4 + b_{i+1} t_i + a_{i+1}, \text{ for } i = 1,2,...,m, \\ 4c_i t_i^3 + b_i &= 4c_{i+1} t_i^3 + b_{i+1} \text{ for } i = 1,2,...,m, \\ \frac{1}{5} c_i (t_i^5 - t_{i-1}^5) + \frac{1}{2} b_i (t_i^2 - t_{i-1}^2) + a_i (t_i - t_{i-1}) &= -\ln\left(\frac{P_i}{P_{i-1}}\right), \; i = 1,2,...,m, \\ c_{m+1} &= 0. \end{aligned}$$

The proof to this theorem can be found in Appendix C.

3.5 Chapter Summary

In this chapter the discussion was around the process in which forward and zero coupon curves are constructed. A look at the mathematics in which the zero and forward curves a based leading to how this is implemented using interpolation methods. To finish of the chapter a discussion how best to improve accuracy and smoothness in a yield curve detailed.

Chapter 4

Duration

It is important to compute the change in value of fixed income instruments due to changes in interest rates that they are based on. The key to measuring interest-rate risk is the accuracy of the estimate of the value of position after an adverse rate change. In this chapter the discussion will be around Duration and convexity [7] this is the measure that will be compared to Key Rate Duration the model.

The *Duration* of a bond, as its name implies, is a measure of how long on average the holder of the bond has to wait before receiving cash payments. A zero-coupon bond that lasts n years has a duration of n years. However, a coupon-bearing bond lasting n years has a duration of less than n years, because the holder receives some of the cash payments prior to year n [10].

Suppose there exists a bond that provides the holder with cash flows c_i at time t_i where $1 \leq i \leq n$. The bond yield y(continuously compounded) and bond price B are related by

$$B = \sum_{i=1}^{n} c_i \exp(-yt_i), \tag{4.1}$$

The Duration of the bond, D, is defined as

$$D = \sum_{i=1}^{n} t_i \left(\frac{c_i \exp(-yt_i)}{B} \right),$$

When a small change δ in the yield is considered, it is approximately true that

$$\Delta B = \frac{dB}{dy} \Delta y, \tag{4.2}$$

When differentiated it becomes

$$\Delta B = -\Delta y \sum_{i=1}^{n} c_i t_i \exp(-yt_i),$$

As can be observed the relationship between yield and bond price is negative which is analogous to them being inversely proportional to each other. With the use of Equation (4.1) the relationship for duration and bond price is

$$\Delta B = -BD\Delta y,$$

this can then be written as

$$\frac{\Delta B}{B} = -D\Delta y. \tag{4.3}$$

Equation (4.3) is an approximate relationship between percentage changes in a bond price and changes in its yield. It is easy to use and is the reason why duration is a popular measure.

4.1 Modified duration

The preceding analysis is based on the assumption that y is expressed with continuous compounding. If y is expressed with annual compounding, it can be shown that the approximate relationship in Equation (4.3) becomes

$$\Delta B = -\frac{BD\Delta y}{1+y},$$

More generally, if y is expressed with a compounding frequency of m times per year, then

$$\Delta D^* = -\frac{D}{1+\frac{y}{m}},$$

sometimes referred to as the bond's modified duration. It allows the duration relationship to be simplified to

$$\Delta B = -BD^*\Delta y.$$

4.2 Convexity

The duration relationship applies only to small changes in yields. This is illustrated in Figure 4.1 ([7]), which shows the relationship between the percentage change in value and change in yield for two bond portfolios having the same duration.

In Figure 4.2 ([7]) gradients of the two curves are the same at the tangent line. This means that both bond portfolios change in value by the same percentage for small yield changes and is consistent with Equation (4.2). For large yield changes, the portfolios behave differently. Bond A has more curvature in its relationship with yields than Bond B. A factor known as convexity measures this curvature and can be used to improve the relationship.

A measure of convexity is the second derivative of a bond with respect to the yield as shown

$$C = \frac{1}{B}\frac{d^2B}{dy^2} = \sum_{i=1}^{n}\left(\frac{c_i t_i^2 \exp(-yt_i)}{B}\right),$$

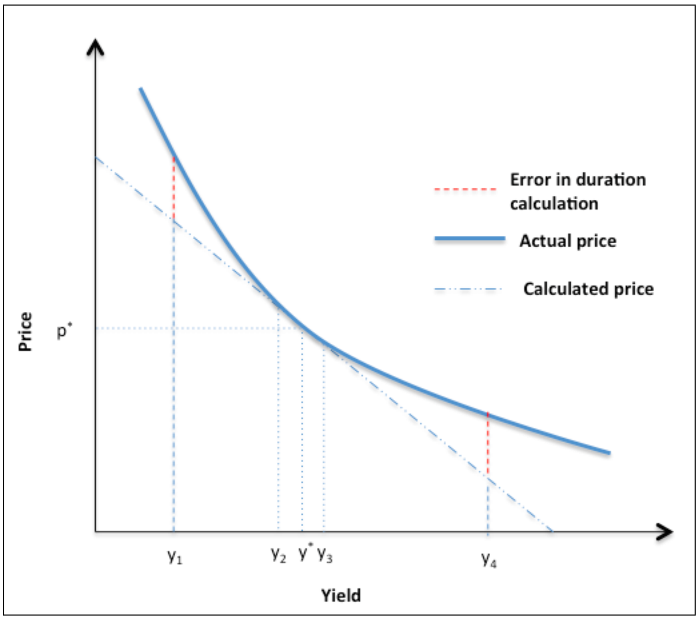

Figure 4.1: Duration error when shift is large

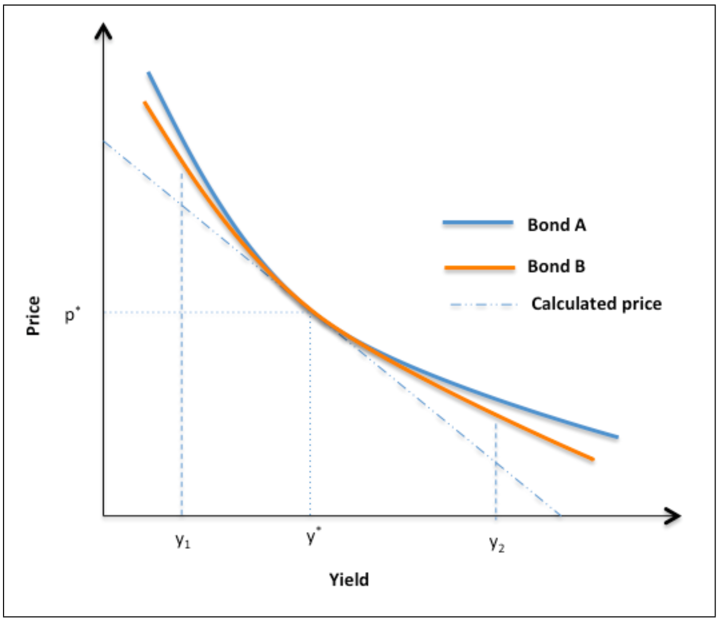

Figure 4.2: Comparison of two bonds and duration calculation

To then compute a change in bond price the following relationship exists,

$$\Delta B = \frac{dB}{dy}\Delta y + \frac{1}{2}\frac{d^2B}{dy^2}\Delta y^2,$$

this then gives us the final argument where the answer is similar to Duration but adjusted for the curvature of the bond yield relationship

$$\frac{\Delta B}{B} = -D\Delta y + \frac{1}{2}C(\Delta y)^2.$$

Despite it offering an improved answer when compared to duration, convexity is still not a good estimator for non-parallel shifts.

4.3 Use of Duration to hedge interest rate exposure

One of the uses of Duration is as a way to hedge against interest rate exposure. To observe this benefit, consider at time 0 a default-free zero coupon bond with a payout at time T. If interest rate is taken to be constant and continuously compounding, the result is the following

$$B = \exp(-rT),$$

where,

- B is the price of the bond,
- r is the constant interest rate per unit time and
- T is the time in the future when the bond matures.

For a coupon paying bond

$$B = \sum_{i=1}^{n} CF_i \exp(-rt_i),$$

where

- CF_i is the ith cash flow at time t_i and
- n is the total number of cash flows.

This leads to defining *Duration* which is the relationship

$$\frac{\frac{\Delta B}{\Delta y}}{B} = -D,$$

With this relationship attained, the assumption of a constant interest rate r is then replaced with a forward interest rate curve and denoted by $f(t)$. This replacement gives

$$B = \exp\left[-\int_0^T f(t)dt\right],$$

For a coupon paying bond

$$B = \sum_{i=1}^{n} CF_i \exp\left[-\int_0^T f(t)dt\right],$$

In this context *Duration* is given by

$$D = \frac{\sum_{i=1}^{n} T_i CF_i \exp\left[-\int_0^{T_i} f(t)dt\right]}{B}.$$

If the price of the bond is differentiated with respect to a parallel shift in the forward curve, the result as r approaches 0 is

$$\frac{\Delta B}{\Delta y} = \sum_{i=1}^{n} T_i CF_i \exp\left[-\int_0^{T_i} f(t)dt\right] = B,$$

Substituting

$$\frac{\frac{\Delta B}{\Delta y}}{B} = -D.$$

This equation shows the benefit and downfall of Duration as a measure of interest rate exposure. This shows that for any forward interest rate curve as long as the shift is small and parallel Duration can be utilised.

4.4 Duration shortcomings

Duration though used widely as being a proxy of interest rate sensitivity and the usefulness in immunising bond portfolios has serious shortcomings.

4.4.1 Using Yield to Maturity for Discounting

The first key shortcoming is that for a set of certain payments, $CF(t)$, and the present value function at time t, $P(t)$, the definition of duration is

$$D = \frac{\sum_{i=1}^{n} t_i CF(t_i)P(t_i)}{\sum_{i=1}^{n} CF(t_i)P(t_i)},$$

this measure is then generally changed into

$$D' = \frac{\sum tCF(t)\exp(-yt)}{\sum CF(t)\exp(-yt)}. \tag{4.4}$$

or a discrete period discounted equivalent.

These two definitions are equivalent if and only if the discount rate implicit in the present value function is the constant y [6]. Theorem 2 [14] was developed to show that when the two definitions differ only the first definition can be used.

Theorem 2. *For infinitesimal shifts in the yield curve, the percentage change in value of any asset with fixed payments is proportional to Macaulay's duration measure as defined in Equation* (4.4) *if and only if the entire yield curve undergoes a uniform additive displacement,* $dy(t) = dy$ *for all* t.

