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Nelson-Siegel

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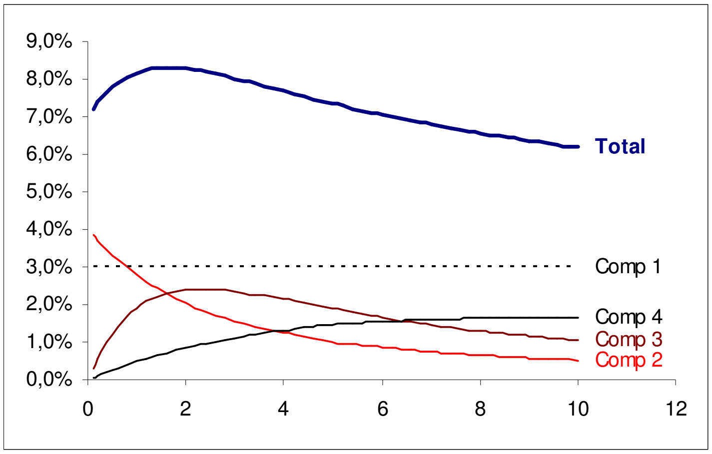
Forward interest rates are interest rates on investment and loans that start at a future date, the settlement date, and last to a date further into the future, the maturity date.

The estimation of the forward rates are discussed by Nelson-Siegel model in this paper and an excel sheet for estimation of the forward curve presented. **Indeed, the forward rate curve is related to the yield curve as the marginal cost curve is to the average cost curve. The forward rate curve can be interpreted as indicating the expected future *time path* of these variables. Therefore forward rates more easily allow a separation of expectations for the short-, medium-, and long-term than the yield curve.**



This is the main advantage of forward rates. Usually estimating forward rates from coupon bonds can then be seen as involving two steps: first implied spot rates are estimated from yields to maturity on coupon bonds, and then implied forward rates are computed from implied spot rates.

Nelson and Siegel (1987) assume that the instantaneous forward rate is the solution to a second-order differential equation with two equal roots. Then Nelson and Siegel's forward rate function can be written

$$f(m;b) = \beta_0 + \beta_1 \exp\left[-\frac{m}{\tau_1}\right] + \beta_2 \frac{m}{\tau_1} \exp\left[-\frac{m}{\tau_1}\right]$$

Where f(m) denotes the instantaneous forward rate f(t,t+m) with time to settlement for a given trade date t. And $b = (\beta_0, \beta_1, \beta_2, \tau_1)$ is a vector of parameters. The forward rate consists of three components. The first is a constant, β_0, the second is an exponential term $\beta_1 \exp\left[-\frac{m}{\tau_1}\right]$ monotonically decreasing (or increasing, if β_1 is negative) toward zero as a function of the time to settlement, and the third is a term which generates a hump-shape (or a U-shape, if β_2 negative) as a function of the time to settlement, $\beta_2 \frac{m}{\tau_1} \exp\left[-\frac{m}{\tau_1}\right]$.

When the time to settlement approaches infinity, the forward rate approaches the constant β_0, and when the time to settlement approaches zero, the forward rate approaches the constant $\beta_0 + \beta_1$.
To increase the flexibility and improve the fit we extend Nelson and

Siegel's function by adding a fourth term, a second hump-shape (or U-shape),

$$\beta_3 \frac{m}{\tau_2} \exp\left[-\frac{m}{\tau_2}\right]$$

with two additional parameters, β_3 and τ_2 (τ_2 must be positive). The function is then

$$f(m;b) = \beta_0 + \beta_1 \exp\left[-\frac{m}{\tau_1}\right] + \beta_2 \frac{m}{\tau_1} \exp\left[-\frac{m}{\tau_1}\right] + \beta_3 \frac{m}{\tau_2} \exp\left[-\frac{m}{\tau_2}\right]$$

Where $b = (\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2)$.

