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Valuing Caps Using Monte Carlo Simulations

Seminar report in Analytical Finance II

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Abstract

In this note we use Monte Carlo simulations to simulate a forward EURIBOR rate in order to price interest rate caps. Under the dynamics of the LIBOR market model (LMM), we use MATLAB to simulate a discretized version.

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1 Introduction

The forward rate is the rate of interest for a period of time in the future, implied by today′s zero rates. A forward rate can imply what the market feels about the future movements in the interest rates. Calculation of forward rates are usually done after a curve of zero rates have been estimated. Estimation of zero rates frequently involves stripping zero coupon bonds and interpolating and/or extrapolating between unknown points. However, alternative methods to generate a forward curve also exist. If one can derive the process of an interest rate, a simulation can be done by using Monte Carlo techniques. Under such a scenario, one usually assume a drift term with an added stochastic part to generate the forward curve.

The LMM model, also known as the Brace, Gatarek and Musiela Moldel (BGM Model) is used to price several interest rate derivatives such as ratchet caps and floors, auto caps and Bermudian swaptions. Each forward rate is modeled as a lognormal process which will lead to that one can employ the famous Black-76 model to value interest rate caps.

A cap or a floor is a derivative on an X-IBOR rate (e.g. LIBOR), and is depending on the path of the interest rate. In order to price such an instrument, one must have accessibility to the term structure of the interest rate. Given that one is able to simulate the X-IBOR curve for the lifetime of the contract, one can thus infer the price using the Black-76 formulas.

2 Theoretical Framework

2.1 Caps and Floors

A popular interest option, usually traded OTC, is an interest rate cap. A cap is designed so that it provides insurance against that interest rate (of a floating rate note) does not rise over a certain level, the *cap rate*. The floating rate, which can follow a X-IBOR rate, is usually reset at three months intervals. This means that the prevailing interest rate will change of the life of the contract. The frequency of the resets days is commonly known as the tenor of the cap.

For a cap with a cap rate of, say 5%, the underlying interest rate is observed at each reset day. If the prevailing rate at that time is below 5% there is no payoff of the cap at the next reset day. However, if the rate is above 5%, the holder of the cap is entitled to receive the difference between the observed market rate and the cap rate divided by the amount of reset days per year. This payment that will occur at the next reset day will be multiplied by the principal value of the cap.

A floor is the reverse of cap where a floor with a *floor rate* of 5% would give the holder payouts if the prevailing rate drops below the floor rate at any reset days. A floor thus acts as a protection that the interest rate does not fall below a certain level.

Since we have reset days at for example every three months, a cap can be valued as a portfolio of interest rate option where the holder either receives the difference of the market rate and the cap rate, or zero. If we let R_k be the EURIBOR rate between the time t_k and t_{k+1} being observed at time t_k with reset dates $t_1, t_2, \ldots, t_n$ where $t_{n+1} = T$. Then the payoff for a cap at time t_{k+1} where $k = 1, 2, \ldots, n$ is

$$L\delta_k \max(R_k - R_K, 0)$$

where $\delta_k = t_{k+1} - t_k$.

A cap can be seen as a portfolio of options on the EURIBOR rate each option call be valued individually. Using market lingo, each option is known as a caplet where a cap includes a basket of caplets. To value a caplet, one can employ the Black-76 formulas (see section 2.2). Similar argumentation will show that a floor containing n different floorlets can be valued was

$$L\delta_k \max(R_K - R_k, 0)$$

2.1.1 Flexi Caps, Auto and Chooser

There are several alternative caps and floors available. One example is a *flexi caps* where the number of exercisable caps is limited. In an *chooser cap* the holder chooses which caplets that can be exercised. For an *auto cap*, only a subset of caplets (not chosen by the holder), can be exercised. Due to the fact that only a subset of caplets can be exercised, the value of a flexi cap (auto and/or chooser) will be less than a standard cap were all caplets can be exercised.

2.2 Black-76 Revised

During the $70'$s, Black [3] extended the groundbreaking work made by Black and Scholes [4] and Merton [8]. Black now presented a framework to price derivatives based on future contracts. Under standard notation where the future price is assumed to follow a lognormal process, we have the forward price of an asset, F, the strike price, K, a risk-free interest rate, r, time to maturity, T, and a volatility of σ. The Black-76 formulas for a plain vanilla European put and call reads

$$c = e^{-rT}\left[F_0 N(d_1) - KN(d_2)\right]$$
$$p = e^{-rT}\left[KN(-d_2) - F_0(-d_1)\right]$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

where the function $N(x)$ is the cumulative probability distribution function for a standardized normal distribution. To price a caplet or a floorlet, one can show that the formulas will now become

