Exotic cap pricing

Seminar report in Analytical Finance II

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Content

Introduction

This paper assesses the valuation process of two exotic interest rate caps with an adjusted Libor market model. We create a valuation model for two exotic barrier caps namely a dual strike and knock out Cap.

What is a Cap?

An interest rate Cap or Floor is a portfolio of European Interest rate call options named caplets, or floorlets respectively which are linearly combined. A Cap can therefore be priced by adding the sum price of all individual caplets and floors as the sum of all floorlets.

The buyer of Caps receives payments at the end of each period in which the interest rate exceeds the agreed strike rate. Similarly an interest rate floor is a derivative contract in which the buyer receives payments at the end of each period in which the interest rate is below the agreed strike rate.

It is a popular financial instruments offered by financial institutions in the over-the-counter (OTC) market. This particular instrument is designed to provide insurance against the rate of interest on floating-rate notes rising above a certain level.
The parameters for such instruments are commonly:

1. Notional Payment / Face Value, denoted as N
2. Cash Flow Dates, denoted as t_i
3. Floating Strike (Interest rate), denoted as R_k
4. Strike Rate (Cap rate), denoted as R_K

At each interest payment date the holder of a Cap decides whether to exercise or let that particular option expire. In an interest rate cap, the seller agrees to compensate the buyer for the amount by which an underlying short-term rate exceeds a specified rate on a series of dates during the life of the contract. Interest rate caps are used often by borrowers in order to hedge against floating rate risk. When the sequence of dates is such that the payoff of the first caplet is known on the date the cap is entered into, the convention is to disregard the first caplet. Prices of caps and floors are generally quoted in the market in terms of Black's implied volatility and therefore they also display variation with skew and smiles.

The payoff of a cap can be observed similar as a cap option where the payoff a cap is $N\delta_k \max(R_k - R_K, 0)$. A floors payoff is $N\delta_k \max(R_K - R_k, 0)$ where R_k is the interest rate is and R_K is the Cap rate. There is a put-call parity relationship between the prices of caps and floors. The put call parity value of a Cap is represented as

$$V_{cap} = V_{floor} + V_{Swap}$$

The swap is an agreement to receive LIBOR and pay a fixed rate of the cap rate R_K.
The cap therefore provides a cash-flow of $LIBOR - R_K$ when Libor > R_K.
The value of a cap, denoted *c*, and floor, *f*, is

$$c = N\delta P(0, t_{k+1})[F_k N(d_1) - R_K N(d_2)]$$

and

$$f = N\delta P(0, t_{k+1})[R_K N(-d_2) - F_k N(-d_1)]$$

where

$$d_1 = \frac{\ln\left(\frac{F_k}{R_K}\right) + \sigma_k^2 t_k/2}{\sigma_k \sqrt{t_k}}$$
$$d_2 = d_1 - \sigma_k \sqrt{t_k}$$

F_k is the forward interest rate at time 0 between t_k and t_{k+1} (δ), σ_k its volatility, and $P(0, t_{k+1})$ is the discount factor.
This formula is derived using Black model.

The valuation of a Cap can also be expressed as the sum of the value of all caplets in a portfolio. The valuation of a single caplet can be expressed as

$$C_t^i = N(1 + \delta R_K)\, \pi(t, \frac{1}{1 + \delta R_K}, t_i - \delta, t_i)$$

where

$\pi(t, K, S, T)$ is the price of a call option on a zero-coupon bond at time t, with the strike price K, expiry T and where the bond expires at time S.

To find the value of the whole Cap before the first reset date at t_0, we sum the valur of the caplets:

$$C_t = \sum_{i=1}^{n} C_t^i = N(1 + \delta K) \sum_{i=1}^{n} \pi\left(t, \frac{1}{1 + \delta K}, t_i - \delta, t_i\right).$$

Examples of caps would be:

1. An agreement to receive a payment for each month the LIBOR rate exceeds 2.5%.
2. An option that matures in 6 months may have a 5% rate for the first four months of its life and a 5.5% rate for the remaining two months.