Proof. An assets value is $V = \sum CF(t)\exp(-y(t)t)$, thus

$$\frac{dV}{V} = \frac{-\sum tCF(t)\exp(-y(t)t)dy(t)}{V} = -Ddy,$$

if $dy(t) = dy$ and proves sufficiency. To prove necessity, its important to recall that the duration of a pure discount bond is its maturity. For two pure discount bonds of maturities t_1 and t_2,

$$\frac{\left(\frac{dP(t_1)}{P(t_1)}\right)}{\left(\frac{dP(t_2)}{P(t_2)}\right)} = \frac{-t_1 dr(t_1)}{-t_2 dr(t_2)}.$$

which equals $\left(\frac{t_1}{t_2}\right)$ only if $dr(t_1) = dr(t_2)$. □

4.4.2 Non-infinitesimal shift

Duration also noted as having key errors as shown in Figure 4.2 this is also illustrated [14] in Theorem 3.

Theorem 3. *A portfolio of positive payments with duration* T *and current value* V *equal to the present value of* \$1 *in T years, will, after a non-infinitesimal shift of size* δ *in the entire yield curve* $r(t)$, *be worth more than the new present value of* \$1 *in T years.*

Proof. To prove this theorem first consider only separate portfolios of two separate payments at times t_1 and t_2 with $t_1 < T < t_2$. The general case follows immediately by induction. Consider two pure discount unit bonds of maturities t_1 and t_2. The prices of these bonds before and after the interest rate shift are denoted by $V_i = \exp(-r(t_i)t_i)$ and $V_i' = \exp(-r'(t_i)t_i)$ where $r'(t)t = r(t) + \delta$. Consider a portfolio holding $n_1, n_2 > 0$ of these bonds with current value V and duration D

$$\begin{aligned} n_1V_1 + n_2V_2 = V \equiv \exp(-r(T)T), \\ \frac{n_1V_1}{V}t_1 + \frac{n_2V_2}{V}t_2 = T, \end{aligned} \tag{4.5}$$

or

$$n_iV_i = \frac{V(t_j - T)}{(t_j - t_i)},$$

and since

$$V_i' = V_i \exp(\delta t_i),$$

the post shift value of the portfolio is

$$V' = n_1 V_1' + n_2 V_2' = n_1 \exp(-\delta t_1) + n_2 \exp(-\delta t_2),$$
$$= \frac{V}{t_2 - t_1} \left[(t_2 - T) \exp(-\delta t_1) + (T - t_1) \exp(-\delta t_2) \right],$$

after substituting for $n_i V_i$. The post shift ratio of the value of this portfolio to that of the pure discount bond of maturity T is

$$Q = \frac{V'}{\exp(-r'(T)T)} = \frac{V' \exp(\delta T)}{V},$$

Evaluating this expression using Equation (4.5) gives

$$Q = \frac{[(t_2 - T) \exp(-\delta t_1) + (T - t_1) \exp(-\delta t_2)]}{t_2 - t_1},$$

with

$$\begin{aligned} \frac{dQ}{d\delta} &= \frac{(t_2 - t)(T - t_1)}{t_2 - t_1} \left[\exp(\delta(T - t_1)) - \exp(\delta(T - t_2)) \right] |^{>0}_{0<} \text{ as } \delta|^{>0}_{0<}, \\ \frac{d^2Q}{d\delta^2} &= \frac{(t_2 - t)(T - t_1)}{t_2 - t_1} \left[(T - t_1) \exp(\delta(T - t_1)) + (t_2 - T) \exp(\delta(T - t_2)) \right] > 0. \end{aligned} \tag{4.6}$$

□

From Equation (4.6) its observed that this ratio reaches a unique minimum at a zero shift where it has the value one. For any non zero shift, the portfolio value will be greater than that of the pure discount bond.

4.4.3 Flat yield curve and parallel shifts

The weakness of Duration when dealing with non-parallel shifts and requiring a flat yield curve were brought to light by Theorem 4 [14].

Theorem 4. *Yields to maturity on all assets with known fixed payments can change by same amount if and only if the yield curve flat (yields to maturity on pure discount bonds of all maturities are the same) and makes parallel shifts[6].*

Proof. Let $y(t)$ denote the yield to maturity on a pure discount bond of maturity t. Assuming that all yields do make an identical change on a sloped yield curve. Consider two maturities, t_1 and t_2 with differing yields. Without loss of generality take $y(t_1) < y(t_2)$. Now consider an asset with payments at times t_1 and t_2 and no other payments, its yield to

$$V = \sum_{i=1}^{2} CF(t_i) \exp(-y(t_i) t_i)) = \sum_{i=1}^{2} CF(t_i) \exp(-y t_i)), \tag{4.7}$$

satisfies $y(t_1) < y < y(t_2)$ since the exponential function is monotonic. If an infinitesimal shift of the assumed type occurs, then $dy = dy(t_1) = dy(t_2)$. This interest rate shift causes a value change of

$$dV = -\sum_{i=1}^{2} CF(t_i) t_i \exp(-r(t_i)t_i) dr(t_i) = -\sum_{i=1}^{2} CF(t_i) \exp(-yt_i)) dy. \tag{4.8}$$

Now multiplying V in Equation (4.7) and adding $\frac{dV}{dy}$ from Equation (4.8)

$$t_1 V - \frac{dV}{dy} = (t_1 - t_2) CF(t_2) \exp(-y(t_2)t_2) = (t_1 - t_2) CF(t_2) \exp(-yt_2).$$

This however can only be true if $y(t_2) = y$ resulting in a contradiction. Thus yields to maturity cannot make identifiable shifts on all of an arbitrary set of assets if the yield curve is not flat.

□

It is this case that this study takes primary interest in as the deficiencies relating to Theorems 2 and 3 are dealt with to some extent with the measure convexity.

4.5 Chapter Summary

Duration is a widely used measure for risk which has good properties in terms of ease of calculation and use for large portfolios. It can also be a good tool for hedging risk. It's main downside is errors that occur due to parallel shifts and non-infinitesimal shifts. In the next chapter an alternative measure that deals with it's shortcoming relating to non-parallel shifts is discussed.

Chapter 5

Key Rate Duration

Though Duration is a good measure of the interest rate risk exposure of a bond or a portfolio, it's main assumption is that the spot yield curve shift is parallel. This leads to nonparallel shifts, factors such as steepness or curvature can lead to estimation errors [11].

Key Rate Duration introduced by [11] is a vector representing the price sensitivity of a fixed income instrument to each key rate change. The key rates normally relate to the main quoted instruments examples being O/N, 1 month and 4 year. The sum of the key rate durations is identical to the duration.

$$D = \sum_{i=1}^{m} KRD_i.$$

where m is the longest maturity normally 30 years.

The principal of Key Rate Duration is that a chosen key rate at t_i is shifted by $\Delta y(t_i)$ but the key rate after t_{i+1} and before t_{i-1} do not shift. Figure 5.1 ([11]) shows how it would look and in some instances has been termed *triangulation*.

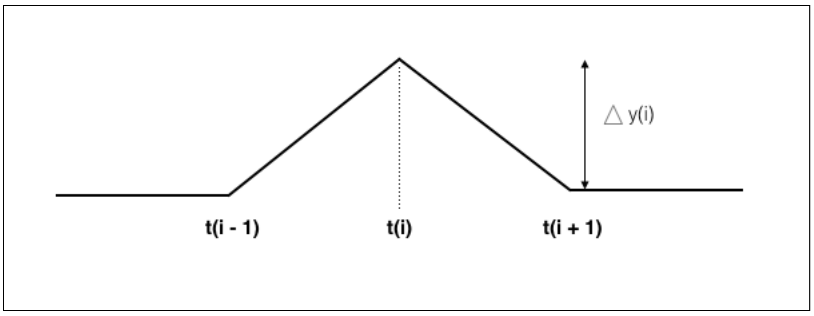

Figure 5.1: Key Rate Duration Shift illustration

The values of the rates $t_{i-1} < t < t_i$ and $t_i < t < t_{i+1}$ affected can then be computed using interpolation methods giving a more accurate view on the impact of a shift on that key rate. To compute KRD we first define $s(t,t_i)$ as the ith basic key rate shift of term t, with the level of shift being $\Delta y(t)$, these rate shifts are $1 \leq i \leq m$. As stated earlier the summation of the shifts gives a value similar to duration and can be shown in Figure 5.2 [11].

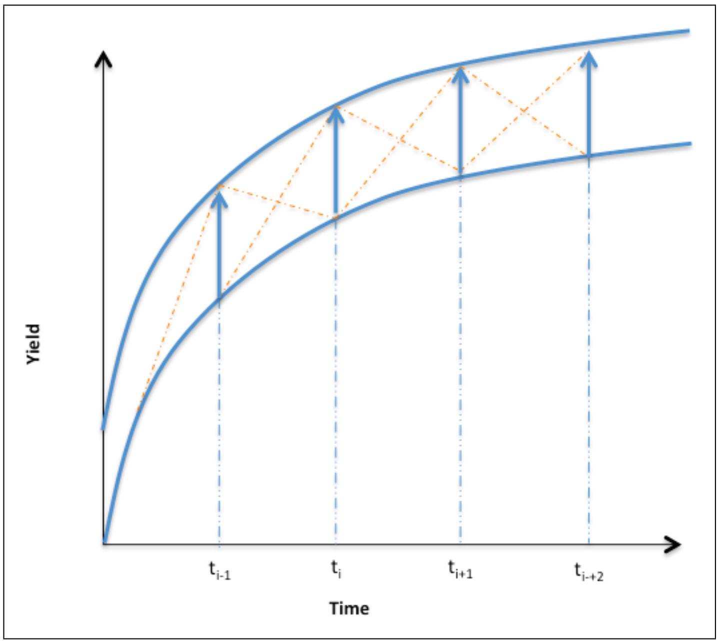

Figure 5.2: Key Rate Duration and Duration Shift relationship

The computation is split into three scenarios

1. $i = 1$ the first key rate normally overnight rate

$$s(t,t_1) = \begin{cases} \Delta y(t_1) & \text{if } t < t_1, \\ \Delta y(t_i)\frac{t_2 - t}{t_2 - t_1} & \text{if } t_1 \leq t \leq t_2, \\ 0 & \text{if } t > t_2. \end{cases}$$

2. $1 < i < m$ these are all the key rates between the first and last

$$s(t,t_i) = \begin{cases} \Delta y(t_i) & \text{if } t < t_{i-1}, \\ \Delta y(t_i)\frac{t_2 - t_{i-1}}{t_i - t_{i-1}} & \text{if } t_{i-1} \leq t \leq t_i, \\ \Delta y(t_i)\frac{t_{i+1} - t}{t_{i+1} - t_i} & \text{if } t_i \leq t \leq t_{i+1}, \\ 0 & \text{if } t > t_{i+1}. \end{cases}$$

3. $i = m$ the last key rate normally 30 year rate

$$s(t,t_m) = \begin{cases} 0 & \text{if } t < t_{m-1}, \\ \Delta y(t_m)\frac{t - t_{m-1}}{t_m - t_{m-1}} & \text{if } t_{m-1} \leq t \leq t_m, \\ \Delta y(t_i) & \text{if } t > t_m. \end{cases}$$

If all these individual non-parallel shifts are done at the key rates the end result is

$$\Delta y(t) = s(t,t_1) + s(t,t_i) + ... + s(t,t_m).$$

Now if when the relationship in Equation (4.3) is used the following is obtained

$$\frac{\Delta B_i}{B} = -KRD_i \Delta y(t_i).$$

The final result obtained is the following

$$\frac{\Delta B}{B} = \sum_{i=1}^{m} KRD_i \Delta y(t_i).$$

5.1 Key Rate Convexities

When a shift occurs to the term structure that is non-infinitesimal, the Key Rate Duration framework extends to deal with the second-order nonlinear effects of the key rate shifts [18].

