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Option prices using Vasicek and CIR

A seminar report in Analytical Finance II

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Abstract:
This report is aimed to describe Vasicek and Cox-Ingersoll-Ross models and estimate there parameters by using Solver in Excel. These parameters are then used to calculate prices of bonds and European options on bonds. The authors also give some background of term structure and derive a general term structure equation.

Introduction:

Interest rates and its dynamics are probably the most challenging to estimate in modern financial theory. The modern fixed income market includes not only bonds but also derivative securities sensitive to interest rates. This derivative market has forced to develop and originate new methods to model the term structure of interest rates. The Vasicek model and Cox-Ingersoll-Ross model described in this report are among well-known models of interest rates which are still actively used for estimation.

In this report the authors are not going to address the ability of Vasicek and CIR models. The objective of this report is how to estimate and use these models. The concept and little bit about mathematics involved behind these models is discussed here.

The authors begin this report with general background and theory of term structure and gradually move to Vaiscek and CIR models. A general term structure is derived where an individual is only required to insert a diffusion process of interest rates and solve it to get solution for pricing of a bond. Estimation of parameters of these models is done by using Solver in Excel. Prices of treasury bills of Sweden are used for estimating these parameters. These parameters are then used to calculate prices of bonds and options. The details of formulas for pricing are discussed in this report.

Background:

The term structure of interest rates (also known as yield curve) is a curve showing relation between yields of securities across different maturity time of similar contract. This curve is constructed by using benchmark zero coupon bonds. Bonds offered by government are considered as benchmark because they have very less probability of default. Short term bonds offered by government usually having time to maturity of 1 to12 months and are called treasury bills. They also have zero coupon and hence preferably used for constructing yield curve. Long term bonds offered by government have 1 to 30 years of maturity and are called treasury bonds. Since government bonds have very less probability of default, their yield is called risk free rate. Coupon bearing bond can also be divided into zero coupon bonds where each coupon acts as zero coupon bond.

The authors are only dealing with Swedish bonds to estimate the parameters of models and using them further for pricing of bonds and options. Swedish treasury bills and treasury bonds have expiry time of 1 to 6 months and 1 to 30 years respectively. This report only deals with European options

Term Structure Equation:

Suppose we denote $P(t,T)$ as price of pure discount bond at time t. Then yield to maturity, denoted by $R(t,T)$, of the bond maturing at time T is given by

$$P(t,T) = exp[-R(t,T).(T-t)]$$

We can write yield to maturity as,

$$R(t,T) = -\frac{\ln[P(t,T)]}{T-t}$$

From theory of finance, we are aware that we can also write yield to maturity as integral of forward rates. Thus,

$$R(t,T) = \frac{1}{T-t}\int_t^T f(t,s)ds$$

Hence we write price of pure discount form as,

$$f(t,T) = -\frac{\partial}{\partial s}\ln[P(t,T)]$$

Traditionally, to specify structure of interest rates we begin with assuming that it follows a diffusion process:

$$dr = \mu(r,t)dt + \sigma(t,T)dW(t)$$

Here $\mu(r,t)$ is a deterministic component, $\sigma(t,T)$ explains randomness of the process and W is a Wiener process. Given this assumption, price of a pure discount bond is a function of interest rate r, current time period t and time to maturity T. We denote price function as $P(r,t,T)$. Applying Ito's lemma on this:

$$dp = \frac{\partial p}{\partial t}dt + \frac{\partial p}{\partial r}dr + \frac{1}{2}\frac{\partial^2 p}{\partial r^2}(dr)^2$$

Substituting dr and $(dr)^2$ in above equation we get:

$$dp = [P_t + \mu(t,T)P_r + \frac{\sigma^2(t,T)}{2}P_{rr}]dt + \sigma(t,T)P_r dW$$

Where subscript denote the appropriate differential with respect to price. Dividing above equation by dt and taking expectation we get:

$$E\left(\frac{dp}{dt}\right) = P_t + \mu(t,T)P_r + \frac{\sigma^2(t,T)}{2}P_{rr}$$

In context of equilibrium pricing model this expected value must be equal to price times risk free interest rate r adjusted with risk premium λ.