Pricing caps

In order to calculate the price of dual strike and knock-out caps, we will simulate Libor rate, and then we will use Monte Carlo simulation to evaluate the caps. We are going to use C++.

The code for the price of a standard cap is:

```
double cap_price(int& N, double& dT, double& v, double& K, double& notional, int& n)
{
   for (i=0; i<=1000; i++)
   {
      ones[i][1]=0.5; // initial Libor rates
   }

   for (l=1; l<=N; l++) // calculate N cap prices (Monte Carlo)
   {
      W[1] = 0;

      for (k=1; k<=n; k++)
      {
         T[k] = k*dT;
         srand(time(NULL));
         alea=(rand() % (1001))/1000.0;
         if (k<n+1) W[k+1] = W[k] + sqrt(dT)*alea; // Wiener process simulation
      }

      for (i=1; i<=n+1; i++)
      {
         L[i][1]=ones[n+1][1]; // initial Libor
      }

      for (i=n+1; i>=1; i--)
      {
         for (k=1; k<=i-1; k++)
```

```
        {
           for (j=i+1; j<=n; j++)
           {
              sum = sum + (dT*v*L[j][k])/(1+dT*L[j][k]);
           }

           L[i][k+1] = L[i][k] - sum * v * L[i][k] * dT + v * L[i][k] * (W[k+1] - W[k]);
        }
     }
     for (k=1; k<=n+1; k++)
     {
        prod = 1;

        for (j=k; j<=n+1; j++)
        {
           prod = prod*(1/(1 + dT*L[j][k]));
        }

        B[n+1][k] = prod; //calculation of the numeraires
     }

     V = 0;

     for (i=1; i<=n-1; i++)
     {
        if (L[i][i]-K > 0) C[i+1] = (L[i][i] - K)/B[n+1][i+1];

        //calculation of the numeraire rebased
        //caplet payoffs
     }

     for (i=1; i<=n-1; i++)
     {
        V = V + C[i+1]; //a single numeraire rebased cap value
     }

     if (l==1) total = V;
     else total = total + V;
  }

  average = total/N;  //final estimate numeraire rebased
                 //value of the cap

  return notional*average;

}
```

Steps:

1) Put initial Libor rates.
2) Simulate Wiener process.
3) Calculate Libor.
4) Calculate caplet and cap prices.
5) Do average (Monte Carlo).

In order to calculate Libor, we use the discretization of the following SDE that Libor satisfies:

$$dL_i(t) = -\left(\sum_{j=i+1}^{n} \frac{\delta_j \sigma_j(t) L_j(t)}{1+\delta_j L_j(t)}\right) \sigma_i(t) L_i(t) dt + \sigma_i(t) L_i(t) dW^{n+1},\ i = 1, \dots, n-1$$

which is

$$L_i(T_{k+1}) = L_i(T_k) - \left(\sum_{j=i+1}^{n} \frac{\delta_j \sigma_j(T_k) L_j(T_k)}{1+\delta_j L_j(T_k)}\right) \sigma_i(T_k) L_i(T_k) \Delta T + \sigma_i(T_k) L_i(T_k) \left(W_{T_{k+1}}^{n+1} - W_{T_k}^{n+1}\right)$$

where $L_i(T_{k+1})$ is the forward Libor rate, δ_i is $t_{i+1} - t_i$, and W is the Wiener process.

We have also used that $C_i(t) / B_{n+1}(t)$ is martingale under the *Terminal measure*, where $C_i(t)$ is the present value of a caplet at time t, and $B_{n+1}(t)$ is the bond price.

In the calculations, we have supposed that δ and the volatility are constants.

Exotic Barrier Caps

In addition to the plain vanilla caps there are several contracts traded on the international OTC market where the cash flow are similar to a plain vanilla cap, though the contract deviates in one or more aspects and, therefore, the pricing of a cap will look considerably different depending on what parameters are being changed. An option whose payoff depends if the underlying asset has reached a predetermined level or barrier is known as a **barrier option**.
This paper focuses on knock-out, dual strike and sticky caps.

