# BOOTSTRAP A GOVERNMENT ZERO CURVE

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# Content

**Bootstrap a Government zero curve**

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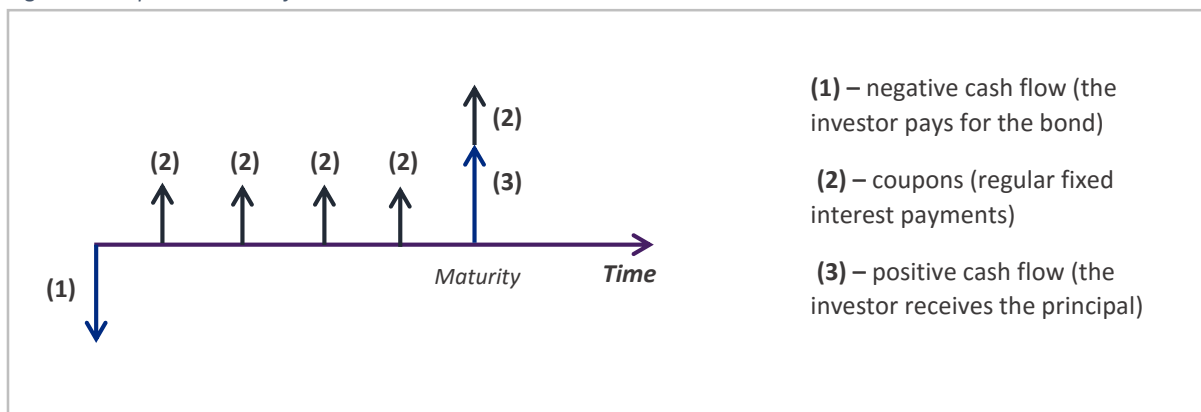**MÄLARDALENS HÖGSKOLA**
**ESKILSTUNA VÄSTERÅS**

## Bonds

Bonds are one of the simplest fixed-income securities. When a bond is issued, the investor is essentially lending money to the issuer in exchange of interest payments and the promise of repayment of the principal at a future date. Most bonds also trade in the secondary market, like stocks and other securities. Understanding the pricing of a bond and, in particular, the relative valuation of different bond issues is very important for investors and traders.

There are 3 parameters that define the structure of cash flows that the investor expects to receive if the bond was held to maturity:

- ✓ Principal (notional amount of the loan)
- ✓ Maturity
- ✓ Interest payment

The interpretation of these cash flows are shown in the *Figure 1*.

Figure 1 *"Coupon bond cash flow"*



**(1)** – negative cash flow (the investor pays for the bond)

**(2)** – coupons (regular fixed interest payments)

**(3)** – positive cash flow (the investor receives the principal)

When a trader buys a bond, the price is normally given as the clean price. The dirty price is equal to the clean price plus the accrued interest since the last coupon. Therefore, the price of the bond as function of the yield-to-maturity can be written as:

$$P = \frac{N}{(1 + ytm)^T} + \sum_{i=1}^{n} \frac{C}{(1 + ytm)^{t_i}}$$

*where*

*N* – the face value expressed as a percentage of the nominal amount
*T* – time to maturity
*C* – coupon payments expresses as a percentage of the nominal amount
$t_i$ - the number of days to the payments divided by the number of years (using 30 days per months and 360 days per year)
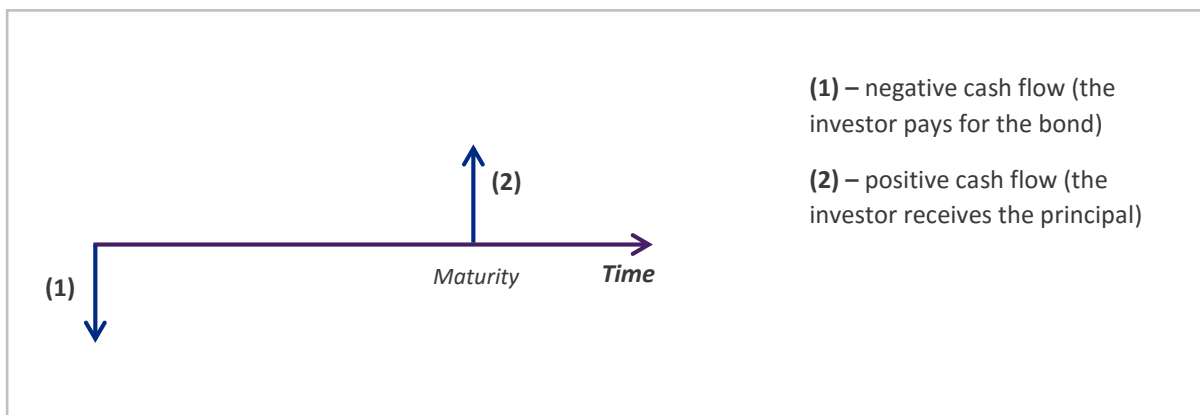*ytm* – the quoted yield