$$r(1+\lambda)P = P_t + \mu(t,T)P_r + \frac{\sigma^2(t,T)}{2}P_{rr}$$

We can further simplify above equation as,

$$P_t + \mu(t,T)P_r + \frac{\sigma^2(t,T)}{2}P_{rr} - r(1+\lambda)P = 0$$

Usually Merton (1971, 1973) result is used to price risk premium. Merton's papers show that ratio of risk premium to standard deviation is constant when the primary function is in logarithmic form. Hence we write,

$$\frac{E(R_i) - r}{\sigma_{R_i}} = \frac{\lambda}{\sigma_{R_i}} = k$$

where $E(R_i)$ is expected return of asset i and σ_{R_i} is standard deviation of returns on asset i. The instantaneous rate of return of a bond is,

$$E(R) = \frac{P + dP}{P} = 1 + \frac{dP}{P}$$

Applying ito's lemma on standard deviation on returns,

$$\sigma_{R_i} = \frac{r\sigma(t,T)}{P}P_r$$

It thus follows that

$$\lambda = \frac{kr\sigma(t,T)}{P}P_r$$

Inserting above expression in term structure equation above,

$$P_t + \mu(t,T)P_r + \frac{\sigma^2(t,T)}{2}P_{rr} - rp - kr\sigma(t,T)P_r \ = 0$$

This is the final simplified term structure equation. Now the only thing remains is to find solution of above equations by using a solvable diffusion process of interest rate.

Now the authors move further moves to explain Vasicek model.

Vasicek Model:

Description:

The diffusion process described by Vasicek model is,

$$dr = a(b - r)dt + \sigma dV$$

This model assumes that short rate is normally distributes and has so called mean reverting process (under Q). Here b is long-term mean rate and a is the measure how fast short rate will reach the long term mean rate. We put this diffusion process in term structure equation and find a formula for pricing for pure discount bond.

$$P(0,T) = exp(A(0,T) - B(0,T)r(0)$$

Where,

$$B(0,T) = \frac{1}{a}[1 - e^{-aT}]$$

$$A(0,T) = -\frac{b}{a^2}[T - B] + \frac{\sigma^2}{2a^2}[T - 2B + \frac{1}{2a}(1 - e^{-2aT})]$$

Price of a pure discount bond can be calculated by above formula. Price of a coupon bond can also be calculated by dividing the bond into discount bonds where each coupon and face value acts as a discount bond.

The price of an option on pure discount bond is given by following formulas,

$$C(0,T,K,S) = LP(0,S)N(h) - KP(0,T)N(h - \sigma_P)$$

$$P(0,T,K,S) = LP(0,T)N(-h + \sigma_P) - LP(0,S)N(-h)$$

Where,

$$\begin{cases} h = \frac{1}{\sigma_P}\ln\left(\frac{L.P(0,S)}{K.P(0,T)}\right) + \frac{\sigma_P}{2} \\ \sigma_P = \frac{\sigma}{a}(1 - e^{-a(S-T)})\sqrt{\frac{1 - e^{-2aT}}{2a}} \end{cases}$$

We can also calculate prices of options on coupon bearing bond by dividing the option on whole bond into options on each coupon and the face value where each coupon and the face value act as discount bond.

Estimation of parameters:

Price of a discount bond in Vasicek model at current time period ($t = 0$) is given by,

$$P(0,T) = exp[A(0,T) - B(0,T)r(0)]$$

First of all we need to estimate $r(0)$. This interest rate is estimated by using treasury bills of Swedish government. Currently we have prices of five treasury bills maturing within next six months. We regress yield observed form these treasury bills against time to maturity. A screenshot of our working on excel is shown below.