Knock-Out Caps

A knock-out option is a type of option that ceases to exist when the underlying asset reaches a certain trigger. Similarly a **knock-out cap** will at any time give the standard payoff unless the floating rate l(t+δ, t) during the period [t_i- δ, t_i] has exceeded a certain level. In that case, the payoff is zero. A knock-out cap therefore acts like a standard cap if $R_k < l$ but is terminated when $R_k \geq l$. The payoff for a knock-out option can be written as

$$If\ R_k < l = N\delta * Max(R_k - R_K, 0)$$
$$Else: 0$$

The price of a knock-out cap is cheaper than the price of a standard cap. For this reason, an investor would like to buy this exotic cap in order to get a cheaper cap but he loses premium if Libor is high. An investor would like to sell a knock-out because he will be protected against higher Libor, but, on the other hand, the price will be lower than if he would sell a standard cap.

Example: *An investor is very profitable if the Libor is greater than 15%. The rate is currently at 8%. To protect himself the investor purchases a knock-out cap, with trigger equal 15%. It is cheaper than buy a standard cap, and if the Libor is greater than 15%, he does not get payments from the cap, but still manages to be in the money.*

Dual strike Caps

A dual strike option can be referred as an option where the payoff is defined in the max payoff of two options with two different underlying and strikes.
The payoff of a dual strike call would hence be

$$Max(S_1 - K_1, S_2 - K_2, 0)$$

A dual strike cap is a modified barrier interest rate cap and is often referred to as an N-cap or as an indexed strike cap. This particular type of instrument cap strike $R_{K2} > R_{K1}$, where $R_{K1}$is the cap rate for a vanilla cap, an upper strike depending on the underlying floating rate l(t+δ, t), and a trigger at a pre-specified trigger level *l*.
The payout of a dual strike cap is hence

$$L(t + \delta_k, t)\ Max(R_k - R_K, 0)$$

Where R_K= K_1 if $R_k < l$ *and* $R_K = K_2$ *if* $R_k \geq l$. Hence the Dual strike cap work as one out of two different caps depending on whether or not the trigger has been activated. Due to the change in its strike rate, dual strike caps are cheaper than ordinary caps.
For this reason, an investor would like to purchase a dual strike cap in order to get a cheaper cap, but getting a lower premium if the Libor rate is high. An investor would like to sell it because he will be protected against higher Libor, but, on the other hand, the price will be lower than if he would sell a standard cap.

Example: *If an investor buys an N-cap with first strike rate equal to 5%, trigger at 10% and second strike rate at 14%. If Libor is lower than 5% in a payment date, then the payoff is zero. If Libor is between 5% and 10%, the payoff is Libor-5, since the strike rate will be 5%. If Libor is between 10% and 14%, the payoff is zero, since now the strike rate is 14% (in a standard cap, the payoff would be Libor-5). Finally, if Libor is greater than 14%, the payoff is Libor-14 (in a standard cap, the payoff would be Libor-5).*

In order to simulate dual strike and knock-out caps, we just need to modify slightly the cap pricing program, since standard caps and those exotic caps are really similar.

Sticky Caps

A **sticky cap** is like a standard cap, the only difference is its strike rate, which is given by
K_1 = min [K, m] and K_i = min [min {K_{i-1}, L_{i-1}} + X, m], for i>1, where X is the spread, and m is some level.

For instance, if we have s sticky cap with spread equal to 2%, level equal 10%, if the previous strike rate was 5%, and the previous Libor was 6%, then the today's strike rate is min[min {5, 6} + 2, 10] = min [7, 10] = 7%.

Each payment in the sticky cap depends on all previous payments. Then, the cap is said to be path-dependent.

Other exotic caps

Bounded cap: it is like a standard cap, but when the sum of payments received so far is greater than some specified level, then the payoff will be zero.

Ratchet cap: it is like a standard cap, the only difference is its strike rate, which is given by
K_1 = min [K, m] and K_i = min [K_{i-1} + X, m], for i>1, where X is the spread, and m is some level.

Flexi cap (or auto cap): it is like a standard cap, but only the first n in-the-money caplets are exercised (n is lower than the number of cash flows).

Chooser cap: it is a flexi cap where the holder can choose the caplets to exercise.