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**Zero-coupon bonds**

A zero-coupon bond (also called a "pure discount bond") is a bond that has a single payment of principal at maturity, without any intermediate interest payments (coupons). As zero-coupon bonds offer the entire payment at maturity, they tend to fluctuate in prices much more than coupon bonds. Examples of zero-coupon bonds include U.S. Treasury bills, U.S. savings bonds, long-term zero-coupon bonds and any other type of coupon bond that has been stripped of its coupons.

The cash flow of zero-coupon bond is shown in the *Figure 2*.

Figure 2 *"Zero-coupon bond cash flow"*



**(1)** – negative cash flow (the investor pays for the bond)

**(2)** – positive cash flow (the investor receives the principal)

Obviously, zero-coupon bond's dirty price is given by the quoted yield as:

$$P = \frac{N}{(1 + ytm)^T}$$

*where*

$N$ – the face value expressed as a percentage of the nominal amount
$T$ – time to maturity
$ytm$ – the quoted yield

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## The main idea of bootstrapping

Because the Treasury bills offered by the government are not available for every time period, the bootstrapping method is used to fill in the missing figures in order to derive the yield curve. The bootstrap method uses interpolation to determine the yields for Treasury zero-coupon securities with various maturities.

In general, bootstrapping is a procedure, repeated over and over again, to convert the yield on coupon bonds into yields on equivalent zero-coupon bonds.

### Linear interpolation/extrapolation

To bootstrap missing values for the Treasury bills, linear interpolation/extrapolation will be used. This is very simple method how to evaluate missing data, however one disadvantage of this method can be found – the curve can get sharp angles in the intersection points (which gives jumps when the forward rates will be calculated).

Knowing the meaning of prefixes "extra-" *(outside)* and "inter-" *(in between)* it is not hard to understand that interpolation is used to estimate the missing value within the range of available data, but extrapolation is used to estimate the value beyond the range of data set.

In order to interpolate/extrapolate the yield in the time *t,* we use the following formula:

$$Y_i(t) = Y_i + \frac{Y_{i+1} - Y_i}{T_{i+1} - T_i} * (t - T_i)$$

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## The spot rate

The spot rate or short rate is defined as the theoretical profit given by a zero coupon bond. This rate is used to calculate the amount we will get in the future (at time $t_1$) if we invest value X today (at time $t_0$):

$$X_{t_1} = X_{t_0} * (1 + r_{spot})^{t_1}$$

The spot rate is calculated by bootstrapping of by fitting the yield curve.

## The forward rate

From the yield curve describing the interest rates that apply between the current date and a future date, it is possible to determine an implied forward rate, i.e. the rate that "should" apply between two future dates. The formula for implied forward rates is based on an arbitrage argument, where the rate for a specific nominal amount between two future dates can be locked in by borrowing and lending at the current rates to the future dates.

A projection of the future interest rate, from one time to another, is calculated from the spot rate or a yield curve. The relationship between the spot rate and the forward rate is given by:

$$r_{t_2-t_1}^{forward} = \left( \frac{(1 + r_{t_2}^{spot})^{t_2}}{(1 + r_{t_1}^{spot})^{t_1}} \right)^{\frac{1}{t_2-t_1}} - 1$$

## The discount function

The discount function $p(t_0, t)$ describes the present value at time $t_0$ of a unit cash flow at time $t$. This is a fundamental function and the discount factors are non-random and equal in all banks due to arbitrage conditions.

Usually $t_0$ is the current time ($t_0 = 0$), but the remaining variable $t$ refers to the time between $t_0$ and $t$. The discount function is used as the base for all other interest rates. This function also represents the value of a zero-coupon bond at time $t_0$ with maturity $t$. At maturity a zero-coupon bond pays one cash unit, therefore $p(t, t) = 1$. At the time t = 0, the discount function always has the value 1 ($p(0, 0) = 1$).