Today	12/11/2012			
Face Value	100			
Bond	**Maturity**	**Price**	**T-t (yrs)**	**r**
RGKT1212	12/19/2012	99.98	0.022222	0.009002
RGKT1301	1/19/2013	99.91	0.105556	0.008534
RGKT1302	2/20/2013	99.815	0.191667	0.00967
RGKT1303	3/20/2013	99.75	0.275	0.009114
RGKT1306	6/19/2013	99.553	0.522222	0.008598
Hence				
r(0)	**0.009106**			

Picture: Screenshot of the extrapolation of $r(0)$

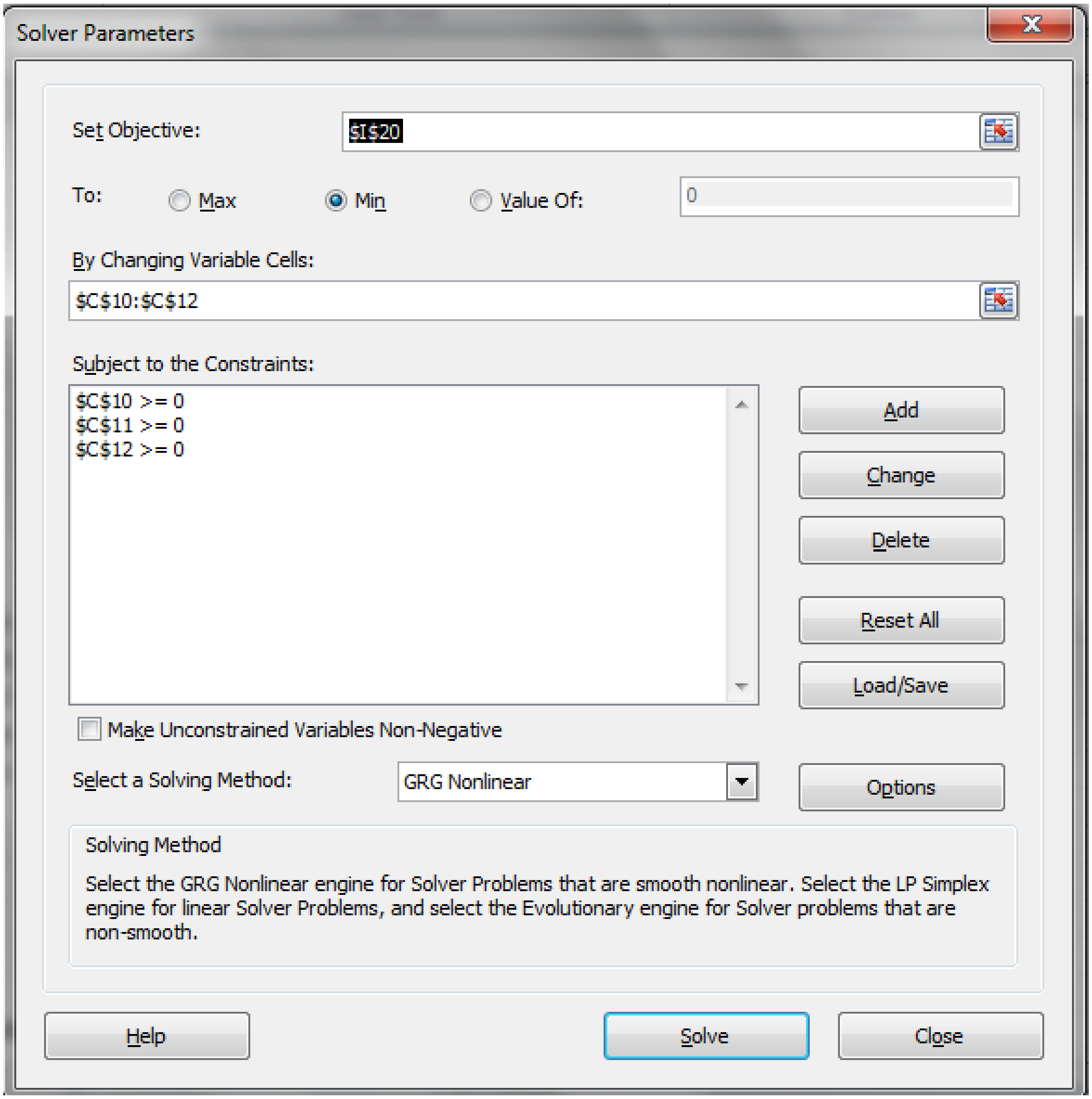
SUMMARY OUTPUT								
Regression Statistics								
Multiple F	0.230272							
R Square	0.053025							
Adjusted I	-0.26263							
Standard E	0.000515							
Observati	5							
ANOVA								
	df	*SS*	*MS*	*F*	*gnificance F*			
Regressio	1	4.45E-08	4.45E-08	0.167982	0.709421			
Residual	3	7.95E-07	2.65E-07					
Total	4	8.39E-07						
	Coefficients	*andard Err*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*ower 95.0%*	*pper 95.0%*
Intercept	0.009106	0.000378	24.10756	0.000156	0.007904	0.010308	0.007904	0.010308
X Variable	-1.5E-06	3.73E-06	-0.40986	0.709421	-1.3E-05	1.03E-05	-1.3E-05	1.03E-05