These are given as key rate convexities and defined as

$$KRC_{i,j} = KRC_{j,i} = \frac{1}{P} \frac{\partial^2 P}{\partial y(t_i) \partial y(t_j)},$$

for every pair of key rates. The set of key rate convexities can be represented by a symmetric matrix of dimension m

$$KRC = \begin{bmatrix} KRC_{1,1} & KRC_{1,2} & \dots & KRC_{1,m} \\ KRC_{2,1} & KRC_{2,2} & \dots & KRC_{2,m} \\ \vdots & \vdots & & \vdots \\ KRC_{m,1} & KRC_{m,2} & \dots & KRC_{m,m} \end{bmatrix}.$$

The relationship between convexity and key rate convexities is given by

$$CON = \sum_{i=1}^{m} \sum_{j=1}^{m} KRC_{i,j}.$$

5.2 Key Rate Durations and Value at Risk Analysis

Value at risk is defined as the maximum loss in the portfolio value at a given level of confidence over a given horizon. Given a multivariate normal distribution for the key rate changes, the portfolio return is distributed normally under a linear approximation, with a mean equal to [18]

$$\mu_R = \sum_{i=1}^{M} KRD_i \times \mu_{\Delta y(i)},$$

and variance equal to

$$\sigma_R^2 = \sum_{i=1}^{M} \sum_{j=1}^{M} KRD_i \times KRD_j \times cov[\Delta y(i), \Delta y(j)], \tag{5.1}$$

where $\mu_{\Delta y(i)}$ is the mean change in the ith key rate and $cov[\Delta y(i), \Delta y(j)]$ is the covariance between changes in the ith and the jth key rates. The Value at Risk of the portfolio at c percent confidence level is given as

$$VaR_c = -V_0(\mu_R - z_c\sigma_R),$$

where z_c is c percentile of a standard normal distribution. If the holding period of the VaR is very small, the expected return and express VaR simply as

$$VaR_c = V_0 z_c \sigma_R, \tag{5.2}$$

substituting Equation (5.1) into Equation (5.2)

$$VaR_c = V_0 z_c \sqrt{\sum_{i=1}^{M}\sum_{j=1}^{M} KRD_i \times KRD_j \times cov[\Delta y(i), \Delta y(j)]}. \tag{5.3}$$

The VaR solution given in Equation 5.3 doesn't not apply when key rate changes are not normally distributed.

5.3 Limitations of KRD

5.3.1 Choice of Key Rates

The Key Rate Duration model does not give guidance into which rates are viewed as *Key Rates* this then leads to varying numbers of key rates. In his article Ho[11] recommends using as many as 11 key rates, also risk managers can also base it on the maturity structure of the portfolio. In Sweden and in this study the key rates to be observed are:
$\{1d, 2d, 3d, 1W, 1M, 2M, 3M, 6M, 9M, 1Y, 2Y, 3Y, 4Y, 5Y, 7Y, 10Y, 12Y, 15Y, 20Y, 25Y, 30Y\}$.

5.3.2 Shape of Key rates

The shape of Key shifted curved is a historically implausible shape of Figure 5.3 shows an example of effect on forward rates that the key rate shift has.

In order to address this shortcoming, a natural choice is to focus on the forward rate curve instead of the zero-coupon curve. This method is called the Partial Derivative Approach. This approach uses the relationship

$$Z(t) = \frac{1}{t}\exp(-\int_0^t f(s)ds),$$

As in the section dealing with interpolation it leads to

$$Z(t) = \frac{1}{t}\sum_{i=1}^{t} f(i-1, i).$$

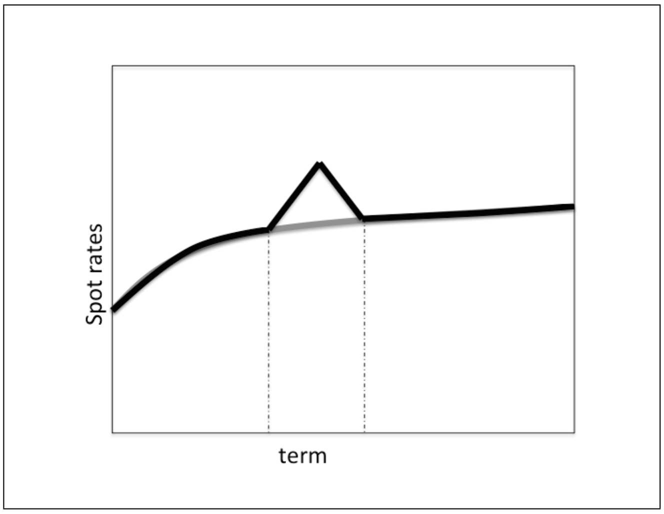

Figure 5.3: Key Rate Duration Shift on spot curve

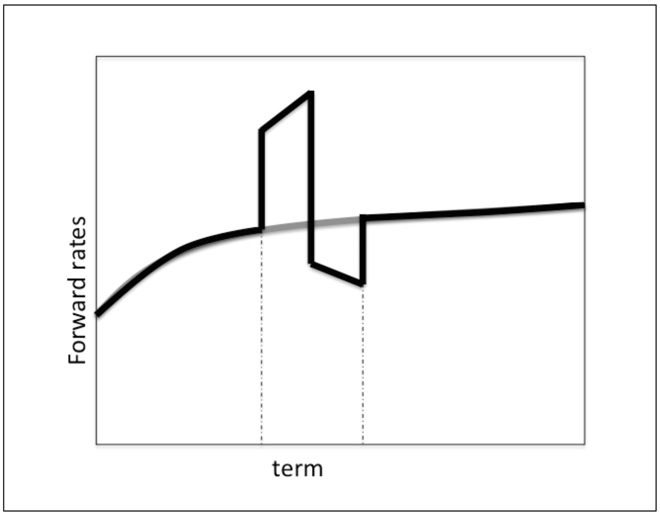

Figure 5.4: Key Rate Duration Shift effect on forward curve

This leads to forward rates being the simple averages of the corresponding forward rates and imply that the present value of a cash flow due at time t is

$$PV = \left(\frac{CF_t}{\exp(\sum_{i=1}^{t} f(i-1,i))} \right).$$

This equation therefore states that the market price is affected by all forward rates preceding the maturity date. The partial duration is then defined as

$$PD_i = \left(-\frac{1}{P} \frac{\partial P}{\partial f(i-1,i)} \right),$$

with

$$Duration = \sum PD_i = \sum KRD_i.$$

Though giving similar outputs the profiles of the key rate duration and partial duration are noticeable different.

5.3.3 Loss of efficiency

Since each key rate change is assumed to be independent of the changes in the rest of key rates, the model deals with movements in the term structure whose probabilities may be too small to worry about.

Yet historical volatilities of interest rates provide useful information about the behaviour of the different segments of the term structure, and the key model disregards this information.

The use of the key rate model for interest rate risk management imposes too severe restrictions on portfolio construction that leads to increased costs and a loss of degrees of freedom [18].

5.4 Chapter Summary

Key rate duration offers a good alternative to duration especially when relating to non-parallel shifts in the term structure. Key rate convexity provides additional accuracy when dealing with non-infinitesimal shifts.

Chapter 6

Implementation

The previous chapters have given a strong theoretical background on fixed income instruments, curve construction and risk calculation. In this chapter these aspects will be conjoined to illustrate the relation between Duration and Key Rate Duration. Using Mathworks Matlab programming software an to demonstrate using actual data this relationship.

6.1 Curve construction

The first aspect will be the construction of the discount rate, zero rate and forward rate curves. The data utilised is obtained from *Swedbank* and based on the following dates:

- first data set 07 July 2015 and
- second data set 04 April 2016.

The interpolation method to be utilised is linear interpolation to be used as starting point. To improve the smoothness as well as the accuracy as discussed in Chapter 3 cubic spline will be utilised. The method utilised to observe the accuracy of the curves is to compare the original quoted rates to the recalculated rates

$$y(t) - y' = \Delta error.$$

where $y(t)$ is the original quote and $y'(t)$ is the recalculated one.

For this bootstrap to be done certain key parameters need to stated

- daycount conversion is $\frac{actual}{365}$ for zero rates,
- cashflow of the swaps is 1 per year and
- if payment dates fall on weekends its moved to the Monday.

All data and calculations are based in Swedish Kronors (SEK).

6.1.1 Data set 1

Table 6.1 represents the first data set to be utilised.

Start Date	Instrument	Maturity	Period	Quote
15/07/07	Deposit	15/07/08	O/N	-0.279
15/07/08	Deposit	15/07/09	T/N	-0.279
15/07/09	Deposit	15/07/16	1W	-0.350
15/07/09	Deposit	15/08/10	1M	-0.340
15/07/09	Deposit	15/09/09	2M	-0.318
15/07/09	Deposit	15/10/09	3M	-0.278
15/09/16	FRA	15/12/16	Sep-15	-0.319667
15/12/16	FRA	16/03/16	Dec-15	-0.323
16/03/16	FRA	16/06/15	Mar-16	-0.284667
16/06/15	FRA	16/09/21	Jun-16	-0.219
16/09/21	FRA	16/12/21	Sep-16	-0.114667
16/12/21	FRA	17/03/15	Dec-16	0.006667
17/03/15	FRA	17/06/21	Mar-17	0.145
17/06/21	FRA	17/09/20	Jun-17	0.286667
17/09/20	FRA	17/12/20	Sep-17	0.4325
17/12/20	FRA	18/03/21	Dec-17	0.580667
18/03/21	FRA	18/06/20	Mar-18	0.727667
18/06/20	FRA	18/09/19	Jun-18	0.881
15/07/09	Swap	19/07/09	4Y	0.3425
15/07/09	Swap	20/07/09	5Y	0.615
15/07/09	Swap	21/07/09	6Y	0.8625
15/07/09	Swap	22/07/09	7Y	1.0775
15/07/09	Swap	23/07/09	8Y	1.2575
15/07/09	Swap	24/07/09	9Y	1.4075
15/07/09	Swap	25/07/09	10Y	1.53
15/07/09	Swap	27/07/09	12Y	1.72
15/07/09	Swap	30/07/09	15Y	1.9175
15/07/09	Swap	35/07/09	20Y	2.100
15/07/09	Swap	40/07/09	25Y	2.180
15/07/09	Swap	45/07/10	30Y	2.210

Table 6.1: Table showing interest data as on 09 July 2015

The forward, discount and zero rate curves that are obtained from this data are the following.