The discount function is monotonically decreasing, which corresponds to the assumption that interest rates are always positive[1]. The discount function never reaches zero since all cash flows, no matter how far in the future they are paid, are always worth more than nothing.

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[1] If there exists such a case when interest rates are negative, the discount function is not monotonically decreasing anymore.

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The discount function has a mathematical relationship to the spot yield curve. The relationship between the discount function and the annually compounded yield curve, using a day count convention that reflects the actual time between time $t_0$ and $t$ measured in years, can be written as:

$$p(t) = \frac{1}{(1 + r_1(t))^t}$$

## Scheme of constructing zero-coupon yield curve

The zero coupon yield curve shows in graphical form the rates of return on zero coupon bonds with different periods to maturity. The main reason for constructing a zero coupon yield curve is for use as a basic tool in determining the price of many fixed income securities.

Market data:

Table 1 *"Swedish government bonds"*

| GOVERNMENT BONDS | | | |
|---|---|---|---|
| **Name** | **Coupon** | **YTM (%)** | **Maturity** |
| RGKB 1050 | 3 | -0,51 | 12.07.2016 |
| RGKB 1051 | 3,75 | -0,44 | 12.08.2017 |
| RGKB 1052 | 4,25 | -0,21 | 12.03.2019 |
| RGKB 1047 | 5 | 0,14 | 01.12.2020 |
| RGKB 1054 | 3,5 | 0,41 | 01.06.2022 |
| RGKB 1057 | 1,5 | 0,66 | 13.11.2023 |
| RGKB 1058 | 2,5 | 0,84 | 12.05.2025 |
| RGKB 1059 | 1 | 1,12 | 12.11.2026 |
| RGKB 1056 | 2,25 | 1,62 | 01.06.2032 |
| RGKB 1053 | 3,5 | 1,99 | 30.03.2039 |

Table 2 *"Swedish bills"*

| TREASURY BILLS | | |
|---|---|---|
| **Name** | **YTM (%)** | **Maturity** |
| SSV 1 månad | -0,43 | 16.12.2015 |
| SSV 2 månader | -0,43 | 20.01.2016 |
| SSV 3 månader | -0,44 | 17.02.2016 |
| SSV 4 månader | -0,44 | 16.03.2016 |
| SSV 5 månader | - | 20.04.2016 |
| SSV 6 månader | - | 18.05.2016 |
| SSV 7 månader | - | 15.06.2016 |