$$\text{caplet} = L\delta_k P(0,t_{k+1})\left[F_k N(d_1) - R_K N(d_2)\right] \tag{1}$$

$$\text{floorlet} = L\delta_k P(0,t_{k+1})\left[R_K N(-d_2) - F_k N(-d_1)\right] \tag{2}$$

where d_1 and d_2 becomes

$$d_1 = \frac{\ln(F_k/R_K) + \sigma_k^2 t_k/2}{\sigma_k\sqrt{t_k}}$$
$$d_2 = \frac{\ln(F_k/R_K) - \sigma_k^2 t_k/2}{\sigma_k\sqrt{t_k}} = d_1 - \sigma_k\sqrt{t_k}$$

where $P(0,t_{k+1})$ is a discount function that are taken into account due to the fact that payments are done in arrears. L is the principal value of the contract (e.g. 1 million EUR).

2.3 The Libor Market Model

The LMM describes the arbitrage-free dynamics of the term structure of interest rate with respect to evolution of the forward rates. Compared to the Heath, Jarrow and Morton Model (1990) [2], the LMM was developed by Brace, Gatarek and Musiela (1997) [1] to overcome the drawbacks of HJM. The main issues with the HJM model is that it is difficult to calibrate and that it is expressed in terms of instantaneous forward rates that are not observable on the market [6].

Under the LMM, assume that $F_k(t)$ is the forward rate between times t_l and t_{k+1} seen at time t (expressed with a compounding period of δ_k on act/act day count). Furthermore, $m(t)$ is the index for next reset date at time t, $\zeta_k(t)$ is volatility of $F_k(t)$ at t and $\nu_k(t)$ is the volatility of zero-coupon bond prices $P(t,t_k)$ at time t. Under a one factor model and in a world that is risk-neutral with respect to $P(t,t_{k+1})$, the forward rate that is governed by the process

$$dF_k(t) = \zeta_k(t)F_k(t)dz$$

Where dz is a Wiener process. In a rolling forward risk-neutral world we have that

$$dF_k(t) = \zeta_k(t)[\nu_{m(t)}(t) - \nu_{k+1}(t)]F_k(t)dt + \zeta_k(t)F_k(t)dz \tag{3}$$

From 3 one can use Itô to come up with the following process of the forward rate

$$\frac{dF_k(t)}{F_k(t)} = \sum_{i=m(t)}^{k} \frac{\delta_i F_i(t)\zeta_i(t)\zeta_k(t)}{1+\delta_i F_i(t)}dt + \zeta_k(t)dz \tag{4}$$

We see that the forward rate evolves with a drift depending on volatility of the forward rate and bond prices with an added stochastic part depending on the volatility of the forward rate.

2.4 Alternative Models for Forward Rates and Short Rates

As previously mentioned, the HJM model is an alternative to LMM. For an account of the HJM model please refer to [2]. Under the HJM model, the instantaneous forward rates evolves under the following process

$$m(t,T,\Omega_t) = s(t,T,\Omega_t)\int_t^T s(t,\tau,\Omega_t)d\tau.$$

In contrast to LMM, which only uses the dynamics of short rates, the HJM model uses the term structure based on the forward rate curve, and therefore is considered more feasible to use when modeling long term interest rate evolution's. For modeling of the short rate, there exist several famous models. Without any presentations, famous short rate model are e.g.

- Vasicek Model
- Cox, Ingersoll, and Ross Model
- Ho-Lee Model
- Hull-White Model

3 Implementation of the LMM Model

In order to employ (4) above, we must first get to a discretized version of it. According to [6], an approximation (where we assume that $F_i(t) = F_i(t_j)$ for $t_j < t < t_{j+1}$) is

$$F_k(t_{k+1}) = F_k(t_j)\exp\left\{\left(\sum_{i=j+1}^{k}\frac{\delta_i F_i(t_j)\Lambda_{i-j-i}\Lambda_{k-j-1}}{1+\delta_i F_i(t_j)} - \frac{\Lambda_{k-j-1}^2}{2}\right)\delta_j + \Lambda_{k-j-i}\varepsilon\right\} \quad (5)$$

where ε is a random sample from a normal distribution with mean zero and standard deviation of one. In MATLAB, we employ a equivalent version of (5), developed by The Numerical Algorithms Group (1997) [7]. This discrete version reads