The spot rate can be derived by integrating the forward rate according to

$$i(t,T) = \frac{\int_{\tau=T}^{T} f(t,\tau)d\tau}{T-\tau}$$

Let *i(m)* denote the spot rate *i(t,t+m)* with time to maturity m, for a given trade date t. It is given by

$$i(m;b) = \beta_0 + \beta_1 \frac{1-\exp\left[-\frac{m}{\tau_1}\right]}{\frac{m}{\tau_1}} + \beta_2 \left[\frac{1-\exp\left[-\frac{m}{\tau_1}\right]}{\frac{m}{\tau_1}} - \exp\left[-\frac{m}{\tau_1}\right]\right] + \beta_3 \left[\frac{1-\exp\left[-\frac{m}{\tau_2}\right]}{\frac{m}{\tau_2}} - \exp\left[-\frac{m}{\tau_2}\right]\right]$$

Where the discount function is given by

$$d(m;b) = \exp\left[-\frac{i(m;b)}{100} m\right]$$

The next step is using a bond pricing formula for finding the bond price over its life-time.

The use of the Pricing formula

Then at this point when we have constructed the Nelson-Siegel formula to calculate the Spot Rate. We begin to price the bond data that we were given. We choose 8 bonds to price which were:

Swed Government Bond 1037
Swed Government Bond 1040
Swed Government Bond 1043
Swed Government Bond 1034
Swed Government Bond 1048
Swed Government Bond 1045
Swed Government Bond 1046
Swed Government Bond 1041

Which head the following data,

Name	Maturity	Yield	Cupon	Next Coup	Price	Clean P.	Duration
1037	2007-08-15	2,685%	8,00%	2007-08-15	106%	103,58%	0,69
1040	2008-05-05	3,030%	6,50%	2007-05-05	109%	104,72%	1,36
1043	2009-01-28	3,250%	5,00%	2007-01-28	108%	103,56%	2,01
1034	2009-04-20	3,345%	9,00%	2007-04-20	118%	112,67%	2,15
1048	2009-12-01	3,430%	4,00%	2007-12-01	102%	101,59%	2,88
1045	2011-03-15	3,605%	5,25%	2007-03-15	110%	106,40%	3,82
1046	2012-10-08	3,730%	5,50%	2007-10-08	110%	109,13%	5,15
1041	2014-05-05	3,810%	6,75%	2007-05-05	123%	118,66%	6,01

Then we applied part of it in the formula

PRICE (settlement, maturity, rate, yield, redemption, frequency, basis)

Then we used basis point 4, since it is used in the Swedish market. So by using this Price function we could calculate the bond prices with the Nelson-Siegel Spot rate.

Which gave us the following Prices

NSE Price	Price
100,58%	106%
97,49%	109%
92,54%	108%
100,88%	118%
87,07%	102%
87,60%	110%
84,72%	110%
94,06%	123%

Here we can se the given prices on the bond against the calculated prices with the NSE (Nelson-Siegel extension) spot rate.

After this has been calculated we needed to estimate the correct parameters for the NSE model by minimizing the sum of the squared bond price errors weighted by (1/Φ):

$$\overset{Min}{\beta_0, \beta_1, \beta_2, \tau_1, \tau_2} \sum \left\{ \left[P_j - P_j^{ENS} \left(\beta_0, \beta_1, \beta_2, \tau_1, \tau_2 \right) \right] / \Phi_j \right\}^2$$

But before doing that we need to calculate Φ = duration*the given Price/(1+yield to maturity). When we have calculated Φ, the only thing left to do is to find the squared bond price errors weighted by (1/Φ), for each bond. Then one should sum all of them, and use solver to obtain the minimum error.