Initially we have 2 data tables: Government bonds *(Table 1)* and Treasury bills *(Table 2).* In fact, there is a lack of T-bill rates, therefore zero curve yield cannot be constructed properly because of missing values. This situation can be solved if we use available coupon bond data and bootstrap the missing values for Treasury bill rates. In order to construct smooth zero-coupon bond yield curve, we can follow the algorithm on the next page.
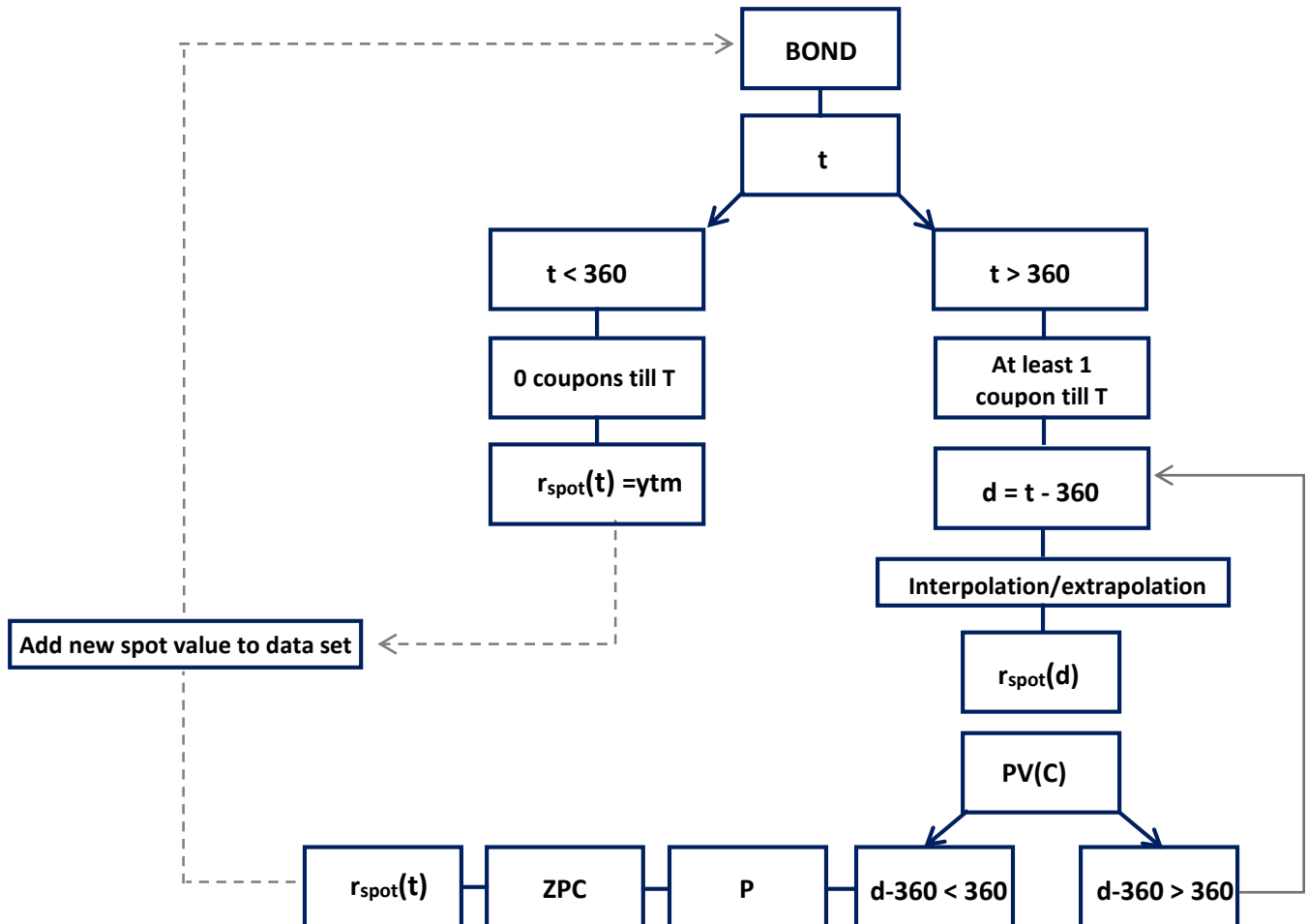
**The algorithm:**

1.  We take the first bond and calculate the time to maturity *(t)* using corresponding maturity date *(T):* $t = T - assumption\ date$
2.  Then we compare, whether time to maturity *(t)* is less than 360 days.
    2.1. If it truth, the bond does not have any coupon payments till maturity except the last one at maturity. Therefore, the spot rate at time *t* will be the same as coupon bond's yield to maturity *(ytm).*
    2.2. This value *($r_{spot}(t)=ytm$)* now can be added to the T-bill table. That means – we have new extra value that can be later plotted.
3.  In the most cases, time to maturity *(t)* is higher than 360 days and then the bond has one coupon payment at maturity *(T)* and at least one coupon till that time.
    3.1 By subtracting 360 days from time to maturity *(T)* we can calculate the day *(d)* when the coupon payment will be made: $d = t - 360$
    3.2 Then we bootstrap the spot rate at date *d* by using linear interpolation or extrapolation.
    3.3 This rate now is used to calculate coupon's present value *PV(C):*
    $$PV(C) = \frac{C}{\left(1 + r(d)\right)^{d/360}}$$
4.  Further, we check existence of other coupon payments. If we imagine timeline, we are moving from the right side to the left side year by year. In other words – from calculated date *d* we subtract one more year.
    4.1. If this value is higher than 360 days, we go to the step *3.2.* and continue the same steps till we don't have coupons anymore (except the very last one at maturity). Otherwise, we go to the step 5.
5.  If we have reached the situation when *d* value is less than 1 year, we can calculate the price of coupon's bond (*P*) using all calculated coupon payments *PV(C). (See the page 3)*
6.  Afterwards, the zero-coupon price is calculated: $ZPC = P - PV(C)$
7.  The previous calculated value allows us to determine our target value - spot rate for the time *t*, which describes involved bond's time to maturity. This value now can be added to the T-bill data set.
8.  We take the next bond from the *Table 1* and start all algorithm from the beginning.
9.  When all possible zero-coupon rates are estimated, we have plenty of data points to construct proper zero-coupon yield curve.