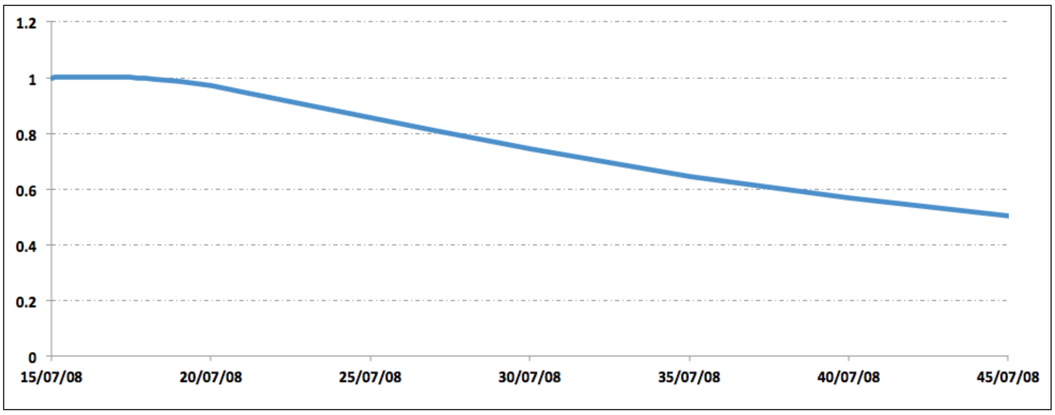

Figure 6.1: Discount rate curve

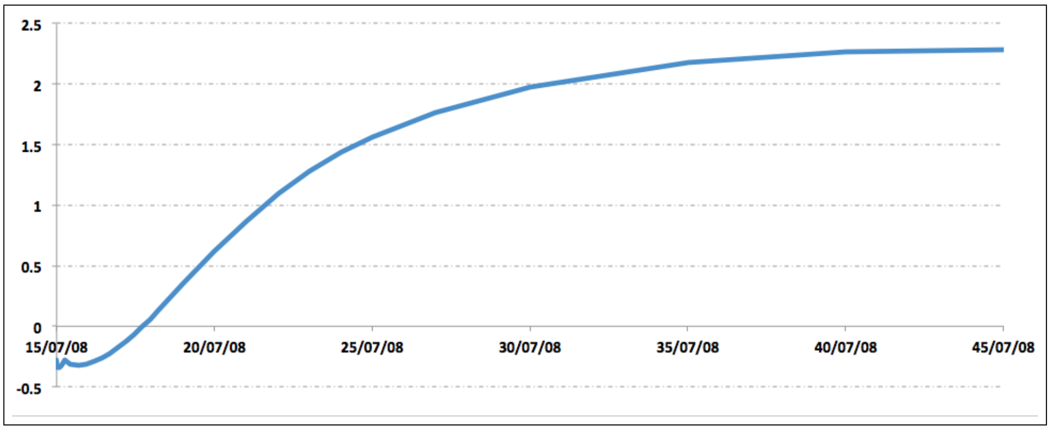

Figure 6.2: Zero rate curve

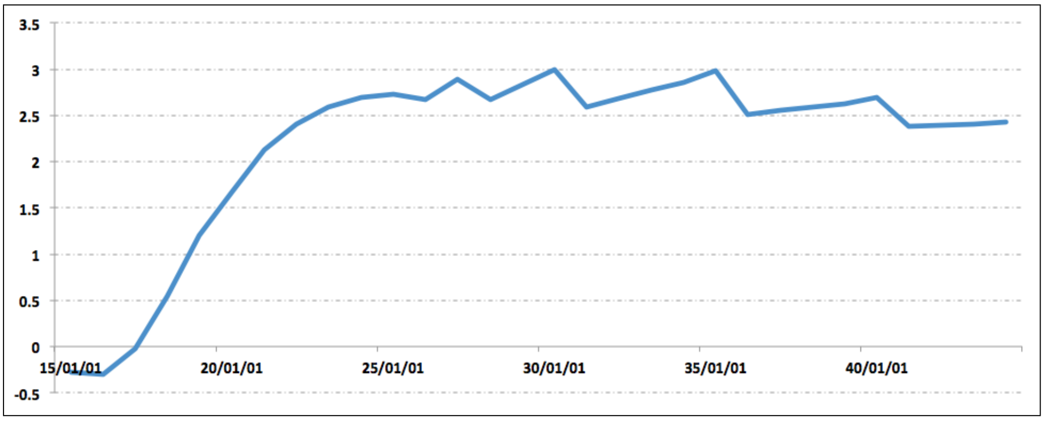

Figure 6.3: Forward rate curve

6.1.2 Data set 2

Table 6.2 represents the second data set to be utilised.

Start Date	Instrument	Maturity	Period	Quote
16/04/04	Deposit	16/04/05	O/N	-0.5400
16/04/05	Deposit	16/04/06	T/N	-0.5390
16/04/06	Deposit	16/04/13	1W	-0.5390
16/04/06	Deposit	16/05/06	1M	-0.5250
16/04/06	Deposit	16/06/07	2M	-0.4890
16/04/06	Deposit	16/07/06	3M	-0.4460
16/06/15	FRA	16/09/21	Jun-16	-0.4750
16/09/21	FRA	16/12/21	Sep-16	-0.4750
16/12/21	FRA	17/03/15	Dec-16	-0.4500
17/03/15	FRA	17/06/21	Mar-17	-0.4000
17/06/21	FRA	17/09/20	Jun-17	-0.3400
17/09/20	FRA	17/12/20	Sep-17	-0.2600
17/12/20	FRA	18/03/21	Dec-17	-0.1600
18/03/21	FRA	18/06/20	Mar-18	-0.0450
18/06/20	FRA	18/09/19	Jun-18	0.0730
16/04/06	Swap	18/03/21	3Y	-0.1800
16/04/06	Swap	18/06/20	4Y	0.0250
16/04/06	Swap	18/09/19	5Y	0.2250
16/04/06	Swap	15/07/28	6Y	0.4530
16/04/06	Swap	16/07/28	7Y	0.6570
16/04/06	Swap	17/07/28	8Y	0.8130
16/04/06	Swap	18/07/28	9Y	0.9580
16/04/06	Swap	19/07/28	10Y	1.0970
16/04/06	Swap	20/07/28	12Y	1.3000
16/04/06	Swap	21/07/28	15Y	1.5400
16/04/06	Swap	23/07/28	20Y	1.7650
16/04/06	Swap	26/07/28	30Y	1.9150

Table 6.2: Table showing interest data as on 06 April 2016

The forward, discount and zero rate curves that are obtained from this data are the following.

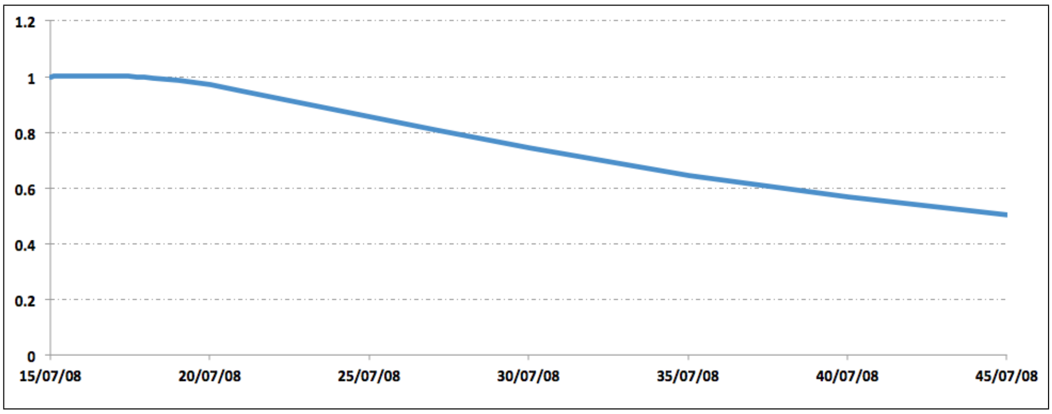

Figure 6.4: Discount rate curve

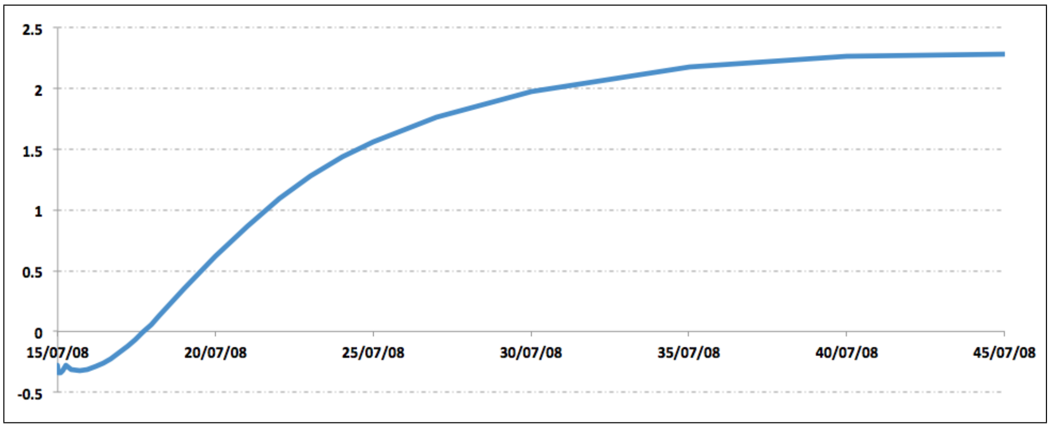

Figure 6.5: Zero rate curve

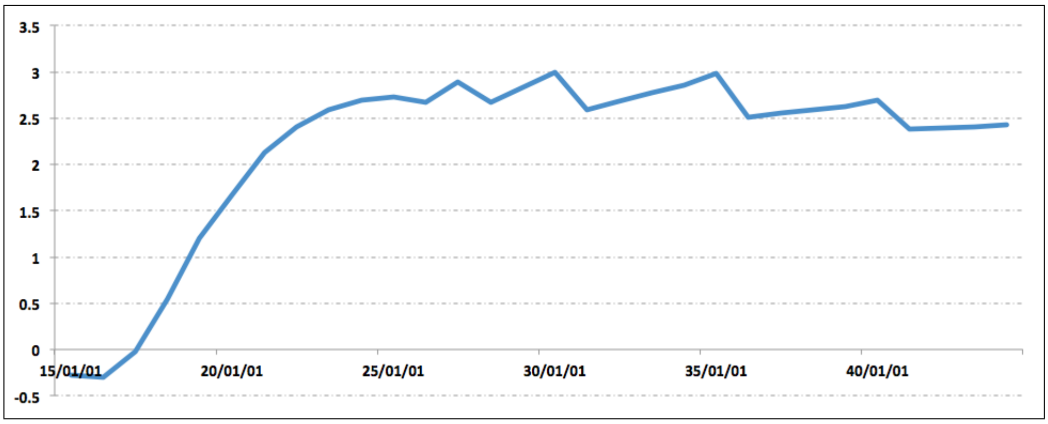

Figure 6.6: Forward rate curve

6.2 Calculation of risk

For the calculation of risk to show the relationship between Duration and Key Rate Duration, Forward Rate Agreements and Swaps are the instruments to be analysed. For both data sets the data and instruments the risk parameters are as follows

1. notional of SEK $1,500,000,000$ and
2. delta of 1 basis point or 0.01%.

