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Svensson (1994) model and the Nelson & Siegel (1987) model

Analytical Finance 2
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Introduction

Various methods exist for estimating zero-coupon yield curves. The most adopted methods are either the Nelson & Siegel (1987) method or the extended version suggested by Svensson (1994), We are going to illustrates the application of the Nelson and Siegel model, and the Svensson model in deriving the zero-coupon yield curve.
The definition of yield rate or the spot interest rate, also called yield to maturity (YMT) is the True rate of return of an investor would receive if the security were held to maturity. If expressed as a function of maturity is known as term structure of interest rates.

Yield curve is the graphical plotting of the yield rate function. The yield curve is one of the most important indicator of the level and changes in interest rates in the economy and hence the interest in studying as well as accurately modeling it.

The yield to maturity (YTM), the single discount rate on an investment that makes the sum of the present value of all cash flows equal to the current price of the investment, has been a common measure of the rate of return. However, using a single discount rate at different time periods is problematic because it assumes that all future cash flows from coupon payments will be reinvested at the derived YTM.
This assumption neglects the reinvestment risk that creates investment uncertainty over the entire investment horizon. Another shortcoming of YTM is that the yields of bonds of the maturity depend on the patterns of their cash flows, which is often referred to as the coupon effect. As a result, the YTM of a coupon bond is not a good measure of the pure price of time and not the most appropriate yield measure in the term structure analysis.

On the other hand, zero-coupon securities eliminate the exposure to reinvestment risks as there is no cash flow to reinvest. The yields on the zero-coupon securities, called the *spot rate*, are not affected by the coupon effect since there are no coupon payments. Also, unlike the yield to maturity, securities having the same maturity have theoretically the same spot rates, which provide the *pure* price of time. As a result, it is preferable to work with zero-coupon yield curves rather than YTM when analyzing the yield curve.
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Models for Fitting:

Svensson (1994) model and the Nelson and Siegel (1987) model

The Svensson (1994) model and the Nelson and Siegel (1987)(i.e. a restricted version of Svensson's model) is highly preferred because of its thrifty nature to fit the implied forward rates from the raw data. Specifically, the Svensson model assumes

$$f(m,\beta) = \beta_0 + \beta_1 \exp\left(\frac{-m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(\frac{-m}{\tau_1}\right) + \beta_3 \frac{m}{\tau_2} \exp\left(\frac{-m}{\tau_2}\right)$$

The parameters:

β_0 is non-negative value and is an asymptotic value f (m);

β_1 determines initial value of curve in terms of deviation from asymptote;

β_2 decides the direction and magnitude of the hump.if its positive a hump occurs at τ_1 whereas if it is negative, the depression occurs at τ_1.

τ_1 a positive non-negative parameter that determines the position of the first hump or the depression shape on the curve.

τ_2 a positive non-negative parameter that determines the position of the second hump or the depression on the curve.

β_3 decides the direction and magnitude of the hump

m =time to maturity,

$\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2)$ is the parameters to estimated,

and β_0, τ_1, τ_2 and $\beta_0 + \beta_1$ are assumed to be greater than 0.

The spot rate $s(m,\beta)$ (.i.e.zero coupon yield for knowing the time value of money is the integral of the instantaneous forward rate

$$s(m,\beta)=\beta_0+\beta_1*\left[\frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}}\right]+\beta_2*\left[\frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}}-\exp\left(-\frac{m}{\tau_1}\right)\right]$$

$$+\beta_3*\left[\frac{1-\exp\left(-\frac{m}{\tau_2}\right)}{\frac{m}{\tau_2}}-\exp\left(-\frac{m}{\tau_2}\right)\right] \qquad (2)$$

From equation (2) we can see that the zero-coupon yield $s(m,\beta)$ depends on the time to maturity (m) of the bond on the set (β). Given any spot rate $s(m,\beta)$, the discount factor $d(m,\beta)$ *is* used to obtain the present value of future cash flows is given as:

$$d(m,\beta) = \exp\left(-\frac{s(m,\beta)}{100}m\right) \qquad (3)$$

For a T-years coupon-bearing bond, its price $p^e(m,\beta)$ is approximated by the sum of the discounted semi-annual coupon payments (C_k) and the final principal (V) as follows:

$$P^e(\beta)=c_1*n*d(n,\beta)+\sum_{k=2}^{[2T]}c_k*d(n+0.5*(k-1))+V*d(T,\beta) \qquad (4)$$

Where [2T] = 2T if 2T is an integer,
[2T] = (integral part of 2T) +1 if [2T] is a non-integer,
n = number of years from trading date to the first coupon payment,
c_k = k_{th} coupon payment.

To get the parameters ($\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2$), the approximated prices p^e (in terms of these parameters) are compared with the observed prices of all outstanding EFBNs.These parameters are estimated by minimizing the sum of squared bond-price errors weighted by $(1/\Phi)^2$:

$$\underset{\beta_0,\beta_1,\beta_2,\beta_3,\tau_1,\tau_2}{Min} \quad \sum_{j=1}^{n} \left\{ \left[P_j - P_j^{e}\left(\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2\right) \right] / \Phi_j \right\}^2 \qquad (5)$$

P_j = Observed price of bond *j*,
n = Total number of bonds outstanding and
Φj = equals the duration of bond *j*.