$$\tilde{L}^i_{T_{j+1}} = \tilde{L}^i_{T_j}\exp\left\{\sigma^i\left(\sum_{k=j+1}^{n}\frac{\tau_k L^k_{T_j}\sigma^k}{1+\tau_k L^k_{T_j}} - \frac{1}{2}\sigma^i\right)\tau_j + \sigma^i\sqrt{\tau_j}Z_j\right\} \quad (6)$$

where in (6), we see that $\tilde{L}^i_{T_j}$, $\forall\, i,j$, is a recursively simulated value whereas $L^k_{T_j}$, $\forall\, k,j$ is a known value, which we got from bootstrapping spot rates from stripped EURIBOR notes, bills and bonds. The result of each evolution is a curve seen in Figure 1, simulated by the function in Appendix A.2. The volatility is for practical reasons considered constant, but if desired, one can obtain and use a volatility surface from actual market-priced caplets if one inverts the Black-76 formula and solve for volatilities [5].

Since this simulated curve just represents one iteration, we simulate this curve 100 000 times and take the mean curve by the simulation function in Appendix A.2, which is fairly close to the bootstrapped curve.

The discrete version of LMM in (6) only considers values of the interest rate at reset dates, i.e the simulated values are based on the time increments $\tau_i = T_{i+1} - T_i$, $\forall\, i$, and gives a finite set of valuation points. The deterministic part of the model is a summation of a ratio of known rates and the diffusion we consider constant, as mentioned above.

4 Results

4.1 A Forward Rate Simulation

Letting MATLAB simulate equation (6) , we plot the trajectory of one simulation in Figure 1 below.

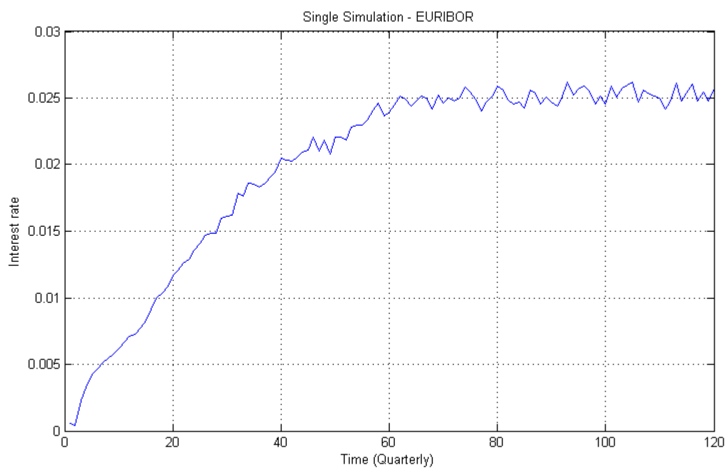

Figure 1: **Single Simulation Example:**
We use the bootstrapped EURIBOR spot rates and simulate one trajectory, using Equation (6).

From Figure 1, we can see that the simulated forward follows a pattern that is commonly observed for EURIBOR rates were we expect a higher forward longer into the future we get.

4.2 Valuation of Caplets Under one Simulation

Under one simulation of the forward curve we will value a cap. We let the cap rate be fixed at 0.4% on a EURIBOR rate. The time between each reset of the EURIBOR rate is 3 months and the principal value equals 1 M EUR. For simplicity, we assume a constant volatility of 25%. The lifetime of the contract is 10 years.

We get the following matrix of forward rates

$$\tilde{\mathbf{L}}(t_0, T_{40}) = \frac{1}{100} \begin{bmatrix} 0.33 & 0.41 & 0.448 & 0.485 & 0.521 & 0.556 & 0.601 & 0.645 & 0.689 & 0.733 \\ 0.788 & 0.844 & 0.899 & 0.954 & 1.011 & 1.065 & 1.121 & 1.177 & 1.226 & 1.274 \\ 1.324 & 1.371 & 1.415 & 1.457 & 1.501 & 1.544 & 1.593 & 1.643 & 1.692 & 1.741 \\ 1.766 & 1.788 & 1.814 & 1.836 & 1.869 & 1.901 & 1.932 & 1.962 & 1.995 & 2.026 \end{bmatrix}$$

By using (1) in order to value each caplet independently, use quarterly reset dates and a notional of 1 Million EUR, we come up with the following matrix of values for each caplet:

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0.00062 & 0.1342 & 0.6036 & 1.018 & 1.501 & 2.0485 & 2.7255 & 3.653 \\ 4.659 & 5.71 & 6.941 & 8.781 & 10.638 & 12.51 & 14.5 & 17.18 & 19.69 & 22.12 \\ 24.52 & 27.7 & 30.44 & 32.96 & 35.76 & 38.9 & 41.78 & 44.18 & 46.82 & 51.47 \\ 55.25 & 58.42 & 60.84 & 64.15 & 66.15 & 67.43 & 68.98 & 73.24 & 76.1 & 78.01 \end{bmatrix}$$

Since the value of a cap is the value of all the caplets, we simply sum the values of all the caplets. The resulting value for a vanilla Cap with all caplets included is then 1177.67 EUR. For protection of 1 M EUR you pay $\sim 0.118\%$ for the contract with quarterly maturities ranging over 10 years.