We then have the following:

Issuer	Coupon	Maturity	Clean	Dirty	NSE Price	Duration	Price	Yield	Φ	Squared bond price error
Swed Government Bond 1037	8,00%	2007-08-15	103,58%	105,91%	100,58%	0,69	106%	2,69%	0,712	0,5843%
Swed Government Bond 1040	6,50%	2008-05-05	104,72%	105,17%	97,49%	1,36	109%	3,03%	1,432	0,5922%
Swed Government Bond 1043	5,00%	2009-01-28	103,56%	105,26%	92,54%	2,01	108%	3,25%	2,099	0,5305%
Swed Government Bond 1034	9,00%	2009-04-20	112,67%	113,67%	100,88%	2,15	118%	3,35%	2,461	0,5009%
Swed Government Bond 1048	4,00%	2009-12-01	101,59%	103,58%	87,07%	2,88	102%	3,43%	2,830	0,2650%
Swed Government Bond 1045	5,25%	2011-03-15	106,40%	107,49%	87,60%	3,82	110%	3,61%	4,063	0,3090%
Swed Government Bond 1046	5,50%	2012-10-08	109,13%	109,92%	84,72%	5,15	110%	3,73%	5,461	0,2143%
Swed Government Bond 1041	6,75%	2014-05-05	118,66%	119,13%	94,06%	6,01	123%	3,81%	7,098	0,1616%
									sum	3,1578%

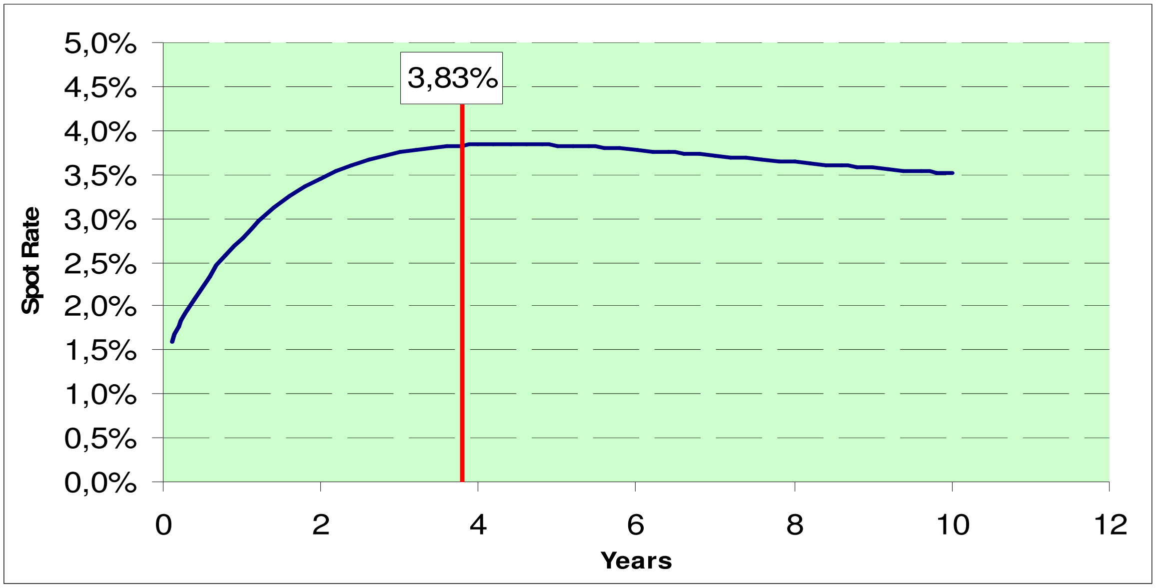
So the target cell to be minimized will be the sum 3, 1578% by using solver on excel or the solver button. Where the solver has the following constraints:

Beta 1 > 0
Beta 1 + Beta 2 > 0
Beta > 0

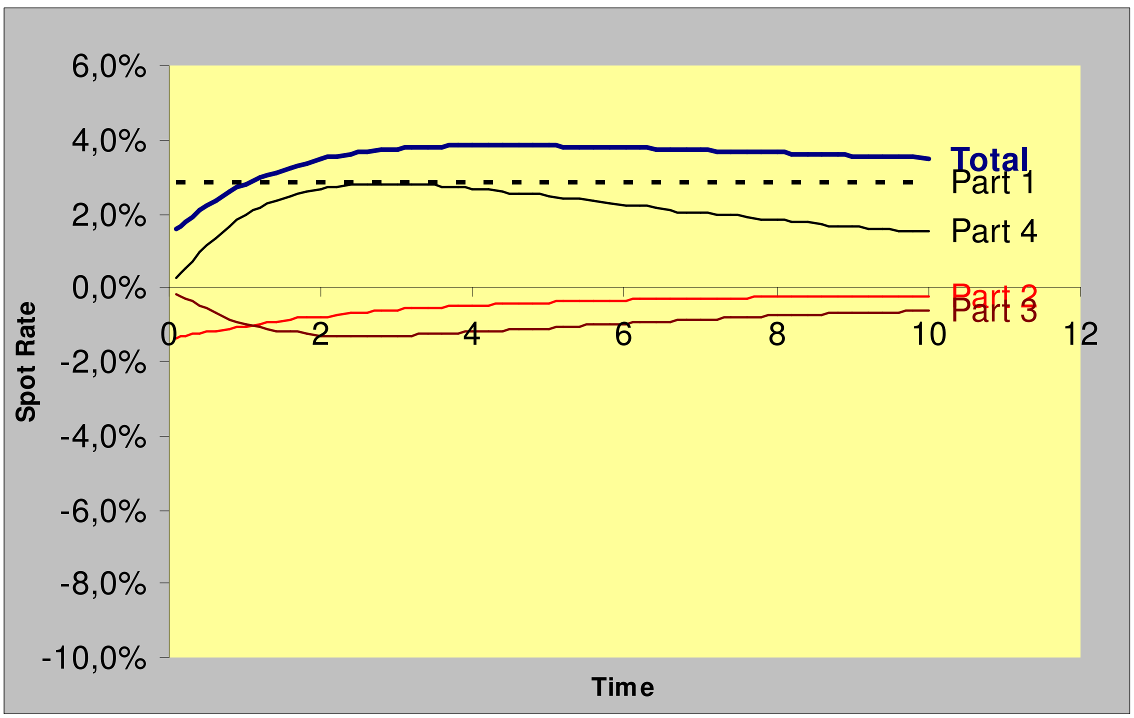
Then when the minimization has been calculated, we will have following value of the NSE spot rate and the error for the sum of the bonds, and the following graphs:

Issuer	Coupon	Maturity	Clean	Dirty	NSE Price	Duration	Price	Yield	Φ	Squared bond price error
Swed Government Bond 1037	8,00%	2007-08-15	103,58%	105,91%	106,12%	0,69	106%	2,69%	0,712	0,0002%
Swed Government Bond 1040	6,50%	2008-05-05	104,72%	105,17%	108,27%	1,36	109%	3,03%	1,432	0,0003%
Swed Government Bond 1043	5,00%	2009-01-28	103,56%	105,26%	107,13%	2,01	108%	3,25%	2,099	0,0011%
Swed Government Bond 1034	9,00%	2009-04-20	112,67%	113,67%	117,56%	2,15	118%	3,35%	2,461	0,0009%
Swed Government Bond 1048	4,00%	2009-12-01	101,59%	103,58%	104,55%	2,88	102%	3,43%	2,830	0,0106%
Swed Government Bond 1045	5,25%	2011-03-15	106,40%	107,49%	108,98%	3,82	110%	3,61%	4,063	0,0009%
Swed Government Bond 1046	5,50%	2012-10-08	109,13%	109,92%	109,31%	5,15	110%	3,73%	5,461	0,0002%
Swed Government Bond 1041	6,75%	2014-05-05	118,66%	119,13%	122,95%	6,01	123%	3,81%	7,098	0,0000%
									sum	0,0142%

Spot rate at time t	$r_{t,i}$	3,83%



3,83%
5,0%
4,5%
4,0%
3,5%
3,0%
2,5%
2,0%
1,5%
1,0%
0,5%
0,0%
Spot Rate
0
2
4
6
8
10
12
Years



6,0%
4,0%
2,0%
0,0%
-2,0%
-4,0%
-6,0%
-8,0%
-10,0%
Spot Rate
Total
Part 1
Part 4
Part 2
Part 3
0
2
4
6
8
10
12
Time