6.2.1 FRA calculation

To determine the FRA risk per key rate the required inputs are the discount rate, the forward rate as determined by the formula

$$f_{t_1,t_2} = \frac{-365 \times \ln\left(\frac{D_{t_2}}{D_{t_1}}\right)}{d_{t_2} - d_{t_1}},$$

and with the given notional amount the floating rate note cost as obtained by

$$\text{FRN cost} = f_{t_1,t_2} \times \left(\frac{d_{t_2} - d_{t_1}}{365}\right) \times \text{notional} \times D_{t_1},$$

This cost calculation is first done on the unshifted curve to determine the current cost of the FRA when the contract is enter into. From there each key rate is shifted by a delta of 1 basis point. If the shift on the key rate affects the instrument changes will occur in the valuation and that changed is the risk and is stated as

$$risk_i = \Delta FRN\ cost_i,$$

This is done for all the key rates and the delta noted, leading to a final risk that is stated as

$$\text{RISK} = \sum_{i=1}^{m} risk_i,$$

To determine the risk using the parallel shift method, the first step is to obtain the present value of the quoted instrument PV and then shift by a basis point and bootstrap the curve. This gives a new value for the present value of the instrument PV'. The risk is then

$$\text{RISK} = PV - PV'.$$

and therefore $PV - PV' = \sum_{i=1}^{m} risk_i$. From data set 1 Table 6.3 shows the results for FRAs are

The following illustration of what data in the Table 6.3 states using the first line of data

Considering that the notional being used is SEK $1,500,000,000$ and error is at times as low as SEK 5.09 proves the relation $Duration = \sum KeyRateDurations$. What is also important to note is the FRA relating to 16/03/16 shows a negative risk of -47,399kr which would not have been accounted for using the Trader's calculation.

FRA	Trader	Nodes							Delta
		2M	3M	6M	9M	1Y	2Y	3Y	
16/09/15	37,479	-19,449	5,481	51,409	0	0	0	0	36.32
16/12/15	37,510	0	-15,205	-27,379	80,088	0	0	0	5.09
16/03/16	37,533	0	0	-24,029	-47,399	108,984	0	0	-24.14
15/06/16	40,439	0	0	0	-32,703	34,472	38,431	0	-12.71
21/09/16	37,550	0	0	0	0	-25,159	62,714	0	-5.89
21/12/16	34,650	0	0	0	0	-39,859	74,501	0	8.16
21/03/17	40,394	0	0	0	0	-66,670	107,031	0	33.29
21/06/17	37,470	0	0	0	0	-12,125	-19,779	69,200	173.25
20/09/17	37,415	0	0	0	0	0	-62,746	99,864	296.18
20/12/17	37,346	0	0	0	0	0	-81,334	118,251	429.45
21/03/18	4,256	0	0	0	0	0	-9,978	13,646	588.19

Table 6.3: Table showing risk buckets per node for data set 1

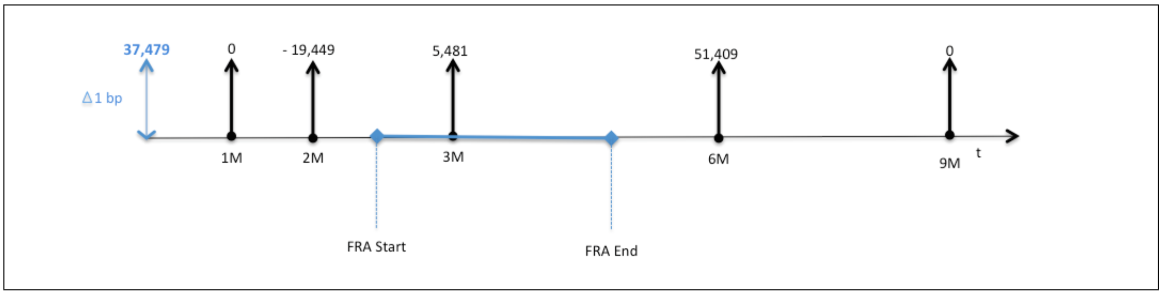

Figure 6.7: FRA risk calculation using data set 1

From data set 2 Table 6.4 shows the results for FRAs and Figure 6.8 shows an illustration of the second data.

		Nodes							
FRA	Trader	2M	3M	6M	9M	1Y	2Y	3Y	Delta
16/09/15	40,414	-18,734	964	60,048	0	0	0	0	64.72
16/12/15	37,569	0	-9,882	-44,692	92,140	0	0	0	3.38
16/03/16	34,710	0	0	-1,535	6,088	11,100	0	0	-43.99
15/06/16	40,541	0	0	-31,267	32,783	39,079	0	0	-53.14
21/09/16	37,669	0	0	0	0	-25,159	63,0761	0	-75.44
21/12/16	37,686	0	0	0	0	-44,148	81,922	0	-88.16
21/03/17	37,692	0	0	0	0	-62,991	100,779	0	-94.85
21/06/17	37,685	0	0	0	0	-11,371	-21,444	70,485	14.70
20/09/17	37,667	0	0	0	0	0	-63,044	100,610	101.38

Table 6.4: Table showing risk buckets per node for data set 2

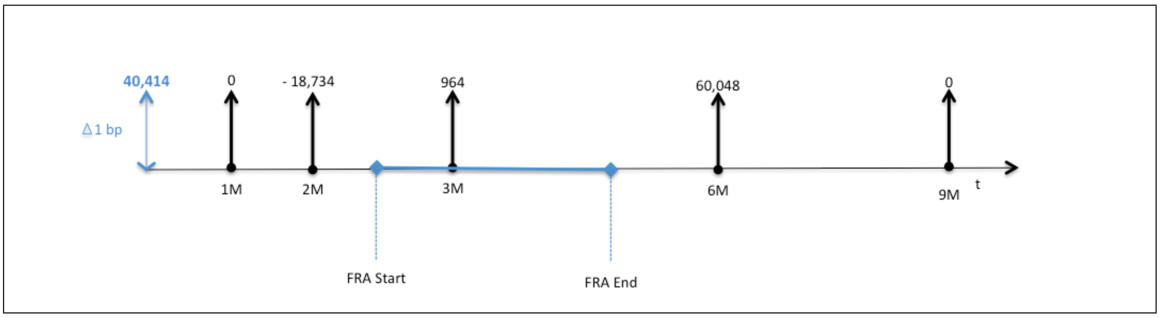

Figure 6.8: FRA risk calculation using data set 2

6.2.2 Swaps Calculation

For the swaps the initial step is to illustrate the swap calculation utilising a 15Y swap from data set 1 and calculating the new swap price.

15Y Swap example

Initial swap rate 1.9175% and at contract start the $Swap_{floating} = Swap_{fixed}$. The new value will depict the fact that $Swap_{floating} \neq Swap_{fixed}$. Table 6.5 shows the results

As can be seen the swap now has a price as compared to the initial case where the price was $SEK0$.

For the pricing of the swap under parallel conditions the procedure applied is similar to that of a Forward Rate Agreement. The comparison is as such between PV and PV'. For the key rate approach the values depend on a number of cash flows as the shifts will affect the cash

No. of cashflows	New Swap Rate	Delta
1	1.4064	0.5111
2	1.4551	0.4624
4	1.4500	0.4675

Table 6.5: Repriced swap price

flows. This effect will also be analysed in the results to be obtained. As the shifts take place on the zero curve it will have an effect on the discount factors and cause a change in swap rate as calculated here

$$\text{Swap rate} = \frac{D_{T/N} - D_{t_1}}{\sum_{i=1}^{n} \delta_i D_{t_i}},$$

the new swap rate is compared to previous one and a delta obtained

$$risk_i = \Delta \text{Swap rate}_i \times notional.$$

and this then becomes $PV - PV' = \Delta \text{Swap rate}_i \times notional.$

The two swaps utilised for the calculation were the 4Y and 5Y for the first data set. There was also a change in the number of cash flows to observe changes if any regarding certain risk nodes.

From data set 1 the results for Swaps are shown in Table 6.6.

		Nodes									
Swap	Cashflow	Trader	6M	9M	1Y	2Y	3Y	4Y	5Y	7Y	Delta
4Y	1	145,849	0	0	130	261	389	146,024	804	0	-1,761
4Y	2	145,849	32	0	114	260	389	145,719	802	0	-1,468
4Y	4	145,849	16	24	101	260	388	145,578	802	0	-1,329
5Y	1	140,227	0	0	180	360	538	709	142,709	390	-4663
5Y	2	140,227	45	0	157	359	535	706	142,217	390	-4184
5Y	4	140,227	22	34	140	359	535	704	141,986	389	-3954

Table 6.6: Table showing risk buckets per node for data set 1

The node that seems to have most effect one closest to the maturity of the swap. This is due to the significant change that happens to D_T. This is all illustrated in Figure 6.9.

The two swaps utilised for the calculation were the 3Y and 4Y for the second data set. From data set 1 the results for Swaps are shown in Table 6.7.

In Figure 6.10 an illustration for the second data set is shown.

The delta values shown have significantly lower values than the ones in the previous data set. Similar to the previous data set the biggest risk bucket is at the node closest to the maturity of the instrument.