4.3 Multiple Forward Rate Simulations

Repeating the simulation multiple times, results of 300 simulations can be found in Figure 2. Here one can see that at the simulated forward rates follow a certain pattern while diverging from the bootstrapped curve due to the volatility and the Wiener process. Due to the nature of the LMM model, we get theoretical curves very similar to the "actual" bootstrapped curve, as shown in red color.

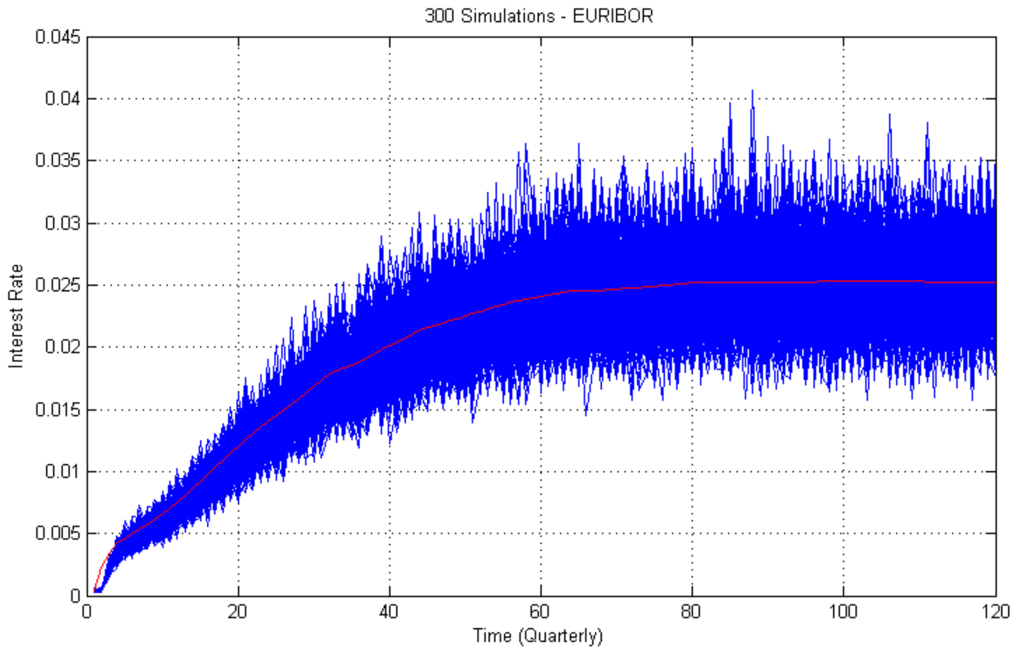

Figure 2: **Multiple Curves Simulation**
The Figure shows 300 simulations in blue color, with the bootstrapped curve in red color.

4.4 Flexible Caps, Auto and Chooser Simulations

We now perform the same simulation as in section 4.3 for tweaked version of caps. The possibilities of different contracts are endless so we just examine some simple examples. As one example, lets suppose we are interested in just having a Chooser Cap with yearly maturities, for that contract, the buyer would pay for $C_4, C_8, C_{12}, \ldots, C_{40}$, which is a basket of ten caplets. The value would be $\sum_{i=1*4}^{10} C_i = 261.73$ EUR.

Another example would be an Flexible Cap specified by the seller that is limited to protection in the last ten caplets, for that the buyer would pay $\sum_{i=1+30}^{10} C_i = 668.66$ EUR.

Suppose we have an Auto Cap, only the ten first caplets are exercised. In that case, the price of the Cap would be $\sum_{i=1}^{10} C_i = 11.6844$ EUR.

5 Discussion & Conclusion

In this note, we use Monte Carlo simulations to simulate a forward EURIBOR rate which is used to find the theoretical value of different interest rate cap derivatives. In order to simulate the forward curve, we employ a discrete version of the LMM from [7]. Understanding the complex dynamics of the forward rate is a challenge. The result from our MATLAB programs are appealing when they are similar to a bootstrapped curve from real market data.