Picture: Regression of r on time to maturity

Now pricing formula only requires to estimate a, b and σ. There are certain estimation methods used by different researchers but here we use a simple method on Excel which doesn't involve any coding. The approach is to assume some value of a,b and sigma ourselves and then use Solver to give optimal value of parameters by minimizing the square of difference between our calculated price from Vasicek model and observed price in market. The working on excel is shown below in screenshots,

Today	12/11/2012						
Face Value	100						
r(0)	0.009106						
Vasicek Model	$dr = a(b - r)dt + \sigma dW$						
Parameters	**Initial Value**	**Estimated value**					
a	0.8	0.26624552					
b	0.2	0.024773877					
σ	0.2	0.195682087					
Bond	**Maturity**	**Observed Price**	**T-t (yrs)**	**B**	**A**	**Calculated Price**	**Error sq.**
RGKT1212	12/19/2012	99.98	0.022222	0.022026	-3.92009E-05	99.97602604	1.58E-05
RGKT1301	1/19/2013	99.91	0.105556	0.101222	-0.000859426	99.82204356	0.007736
RGKT1302	2/20/2013	99.815	0.191667	0.177695	-0.002752356	99.56390863	0.063047
RGKT1303	3/20/2013	99.75	0.275	0.246852	-0.005511755	99.22704433	0.273483
RGKT1306	6/19/2013	99.553	0.522222	0.426864	-0.018369311	97.79895076	3.076689
							3.42097

Picture: Screenshot of the parameter estimation program



Picture: Screenshot of Solver

Finally we get the following result from solver, Objective Cell (Min)

Cell	Name	riginal Val	Final Value
I20	Error sq.	3.42097	0.000122

Variable Cells

Cell	Name	riginal Val	Final Value	Integer
C10	a Initial Value	0.8	0.266246	Contin
C11	b Initial Value	0.2	0.024774	Contin
C12	σ Initial Value	0.2	0.195682	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
C10	a Initial Value	0.266246	C10>=0	Not Binding	0.26624552
C11	b Initial Value	0.024774	C11>=0	Not Binding	0.024773877
C12	σ Initial Value	0.195682	C12>=0	Not Binding	0.195682087

Picture: Screenshot of Solver solution

Above table shows estimated value in column of final value. Here mean reversion rate a is 0.266, long-term mean rate b is 2.477% and volatility is 19.56%. We can put these parameters in pricing formula and get our results. But this must be kept in mind that these estimated parameters are only associated with Swedish bonds.

The disadvantage of this model is that it gives positive probability of negative interest rates. It is possible to get negative real interest rate but here our working only involves nominal rates. Nominal rates always remain positive in the economy.

Cox-Ingersoll-Ross (CIR) model:

Description:

The CIR model prevents the disadvantage of Vasicek model given in above section. The diffusion process described by CIR model is,

$$dr = a(b - r)dt + \sigma\sqrt{r}dV$$

Since the volatility component is a function of interest rate r, it prevents the deficiency of Vasicek model described in above section. This model only gives positive probability for positive interest rates. This model assumes that short rate is normally distributes and has so called mean reverting process (under Q). Here b is long-term mean rate and a is the measure how fast short rate will reach the long term mean rate. We put this diffusion process in term structure equation and find a formula for pricing for pure discount bond.