The graphical interpretation of this algorithm is shown in the *Scheme 1 (Page 9).*

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Scheme 1 *"Bootstrap zero-coupon bond curve"*

```
                              ┌──────────┐
        ┌────────────────────>│   BOND   │
        │                     └──────────┘
        │                     ┌──────────┐
        │                     │    t     │
        │                     └──────────┘
        │                    /            \
        │            ┌──────────┐      ┌──────────┐
        │            │ t < 360  │      │ t > 360  │
        │            └──────────┘      └──────────┘
        │                 │                 │
        │            ┌──────────┐    ┌──────────────┐
        │            │0 coupons │    │  At least 1  │
        │            │ till T   │    │ coupon till T│
        │            └──────────┘    └──────────────┘
        │                 │                 │
        │          ┌─────────────┐   ┌──────────────┐
        │          │r_spot(t)=ytm│   │ d = t - 360  │<──┐
        │          └─────────────┘   └──────────────┘   │
        │                 │                 │           │
   ┌──────────────┐       │        ┌──────────────────┐ │
   │ Add new spot │<──────┘        │Interpolation/    │ │
   │value to data │                │extrapolation     │ │
   │     set      │                └──────────────────┘ │
   └──────────────┘                        │            │
        │                           ┌──────────────┐    │
        │                           │  r_spot(d)   │    │
        │                           └──────────────┘    │
        │                           ┌──────────────┐    │
        │                           │    PV(C)     │    │
        │                           └──────────────┘    │
        │                              /        \       │
 ┌──────────┬──────┬──────┬──────────────┬──────────────┐
 │r_spot(t) │ ZPC  │  P   │ d-360 < 360  │ d-360 > 360  │
 └──────────┴──────┴──────┴──────────────┴──────────────┘
```

**Notations:**

$t$ – *time to maturity*

$T$ – *maturity*

$r_{spot}$ – *zero-coupon rate*

$ytm$ – *yield of the bond*

$d$ – *coupon payment day*

$PV(C)$ – *present value of the coupon*

$P$ – *price of the coupon bond*

$ZPC$ – *price of the zero-coupon bond*

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## The *Python* code

```python
# BOOTSTRAPPING SWEDISH GOVERNMENT BONDS
from scipy.stats import *
from numpy import *
from matplotlib.pyplot import *
from datetime import *


### Parameter Values ###

        ### BONDS ###
#Today
StartD=date(2015,11,18)
N=360.0
Nominal=100.0

#Coupon
C=[3.0, 3.75, 4.25, 5.0, 3.5, 1.5, 2.5, 1.0, 2.25, 3.5 ]

#Yield to maturity (%)
YTM=[-0.0051, -0.0044, -0.0021, 0.0014, 0.0041, 0.0066, 0.0084, 0.0112, 0.0162, 0.0199]

#Maturity date
D1=date(2016,7,12)
D2=date(2017,8,12)
D3=date(2019,3,12)
D4=date(2020,12,1)
D5=date(2022,6,1)
D6=date(2023,11,13)
D7=date(2025,5,12)
D8=date(2026,11,12)
D9=date(2032,6,1)
D10=date(2039,3,30)

        ### BILLS ###

#Days
Days=[30.0,60.0,90.0,120.0]

#Yields
Yields=[-0.0043, -0.0043, -0.0044, -0.0044]

z=zip(Days,Yields)


def bond(Coupon, ytm, Date,z):
    M=Date-StartD # Time to maturity
    M=M.days
    T=M
    PVC=[]
    while T>N:   # while time to maturity is more than year
        T=T-N    # the time when coupon payment will be made
        t=T // N # how many coupon payments have been made
        tt=T%N   # the time when the first coupon payment will be made
        S1=min(Days, key=lambda x:abs(x-T)) # finding the nearest value
        n=Days.index(S1) # the index in the list for the nearest value

        #interpolation/extrapolation
        if S1<T and S1==Days[-1]:
            Y=z[n][1]+(z[n][1]-z[n-1][1])/(z[n][0]-z[n-1][0])*(T-z[n][0])
        elif S1<T:
            Y=z[n][1]+(z[n][1]-z[n+1][1])/(z[n][0]-z[n+1][0])*(T-z[n][0])
```