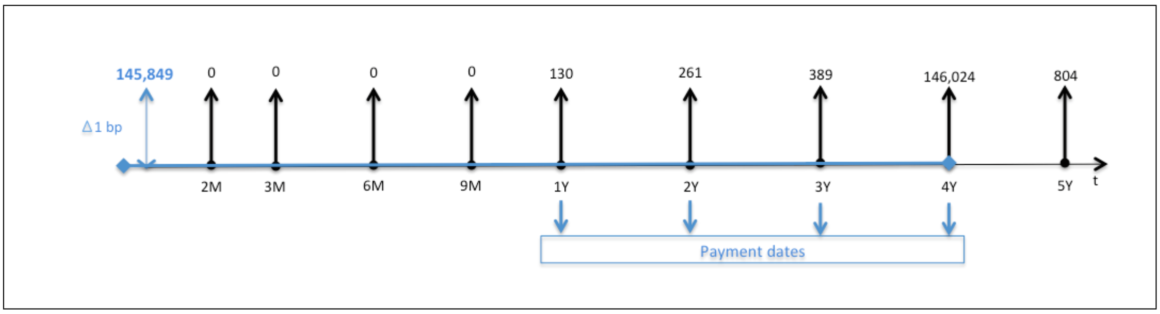

Figure 6.9: Swap risk calculation for data set 1

			Nodes								
Swap	Cashflow	Trader	3M	6M	9M	1Y	2Y	3Y	4Y	5Y	Delta
3Y	1	150,965	0	0	0	-91	-182	149,881	826	0	531.49
3Y	2	150,965	0	-22	0	-79	-182	150,003	827	0	418.78
3Y	4	150,965	-5	-11	-17	-71	-182	150,068	828	0	357.29
4Y	1	149,643	0	0	0	9	18	27	148,436	818	333.23
4Y	2	149,643	0	2	0	8	18	28	148,389	818	379.22
4Y	4	149,643	0	2	0	8	18	28	148,387	818	396.04

Table 6.7: Table showing risk buckets per node for data set 2

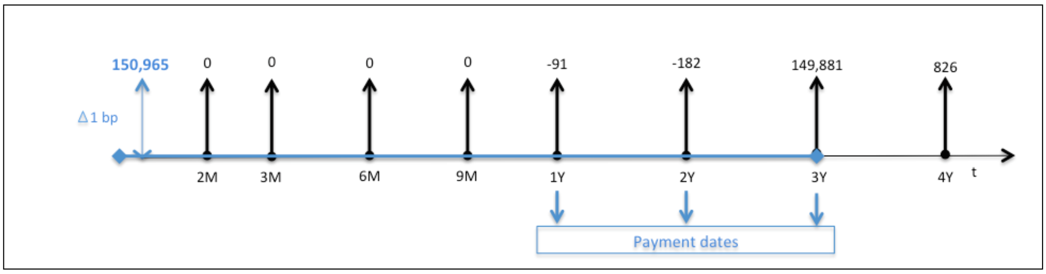

Figure 6.10: Swap risk calculation for data set 2

6.3 Chapter Summary

The two key aims of this study are to communicate the impact of the differing risk amounts are the key rates/nodes and relation between the trader perspective and risk manager. In this chapter it is shown that the risk buckets can have significant risk in them and that when these are summed the relationship $\textit{Trading floor} = \sum_{i=1}^{m} \textit{risk department}_i$ is observed.

Chapter 7

Conclusion

In this paper the aim was to communicate the importance of the approach taken by the risk department and it's linkage to the trading floor perspective. The first step taken was to determine in terms of literature what risk measures have similar properties to those used by the two departments.

It's in this approach that Duration and Key Rate Duration were noted as perfect measures to utilise in this study. As Key Rate Duration shows risk when shifts are in a non-parallel manner. KRD also has the same relationship to Duration as the trader, risk manager perspective.

To illustrate this in Chapter 6 a model is built using Mathworks MATLAB and the relationship is shown and the risk in the differing nodes is also shown. With this in hand a trader can now understand the cause of certain nodes having higher risk buckets and at times significantly larger than that of the overall risk calculated by his/her method. This allows one to foresee risks of interest movements at certain nodes allowing for better risk mitigation.

Further work to be looked into is the optimal number of key rates and which are these key rates as currently it is highly dependant on the person performing the analysis what is deemed a key rate. The other point of interest is to use the forward curve as opposed to zero curve in the measurement of risk.

Chapter 8

Notes on fulfilment of Thesis objectives

This thesis paper is done in order to successfully complete the Masters in Financial Engineering program. The Masters Candidate's thesis has to meet requirements set by the Swedish National Agency for Higher Education to earn a passing grade. In this section the Masters Candidate will show how the work that has been submitted fulfills these requirements.

Objective 1

For Master degree, student should demonstrate knowledge and understanding in the major field of study, including both broad knowledge in the field and substantially deeper knowledge of certain parts of the area as well as insight into current research and development.

Fulfilment: The author has undertaken a diligent study of the literature relating to Fixed Income field. The author begins by giving a detail of the various key instruments that are there in the domain of Fixed Income. The author then looks at the literature relating to the calculating of risk for these over the past 40 years. It is through this review of literature that author notes the common use of Duration as the risk proxy for Fixed Income instruments.

The author then narrows in on the key deficiencies of Duration as risk proxy, with a key focus of the non-parallel interest movements. It is here were Key Rate Duration as an alternative to Duration is introduced and illustrated. In this study the author also notes the limitations of Key Rate Duration as a risk proxy.

The author then models these risk proxies using actual market data to prove the relationships that have been discussed.

Objective 2

For Master Degree, student should demonstrate deeper methodological knowledge in the major field of study.

Fulfilment: The methodology utilised by the author takes into consideration current research into the use of Duration as a risk proxy. The layout of this document is in such a way that a person with a quantitative background can follow arguments put forward and implement the results of the research. The study also gives good methodology for utilisation of empirical methods for the estimation of term structures. The whole study importantly shows how one can go about the process of finding a scientifically based solution to certain real life problems.

Objective 3

For Master degree, student should demonstrate the ability to critically and systematically integrate knowledge and to analyse, assess and deal with complex phenomena, issues and situations even with limited information.

Fulfilment: The author has taken a real life problem and synthesised it into a theoretical problem. The author has then utilised this theoretical framework to explore current research into the problem being studied. The research undertaken is of great importance as currently only one scholarly journal article exists that deals with this phenomena. This shows firstly this is frontier research and also a serious gap in the research relating to risk calculation of Fixed Income instruments.

Objective 4

For Master degree, student should demonstrate the ability to critically, independently and creatively identify and formulate issues and to plan and carry out advanced tasks within specified time frames, thereby contributing to the development of knowledge and to evaluate this work.

Fulfilment: From the onset of the study the author drew up a work plan for the project that was agreed upon with supervisors and was carried out and in submission of this thesis all the time-lines of that work plan have been met.

Objective 5

For Master degree, student should demonstrate ability in both national and international contexts, orally and in writing to present and discuss their conclusions and the knowledge and arguments behind them, in dialogue with different groups.

Fulfilment: The author has fulfilled the written aspect and will on the 10th of June 2016 fulfill the oral aspect of this objective.

Objective 6

For Master degree, student should demonstrate ability in the major field of study make judgments taking into account relevant scientific, social and ethical aspects, and demonstrate an awareness of ethical issues in research and development.

Fulfilment: In many of the post mortems relating to financial crisis of 2008 indications point to the lack of adequate risk modelling that led to poor oversight with the financial industry. This paper aims to show that gaps still do exist as shown in that significantly high risk can be present at certain key rates that the current modelling would not show. This paper will enhance the plethora of safe guards that have been built into the system in a post 2008 era.

In regard to to scientific ethics all sources used in this thesis have been cited and their work has not been claimed as that of the author of this thesis.

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Appendix A

Calculation of Rates

When bootstrapping a zero-coupon curve liquidity plays an important role hence it is important to use highly liquid instruments. In the Swedish market this is typically leads to the following classification of instruments:

1. Cash deposits with the maturities overnight (O/N), tomorrow next (T/N), one week, one month and three months,
2. Forward Rate Agreements for maturities between three months and 4 years as they are more liquid than swaps and
3. Swaps for maturities 4 years to 30 years.

These different methods can then be used to create the zero coupon and forward rate curves.

A.0.1 Cash Deposits

Overnight Rate

The overnight rate is generally the interest rate that large banks use to borrow and lend from one another in the overnight market. The overnight rate is the first rate to be calculated. The first operation is to calculate the discount factor

$$D_{o/n} = \frac{1}{1 + r_{o/n}\frac{d_{o/n}}{360}},$$

this is followed by calculating the zero-coupon rate

$$Z_{o/n} = \frac{-100ln(D_{o/n})}{\frac{d_{o/n}}{365}},$$

and finally the rate calculation to obtain the re-pricing of the instrument. This is important so as to attain the error within the calculation.

$$r_{o/n} = \left(\frac{1}{D_{o/n}} - 1\right)\left(\frac{360}{d_{o/n}}\right). \tag{A.1}$$

The tomorrow next is done to avoid taking physical delivery of the instrument you're trading in. Delivery would normally be due two days after the transaction, so if you want to hold your position for more than a day the tom-next market mechanism is applied to delay delivery. The first operation is to calculate the discount factor

$$D_{t/n} = \frac{D_{o/n}}{1 + r_{t/n}\frac{d_{t/n}}{360}},$$

this is followed by calculating the zero-coupon rate

$$Z_{t/n} = \frac{-100ln(D_{t/n})}{\frac{d_{t/n}}{365}},$$

then finally re-calculate the quoted rate

$$r_{t/n} = \left(\frac{D_{o/n}}{D_{t/n}} - 1\right)\left(\frac{360}{d_{t/n}}\right). \tag{A.2}$$

The next set of cash deposit based maturities are the 1 week, 1 month, 2 months, 3 months. The first operation is to calculate the discount factor

$$D_i = \frac{D_{t/n}}{1 + r_i\frac{d_i}{360}},$$

this is followed by calculating the zero-coupon rate

$$Z_i = \frac{-100ln(D_i)}{\frac{d_i}{365}},$$

then finally re-calculate the quoted rate

$$r_i = \left(\frac{D_{t/n}}{D_i} - 1\right)\left(\frac{360}{d_i}\right). \tag{A.3}$$

A.0.2 FRAs

The set of instruments is the forward rate agreements that have been discussed in Chapter 2. The first operation is to calculate the discount factor

$$D_i = \frac{D_{i-1}}{1 + r_i\frac{d_i}{360}},$$

this is followed by calculating the zero-coupon rate

$$Z_i = \frac{-100ln(D_i)}{\frac{d_i}{365}},$$

then finally re-calculate the quoted rate

$$r_i = \left(\frac{D_{t/n}}{D_i} - 1\right)\left(\frac{360}{d_i}\right). \tag{A.4}$$