Valuation of caps and floors are rather simple in theory. Evaluating more complex versions of caps becomes a challenge when the value of the caps are strongly dependent on which caplets that can/will be exercised. In section 4, we only present some conceivable instruments that in no way reflects the caps traded on the real market. Obviously, we would get more validity if we price "real" instruments. We can conclude that the vanilla cap is more expensive than any basket containing a subset of caplets, such as the Auto-, Flexible- and Chooser Cap.

References

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A Appendix

A.1 Excel Graph

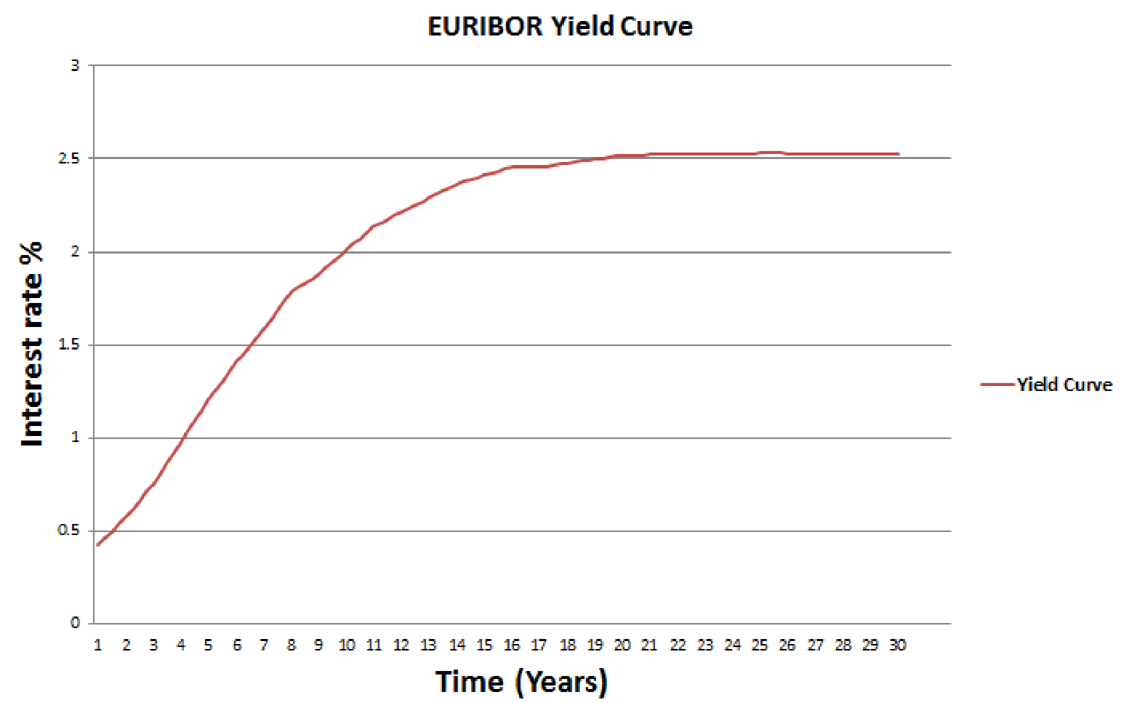

Figure 3: **Bootstrapped curve**
This curve is bootstrapped from 6 month EURIBOR notes, bills and bonds, using extrapolation and interpolation.

A.2 MATLAB Code

Forward rate function

```
% ForwardRate function simulates the Forward rate from t0 up to T

function L = ForwardRate(sigma, L_0, tau, n, L)
 L(1) = L_0(1).*exp((sigma.*(tau.*L_0(1).*sigma)/(1 + tau*L_0(1))...
        - 0.5*sigma).*tau + sigma.*sqrt(tau).*randn);
for i = 2:n
    L(i) = L_0(i-1).*exp((sigma.*(tau.*L_0(i).*sigma)/(1 + tau*L_0(i))...
        - 0.5*sigma).*tau*.01 + sigma.*sqrt(tau*0.01).*randn);
end
```

```
end
```

Simulation

```
function Meanrate = Simulation(tau, sigma, L_0, m, n,L)

for s = 1:m
    L = L+ InterestRates(sigma, L_0, tau, n,L);
end

Meanrate = 1/m*L;
end
```

Main Program

```
clc
clear all
close all
tic
[L_0,BDates,MDates]= Storage();
n = length(L_0);
tau = 30/n;
m = 1;
sigma = 0.25;
g = 1; % No. of graph to be be ploted
for i = 1:1:g
    L = zeros(n,n); % Storage vector for Interest rates
    Meanrate = Simulation(tau, sigma, L_0, m, n, L);
    L = Meanrate;
    L(L==0) = nan;
    plot(L(1,:))
    hold on
    grid on

end
plot(L_0, 'r')
toc
```