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```python
        else:
            Y=z[n][1]+(z[n][1]-z[n-1][1])/(z[n][0]-z[n-1][0])*(T-z[n][0])
            PVC.append(Coupon/(1+Y)**(t+tt/N)) #present value for coupons
    PVC=sum(PVC)
    TT=M//N # how many coupon payments will be made

    Price=(1/(1+ytm)**(M/N))*(Nominal+Coupon*(((1+ytm)**(TT+1)-1)/ytm))#price of the bond
    ZCB=Price-PVC # zero coupon bond
    ZCR=((Nominal+Coupon)/ZCB)**(N/M)-1 # zero coupon rate

    # Appending the calculated rate and days to the list
    S2=min(Days, key=lambda x:abs(x-M))
    n2=Days.index(S2)
    if M>S2:
        z.insert(n2+1,(M,ZCR))
        Days.insert(n2+1,M)
        Yields.insert(n2+1, ZCR)
    else:
        z.insert(n2, (M,ZCR))
        Days.insert(n2, M)
        Yields.insert(n2,ZCR)
    return z

# Calculating spot rate curve using government bonds
a1=bond(C[0], YTM[0], D1,z)
a2=bond(C[1],YTM[1], D2, a1)
a3=bond(C[2], YTM[2], D3,a2)
a4=bond(C[3], YTM[3], D4,a3)
a5=bond(C[4], YTM[4], D5,a4)
a6=bond(C[5], YTM[5], D6,a5)
a7=bond(C[6], YTM[6], D7,a6)
a8=bond(C[7], YTM[7], D8,a7)
a9=bond(C[8], YTM[8], D9,a8)
a10=bond(C[9], YTM[9], D10,a9)


i=len(Yields)-1
F=(((1+Yields[i])**(Days[i]/N))/((1+Yields[i-1])**(Days[i-1]/N)))**(N/float(Days[i]-
Days[i-1]))-1 # forward rate for the last given date


e=25.0 # calculating spot rates for more 10 years
R10=((1+Yields[i])**(Days[i]/N)*(1+F)**((e*N-Days[i])/N))**(1/e)-1
Yields.append(R25)
Days.append(e*N)
r=R25
for k in range(1,10):
    e=e+1
    R=(((1+r)**(e-1))*(1+F))**(1/e)-1
    Yields.append(R)
    Days.append(e*N)
    r=R

s=len(Days)-1
D=[0]*(s+1)
FR=[0]*s

for q in range (0,s+1): # finding the discount function
    D[q]=1/(1+Yields[q])**(Days[q]/N)