In most instances the D_0 is not quantifiable so a stub needs to be created and it's discount factor is D_{stub}. The first step is to do an interpolation on the zero coupon curve to attain the zero rate of the stub

$$Z_{stub} = Z_{t_0} + (Z_{t_1} - Z_{t_0})\left(\frac{T_{stub} - t_0}{t_1 - t_0}\right),$$

the next step is to then calculate the discount factor of the stub

$$D_{stub} = exp(-Z_{stub}\left(\frac{T_{stub}}{365}\right),$$

with this the following relationship is attained and the calculation of D_i can begin

$$D_0 = D_{stub}. \tag{A.5}$$

A.0.3 Swaps

Finally is the swaps for the time periods 4 years to 30 years. The first operation is to calculate the discount factor

$$D_i = \frac{D_{t/n} - r_t \sum_{i=0}^{n} \delta_i D_i}{1 + r_i \delta_i},$$

this is followed by calculating the zero-coupon rate

$$Z_i = \frac{-100 ln(D_i)}{\frac{d_i}{365}},$$

then finally re-calculate the quoted rate

$$r_i = \frac{D_{t/n} - D_i}{\sum_{i=0}^{n} \delta_i D_i}. \tag{A.6}$$

Appendix B

Matlab code

```
clc
clear all;

% To observe decimal accuracy
format long

                                    % INPUTS TO THE PROGRAM

% The extraction of source data from Excel
filename = 'CurveData.xlsx';
sheet = 1;
formatIn = 'dd/mm/yyyy';
% xlRange = 'A2:A31';
% xlRange1 = 'C2:C31';
% xlRange2 = 'E2:E31';
% xlRange3 = 'B2:B31';
%
% xlRange = 'F2:F31';
% xlRange1 = 'H2:H31';
% xlRange2 = 'J2:J31';
% xlRange3 = 'G2:G31';

xlRange = 'L2:L28';
xlRange1 = 'N2:N28';
xlRange2 = 'P2:P28';
xlRange3 = 'M2:M28';

            % THE CONVERSION OF DATA INTO MATLAB READABLE FILES.

% 1. Download all the start dates of the instruments utilised.
Startdates = xlsread(filename,sheet,xlRange);
Startdate = x2mdate(Startdates);

% 2. Download all the maturity dates of the instruments utilised.
Maturitydates =  xlsread(filename,sheet,xlRange1);
Maturitydate =  x2mdate(Maturitydates);

```

```
% 3. Download all the qoutes of the rates of the instruments utilised.
qoute =  xlsread(filename,sheet,xlRange2);

% 4. Download all the type of the instruments utilised 'Deposit' 'FRA' '
    Swap'.
[num,txt,raw] =  xlsread(filename,sheet,xlRange3);
gen = [num,txt,raw];
generator = gen(:,1);


                              % PARAMETERS

% 1. The number of years to be utilised from date of issue to maturity
p = round(((Maturitydate(end)-Startdate(1,1))/365.25));

% 2. The number of cash-flows to be made during the year by the swaps
cf = 1;

% 3. Interpolation method options are: 'linear', 'spline', 'cubic'
inter_method = 'linear';

% 4. Number of elements
n = length(generator);

%  End OF INPUTS
%-------------------------------------------------------------------------
%-------------------------------------------------------------------------

                    % CREATION OF IMPORTANT DATE COUNTERS

% 1. Annual swap dates
for i=2:1:p
    if Startdate(i,1)-Startdate(i-1,1)==0
        theswap = Startdate(i-1);
        break
    end
end
yebo = theswap*ones(p,1);
 swapMdate = zeros(p,1);
for i=1:p
    swapMdate(i,:) = i*365.25;
end

swapMdates = yebo + swapMdate;
ilanga = weekday(swapMdates);

% 2. Ensure that if the day is on a weekend it is moved to a Monday
for i=1:1:p
    if ilanga(i,1)==1
        swapMdates(i,1) = swapMdates(i,1)+1;
    elseif ilanga(i,1)==7
        swapMdates(i,1) = swapMdates(i,1)+2;
```

```
    end
end
datestr(floor(swapMdates));

% 3. Zero coupon day count
zdays = Maturitydate -  Startdate(1,1);

% 4. Discount day count
ddays = Maturitydate - Startdate;

% 5. Swap day count
sdays =  swapMdates - theswap;

% End of day counts
%------------------------------------------------------------------------
%------------------------------------------------------------------------
%                          CREATION OF THE SWAP CURVE

% 1. CASH DEPOSIT COMPONENT
% 1.1 Creation of Overnight rate
% D = zeros(p);
% Z = zeros(p);
% r = zeros(p);
for i = 1:1:n
    if strcmp(generator(i,1),'Deposit')==1

        D(i,:) = 1/(1+0.01*qoute(i,1)*(ddays(i,1)/360));
        Z(i,:) = (-100*log(D(i,1)))/(zdays(i,1)/365);
        r(i,:) = ((1/D(i,1))-1)*(360/ddays(i,1))*100;
        break
    end
end
a=length(r)+1;
for i = a:1:n
    if strcmp(generator(i,1),'Deposit')==1
        D(i,:) = D(1,1)/(1+0.01*qoute(i,1)*(ddays(i,1)/360));
        Z(i,:) = (-100*log(D(i,1)))/(zdays(i,1)/365);
        r(i,:) = ((D(1,1)/D(i,1))-1)*(360/ddays(2,1))*100;
        break
    end
end
b=length(r)+1;
for i = b:1:n
    if strcmp(generator(i,1),'Deposit')==1
            D(i,:) = D(2,1)/(1+0.01*qoute(i,1)*(ddays(i,1)/360));
            Z(i,:) = (-100*log(D(i,1)))/(zdays(i,1)/365);
            r(i,:) = ((D(2,1)/D(i,1))-1)*(360/ddays(i,1))*100;
    end
end
c=length(r)+1;
for i = 1:1:p
    if strcmp(generator(i,1),'FRA')
```

```
            stub = Startdate(i,1)-1;
            break
        end
end
for i=1:p
        if Maturitydate(i,1)-stub>0;
            t(2,1)=Maturitydate(i,1);
            t(1,1)=Maturitydate(i-1,1);
            q(2,1)=Z(i,1);
            q(1,1)=Z(i-1,1);
            break;
        end;
end;
%-------------------------------------------------------------------------

                 % 2. FORWARD RATE AGREEMENT COMPONENT

% 2.1.2 Creation of stub zero coupon rate and discount rate.
    z_stub = interp1(t,q,stub,inter_method);
    D_stub = exp(-0.01*z_stub*((stub - Maturitydate(2,1))/365));

% 2.2 Creation of FRA related rates.
for i = c:1:n
    if strcmp(generator(i,1),'FRA')==1
        D(i,:) = D_stub/(1+(0.01*qoute(i,1)*(ddays(i,1)/360)));
        Z(i,:) = (-100*log(D(i,1)))/(zdays(i,1)/365);
        r(i,:) = ((D_stub/D(i,1))-1)*(360/ddays(i,1))*100;
        break
    end
end
d=length(r)+1;
for i = d:1:n
    if strcmp(generator(i,1),'FRA')==1
        D(i,:) = D(i-1,1)/(1+(0.01*qoute(i,1)*(ddays(i,1)/360)));
        Z(i,:) = (-100*log(D(i,1)))/(zdays(i,1)/365);
        r(i,:) = ((D(i-1,1)/D(i,1))-1)*(360/ddays(i,1))*100;
    end
end
length(D);
%-------------------------------------------------------------------------

                          % 3. SWAP COMPONENT

% 3.1 Determining the first three annual swap rates by interpolation of
    the
% zero coupon curve
r_swap = interp1(Maturitydate,qoute,swapMdates,inter_method,'extrap');

for i = 1:1:n
    if Maturitydate(i,1)-Startdate(i,1)>270
        k = round((Maturitydate(i,1)-Startdate(i,1))/365.25);
        break
```

```
        end
end

for j = 1:k-1
    for i =1:1:n
        if Maturitydate(i,1)-swapMdates(j,1)>0
            t(2,1) = Maturitydate(i,1);
            t(1,1) = Maturitydate(i-1,1);
            z(2,1) = Z(i,1);
            z(1,1) = Z(i-1,1);
            break
        end
    end
    z_swap(j,:) = interp1(t,z,swapMdates(j,1),inter_method);
end

for j = 1:1:k-1
    d_swap(j,1) = exp(z_swap(j,1)*((sdays(j,1))*(1/365))*-0.01);
end

for i = k:1:p
    d_swap(i,:) = (D(2,1) - (r_swap(i,1)*0.01)*sum(d_swap(1:i-1,1)))
   /(1+(1/cf)*r_swap(i,1)*0.01);
    z_swap(i,:) = (-100*log(d_swap(i,1))/(sdays(i,1)/365));
end

f = length(r)+1;
for i = f:1:n
    if strcmp(generator(i,1),'Swap')==1
       swap_mat(i,:) = Maturitydate(i,1);
    end
end

swap_mat=swap_mat(swap_mat~=0);
dd_swap = interp1(swapMdates,d_swap,swap_mat,inter_method,'extrap');
rr_swap = interp1(swapMdates,r_swap,swap_mat,inter_method,'extrap');

for i = f:1:n
    D(i,:) = dd_swap(i+1-f,1);
    Z(i,:) = (-100*log(D(i,1)))/(zdays(i,1)/365);
    r(i,:) = rr_swap(i+1-f,1);
end

error = r - qoute;

                        % CREATION OF FORWARD RATE CURVE

% 1. Creation of the forward rate dates
for i=2:1:p*cf
    if Startdate(i,1)-Startdate(i-1,1)==0
        thefswap = Startdate(i-1);
        break
```

```
    end
end
fyebo = thefswap*ones(p*cf,1);
for i=1:p*cf
    fswapMdate(i,:) = i*365.25*(1/cf);
end
datestr(fyebo);
fswapMdates = fyebo + fswapMdate;
filanga = weekday(fswapMdates);

for i=1:1:p*cf
    if filanga(i,1)==1
        fswapMdates(i,1) = fswapMdates(i,1)+1;
    elseif filanga(i,1)==7
        fswapMdates(i,1) = fswapMdates(i,1)+2;
    end
end

% 2. Calculation of the forward rate discounts

fdswap = interp1(swapMdates,d_swap,fswapMdates,inter_method);

% 3. Calculation of the first two forward rates
f_swap(1,1) = (36500*(log(1)-log(D(1,1))))/(Maturitydate(1,1)-Startdate
    (1,1));
f_swap(2,1) = (36500*(log(D(1,1))-log(fdswap(1,1))))/(fswapMdates(1,1)-
    Maturitydate(1,1));

% 4. Calculation of the remainder of the forward rates
for u = 3:1:p*cf
    f_swap(u,:) = (36500*(log(fdswap(u-2,1))-log(fdswap(u-1,1))))/(
    fswapMdates(u-1,1)-fswapMdates(u-2,1));
end

for u = 2:1:p*cf
    f_swapMdates(1,1) = Maturitydate(1,1);
    f_swapMdates(u,1) = fswapMdates(u-1,1);
end
% % % ---------------------------------------------------------------

% figure
% hold on
% % 1. Graphical representation of the zero coupon curve
% plot(Maturitydate,D,'g');
% % 2. Graphical representation of the discount curve
% plot(Maturitydate,Z,'-.b');
% % 3. Graphical representation of the forward curve
% plot(f_swapMdates,f_swap,'-ro');
% datetick('x',10);
% title('Bootstrapping');
% xlabel('Time');
% ylabel('%');
```

```
% legend('Qoute','Zero-Coupon','Forward','location','northwest');
% % legend('Error','location','northwest');

% Creating the nodes as per Swedish key rate dates

Swe_nodes = [1
     2
     3
     7
     30
     61
     91
     183
     274
     365
     730
     1095
     1460
     1825
     2555
     3650
     4380
     5475
     7300
     9125
     10950];
Swedish_nodes = Swe_nodes + Startdate(1,1);
Zero = interp1(Maturitydate,Z,Swedish_nodes,inter_method);
Discount = interp1(Maturitydate,D,Swedish_nodes,inter_method);

%----------------------------------------------------------------------------
%----------------------------------------------------------------------------

                              % CALCULATION OF RISK

                              % INPUTS TO THE PROGRAM

% 1. Start date of Instrument under analysis
Inst_Startdate = {'09/07/2015'};
Instrument(1,1) = datenum(Inst_Startdate,formatIn);

% 2. Maturity date of Instrument under analysis
Inst_Maturitydate = {'09/07/2030'};
Instrument(2,1) = datenum(Inst_Maturitydate,formatIn);

% 3. Type of instrument
Inst_type = 'Swap';

% 4. Amount of change in basis points
Delta_amount = 0.01*0.01;

% 5. Notional amount change in SEK
```

```
notional = 1500000000;

% 6. Cashflows in swap
cflow = 4;
% ------------------------------------------------------------------------

nodes = length(Swedish_nodes);
%
% Instrument = [Inst_Sd
%      Inst_Md];

Inst = Instrument - Startdate(1,1);
         numcflow = round(((Instrument(2,1) - Instrument(1,1))/365)*cflow);
         for i = 1:1:numcflow
              Instcfdate(i,1) = Instrument(1,1) + 365*(1/cflow)*i;
              Instdays(i,1) = Instcfdate(i,1) - Startdate(1,1);
         end

         risk = interp1(Swedish_nodes,Discount,Instcfdate,'linear')
         (D(2,1) - risk(end))/((1/cflow)*sum(risk(3:numcflow)))

Zero = Zero*0.01;

if strcmp(Inst_type,'FRA')==1
%      1. Calculation of risk relating to FRA
     risk = interp1(Swedish_nodes,Zero,Instrument,'linear');

     for j = 1:1:2
%          disc = zeros(2,1);
         disc(j,:) = exp(-risk(j,1)*(Inst(j,1)/365));
     end

     forward = (-365*log(disc(2,1)/disc(1,1)))/(Instrument(2,1)-Instrument
    (1,1));
     frncost = disc(1,1)*notional*forward*((Instrument(2,1)-Instrument(1,1)
    )/365);

     for i = 1:1:nodes
%          Zero_1 = zeros(21,21);
         Zero_1(:,i) = Zero;
     end

%      size(Zero_1)
Delta_amounts = Delta_amount*eye(21);

     for i = 1:1:nodes
         for k = 1:1:nodes
%          Zero_1 = Zero;
              Zero_1(k,i) = Zero_1(k,i) + Delta_amounts(k,i);
              risk_1(:,i) = interp1(Swedish_nodes,Zero_1(1:nodes,i),
    Instrument,'linear');
              for j = 1:2
```

```
%                   disc_1 = zeros(2,21);
                disc_1(j,i) = exp(-risk_1(j,i)*(Inst(j,1)/365));
            end
%             forward_1 = zeros(1,21);
%             risk_node_1 = zeros(1,21);
            forward_1(:,i) = (-365*log(disc_1(2,i)/disc_1(1,i)))/(Instrument(2,1)-Instrument(1,1));
            frncost_1(:,i) = disc_1(1,i)*notional*forward_1(1,i)*((Instrument(2,1)-Instrument(1,1))/365);
            risk_node_1(:,i) = frncost_1(1,i) - frncost;
        end
    end

    elseif strcmp(Inst_type,'Swap')==1
%     2. Calculation of risk relating to Swap

        numcflow = round(((Instrument(2,1) - Instrument(1,1))/365)*cflow);

        for i = 1:1:numcflow
            Instcfdate(i,1) = Instrument(1,1) + 365*(1/cflow)*i;
            Instdays(i,1) = Instcfdate(i,1) - Startdate(1,1);
        end

        risk = interp1(Swedish_nodes,Zero,Instcfdate,'linear');

        for j = 1:1:numcflow
    %           disc = zeros(numcflow,1);
            disc(j,:) = exp(-risk(j,1)*(Instdays(j,1)/365));
        end

        swaprate = (Discount(2,1) - disc(end))/((1/cflow)*sum(disc));

        for i = 1:1:nodes
    %          Zero_1 = zeros(21,21);
            Zero_1(:,i) = Zero;
        end

        Delta_amounts = Delta_amount*eye(21);

        for i = 1:1:nodes
            for k = 1:1:nodes
    %               Zero_1 = Zero;
                Zero_1(k,i) = Zero_1(k,i) + Delta_amounts(k,i);

                risk_1(:,i) = interp1(Swedish_nodes,Zero_1(1:nodes,i),Instcfdate,'linear');

                for j = 1:numcflow
    %                   disc_1 = zeros(numcflow,1);
                    disc_1(j,i) = exp(-risk_1(j,i)*(Instdays(j,1)/365));
                end