Alternatively:
One can achieve this by minimizing the deviation between estimated and observed yields. In this case, the estimation procedure involves two stages:

(1): a discount function similar to equation (3) is used to compute estimated prices, such that

$$P_j^{e}(\beta) = c_1 * n * d(n,\beta) + \sum_{k=2}^{[2T]} c_k * d(n + 0.5*(k-1)) + V * d(T,\beta) \qquad (6)$$

(2) The estimated yield to maturity for the bond, denoted by $Y_j^e(\beta)$, is estimated from equation(6) by solving the following equation:

$$P_j^{e}(\beta) = \sum_{k=1} coupon * \exp(-Y_j^{e}(\beta)k/2) + V * \exp(-Y_j^{e}(\beta)T) \qquad (7)$$

The parameter set (β) is obtained by minimising the sum of squared yield errors between the observed yield to maturity Y_j and the corresponding estimated yield to maturity $Y_j^e(\beta)$

$$\underset{\beta_0,\beta_1,\beta_2,\beta_3,\tau_1,\tau_2}{Min} \quad \sum_{j=1}^{n} \left[Y_j - Y_j^{e}(\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2) \right]^2 \qquad (8)$$

The optimization is performed by applying non-linear minimisation procedure separately to the sum of squared (weighted) price errors in equation (5), and then to the sum of squared yield errors in equation (8), with different set of initial values, with respect to the constraints on the parameter values. After getting the estimated parameters, the implied forward rate and the zero-coupon yield (spot rate) curve can be computed by substituting these parameters into equations (1) and (2).

For estimation of the Nelson and Siegel’s model, similar procedures can be applied. The only difference is that the third term of both the forward rate in equation (1) and spot rate in equation (2) under Svenson’s model does not exist in Nelson and Siegel’s framework. ie. there are only 4 parameters $\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2$ to be determined under Nelson-Siegel’s model.

CONCLUSION:

The investigation shows that the zero-coupon yield curves for EFBNs can be suitably fitted by both Svensson and Nelson-Siegel models using either price or yield errors minimisation approach. Based on the R^2, the Svensson model, with an additional hump, fits the data slightly better than the Nelson-Siegel model irrespective of which minimisation method is used.

It can be said therefore that both prices and yield errors minimisation methods seem to fit the data well. The yield errors minimisation approach is more cumbersome in computation than the price one as it involves an extra iteration stage in the estimation process. Hence convergence problems occur more frequently and the optimization process is much more sensitive to the choice of initial values. All the same, since the main concern for the yield curve analysis lies in the interest rates, it is better to use the yield errors minimisation approach.

REFERENCE:

1. Lecture note of course Analytical finance 2, Märladalen University by Jan Röman,

2 http://www.info.gov.hk/hkma/eng/research/RM09-2002.pdf

APPENDIX:

Table 1. Estimation Results based on Market Consensus Prices of EFBNs

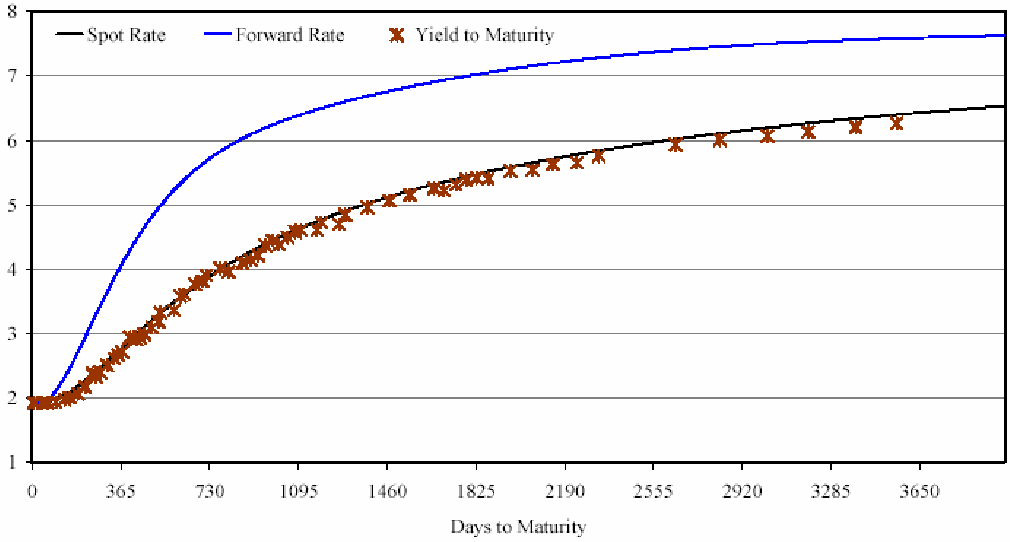
Estimation Method		**Yield Errors Minimisation**
Estimation Parameters	**Svensson**	**Nelson & Siegel**

Estimation Method		**Price Errors Minimisation**
Estimation Parameters	**Svensson**	**Nelson & Siegel**
	11 March 2002	
β_0	7.41	7.05
β_1	-5.41	-5.05
β_2	-5.03	-4.55
β_3	-4.43	-
τ_1	0.44	0.84
τ_2	1.38	-
R^2	0.9999	0.9999
# of observations	32	32

Chart 1. Estimation Results of EFBN Zero-Coupon Yield Curve based on Market Consensus Prices of EFBNs

Estimation Method: Yield Errors Minimisation
11 March 2002

Svensson's Model



Nelson & Siegel's Model