for q in range (0,s): # finding the forward rates
    FR[q]=(((1+Yields[q+1])**(Days[q+1]/N))/((1+Yields[q])**(Days[q]/N)))**
    (1/((Days[q+1]/N-Days[q]/N)))-1
```

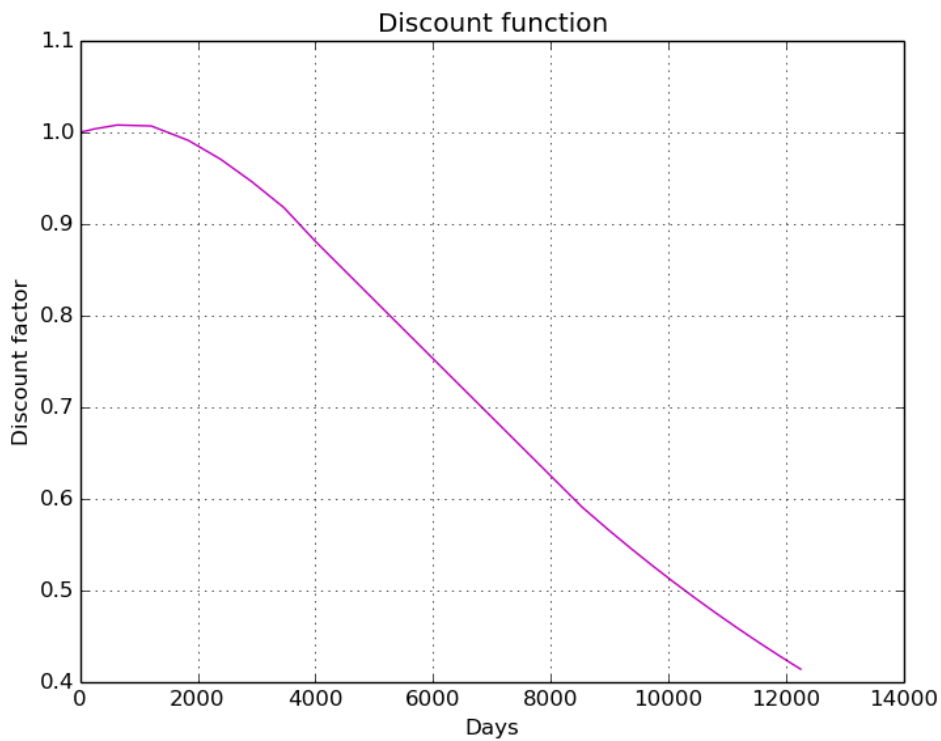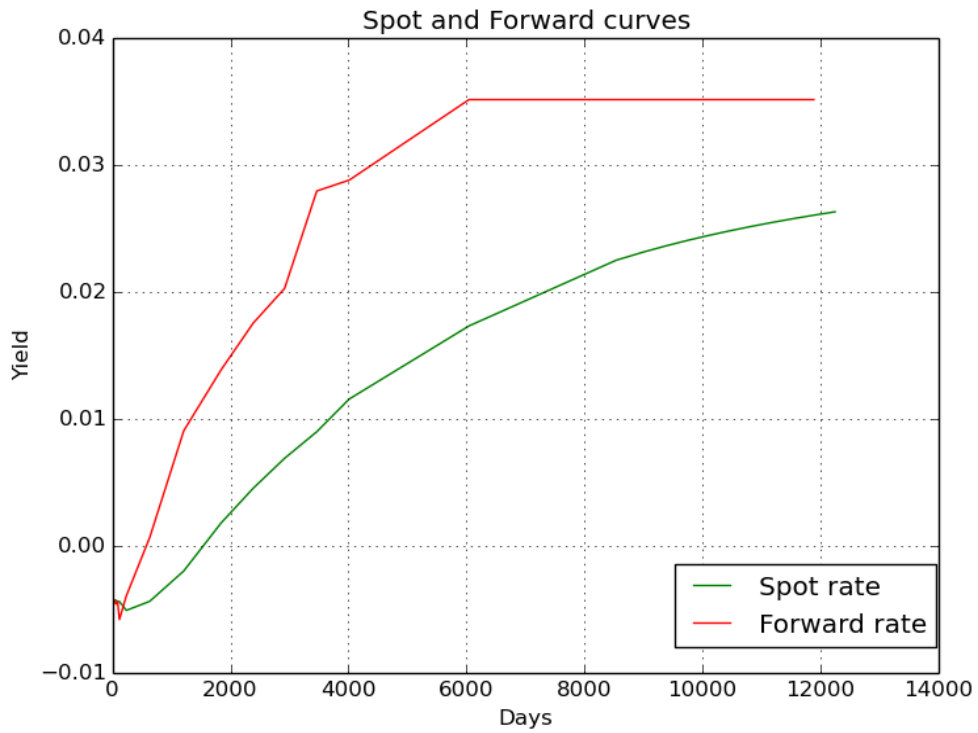*Gabi Balji*
*Liene Dudele*
*Sanda Medniece*

```
### Plotting ###

figure(1)
grid()
plot(Days, Yields, 'g', label="Spot rate")
title('Spot and Forward curves')
xlabel('Days')
ylabel('Yield')

figure(2)
grid()
plot(Days, D, 'm')
title('Discount function')
xlabel('Days')
ylabel('Discount factor')

figure(1)
plot(Days[0:s], FR, 'r', label="Forward rate")
legend(bbox_to_anchor=(0.68, 0.17), loc=2, borderaxespad=0.)

show()
```

## Result plots

# References

1.  Lecture notes ''Analytical finance 2'' by Jan Jan Röman

2.  http://janroman.dhis.org/

3.  www.nasdaqomxnordic.com

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5.  http://www.investopedia.com/