$$P(0,T) = exp(A(0,T) - B(0,T)r(0)$$

Where,

$$B(0,T) = \frac{2(e^{\Upsilon T} - 1)}{(\Upsilon + a)(e^{\Upsilon T} - 1) + 2\Upsilon}$$

$$A(0,T) = \frac{2ab}{\sigma^2} ln\left(\frac{2\Upsilon e^{\frac{(\Upsilon + a)T}{2}}}{(\Upsilon + a)(e^{\Upsilon T} - 1) + 2\Upsilon}\right)$$

$$\Upsilon = \sqrt{a^2 + 2\sigma^2}$$

Price on a coupon bearing bond can be calculated similarly as described above in case of Vasicek model. Price of an option on pure discount bond is given by following formula,

$$C(0,T,K,S) = LP(0,S)\chi_n^2(x_1, v_1, v_2) - KP(0,T)\chi_n^2(x_2, v_1, v_3)$$

Where,

$$x_1 = 2\frac{A(S,T) - lnK}{B(S,T)}\left\{\frac{2\Upsilon}{\sigma^2(e^{\Upsilon T} - 1)} + \frac{a + \Upsilon}{\sigma^2} + B(S,T)\right\}$$

$$x_2 = x_1 - 2[A(S,T) - \ln(K)]$$

$$v_1 = \frac{4b}{a\sigma^2}$$

$$v_2 = \frac{\dfrac{8\Upsilon^2 r e^{\Upsilon T}}{\sigma^4(e^{\Upsilon T}-1)^2}}{\dfrac{2\Upsilon}{\sigma^2(e^{\Upsilon T}-1)} + \dfrac{a+\Upsilon}{\sigma^2} + B(S,T)}$$

$$v_3 = \frac{\dfrac{8\Upsilon^2 r e^{\Upsilon T}}{\sigma^4(e^{\Upsilon T}-1)^2}}{\dfrac{2\Upsilon}{\sigma^2(e^{\Upsilon T}-1)} + \dfrac{a+\Upsilon}{\sigma^2}}$$

Price of a put option can be calculated by using put-call parity.

Estimation of parameters:

Price of a discount bond in CIR model at current time period ($t = 0$) is given by,

$$P(0,T) = exp[A(0,T) - B(0,T)r(0)]$$

Since $r(0)$ is already calculated above, there is no need to same working again. Here also we need only to estimate a, b and σ. The methodology of estimating remains same as in Vasicek model. Hence we only show the result from excel.

Today	12/11/2012						
Face Value	100						
r(0)	0.009106						
CIR	$dr = a(b-r)dt + \sigma\sqrt{r}dV$						
Parameters	**Initial value**	**Estimated value**					
a	0.8	0.163659722					
b	0.2	6.08271E-08					
σ	0.2	0.603950055					
ϒ	0.848528137						
Bond	**Maturity**	**Observed Price**	**T-t(yrs)**	**B**	**A**	**Calculated Price**	**Error sq**
RGKT1212	12/19/2012	99.98	0.022222	0.022026	-3.9E-05	99.97601889	1.5849E-05
RGKT1301	1/19/2013	99.91	0.105556	0.101214	-0.00087	99.8213182	0.00786446
RGKT1302	2/20/2013	99.815	0.191667	0.177655	-0.00279	99.55980441	0.06512479
RGKT1303	3/20/2013	99.75	0.275	0.24674	-0.00563	99.21556956	0.2856159
RGKT1306	6/19/2013	99.553	0.522222	0.426235	-0.01906	97.73225097	3.31512703
							3.67374802

Picture: Screenshot of Parameter estimation program

Interpretation of these estimated parameters remains same as in Vasicek model.

Conclusion:

In this report we gave a background of term structure so that reader can follow the description and estimation of models. We derived a general term structure equation, now an individual only needs to describe a diffusion process of interest rate and put it into term structure equation and solve it to attain his solution. We described two famous models which are actively used in market.

The estimation of parameters can be little bit more accurate by using Matlab or Java programming. But to explain the concept and simple methodology we only kept ourselves restrict to Excel only. The parameters are only valid for Swedish bonds.

Works Cited

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