```

```
                swaprate_1(:,i) = (Discount(2,1) - disc_1(numcflow,i))
    /((1/cflow)*sum(disc_1(1:numcflow,i)));

                risk_node_1(:,i) = (swaprate_1(1,i) - swaprate)*notional*
    disc_1(numcflow,i);
            end
        end
end

% 3. Trader risk calculation
Total_risk = 0.0000934849357*notional;

% 4. Risk management risk calculation
risk_node_1';
tot_risk = sum(risk_node_1);

% 5. Delta between Trader & Risk Manager risk calculation
Total_risk - tot_risk;

                                % THE END
% ------------------------------------------------------------------------
% ------------------------------------------------------------------------
% ------------------------------------------------------------------------
```
MATLAB Code

Appendix C

Proof of Theorem 1

This proof of Theorem 1 taken from work done by McCulloch [17], the pioneer in the use of polynomial splines in fitting the term structure of interest rates.

Proof. Let $f(t)$ be the current forward rate function, so that

$$P(t) = \exp(-\int_0^t f(s)ds), \tag{C.1}$$

is the price of a discount bond maturing at time t.
The maximum smoothness term structure is a function f with a continuous derivative that satisfies the optimisation problem

$$\min \int_0^T (f''(s))^2 ds, \tag{C.2}$$

subject to the constraints

$$\int_0^{t_i} f(s)ds = -\ln P_i, \tag{C.3}$$

Here the $P_i = P(t_i)$, for $i = 1,2,...,m$ are given prices of discount bonds with maturities $0 < t_1 < t_2 < ... < t_m < T$. Intergrating twice by parts the following is obtained

$$\int_0^t f(s)ds = tf(t) - \frac{1}{2}t^2 f'(t) + \frac{1}{2}\int_0^t s^2 f''(s)ds, \tag{C.4}$$

Let $g(t) = f''(t)$, for $0 \leq t \leq T$

$$Q_i = t_i f(t_i) - \frac{1}{2}t_i^2 f''(t_i), \tag{C.5}$$

for $i = 1,2,...,m$ and define the step function

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases}$$

The optimisation problem can then be written as

$$\min \int_0^T (g^2(s))ds, \tag{C.6}$$

subject to

$$\frac{1}{2}\int_0^t s^2 u(t_i - s)g(s)ds = -\ln P_i - Q_i, \tag{C.7}$$

and by letting λ_i for $i = 1, 2, ..., m$ be the Lagrange multipliers corresponding to the constraints, the objective function becomes

$$\min Z[g] = \int_0^T g^2(s) + \frac{1}{2}\sum_{i=1}^{m} \lambda_i (\int_0^T s^2 u(t_i - s)g(s)ds + \ln P_i + Q_i). \tag{C.8}$$

According to the calculus of variations, if the function g is a solution to (C.8), then

$$\frac{d}{d\varepsilon} Z[g + \varepsilon h]|_{\varepsilon=0} = 0, \tag{C.9}$$

for any function *h(t)* defined on [0,T]. We get

$$\frac{d}{d\varepsilon} Z[g + \varepsilon h]|_{\varepsilon=0} = 2\int_0^T [g(s) + \frac{1}{4}s^2 \sum_{i=1}^{m} \lambda_i u(t_i - s)]h(s)ds, \tag{C.10}$$

To ensure that the intergral is zero for any function of h, the following must occur

$$g(t) + \frac{1}{4}t^2 \sum_{i=1}^{m} \lambda_i u(t_i - t) = 0, \tag{C.11}$$

for all $0 \leq t \leq T$. This means that

$$g(t) = 12c_i t^2, \tag{C.12}$$

for $t_{i-1} < t \leq t_i$ and $i = 1, 2, ..., m$, where,

$$c_i = -\frac{1}{48}\sum_{j=i}^{m} \lambda_i, \tag{C.13}$$

$$c_{m+1} = 0$$

and define t_0, $t_{m+1} = T$

$$f(t) = c_i t^4 + b_i t + a_i, \tag{C.14}$$

continuity of f and f' then implies that

$$c_i t_i^4 + b_i t_i + a_i = c_{i+1} t_i^4 + b_{i+1} t_i + a_{i+1}, \tag{C.15}$$

$$4c_i t_i^3 + b_i = 4c_{i+1} t_i^3 + b_{i+1}, \tag{C.16}$$

the constraints become

$$\frac{1}{5}c_i(t_i^5 - t_{i-1}^5) + \frac{1}{2}b_i(t_i^2 - t_{i-1}^2) + a_i(t_i - t_{i-1}) = -\ln(\frac{P_i}{P_{i-1}}), \tag{C.17}$$

where we define $P_0 = 1$. □