Seminar Report

Vasicek and CIR model

MÄLARDALEN UNIVERSITY
Department of Mathematics and Physics

Instructor: Jan Röman

Team members:
Rui Fan
Chizheng Miao
Zitao Yang
Qi Seven Zhao

Tasks

- Given the Swedish Bond prices to create a zero coupon curve fitted to the Vasicek and the CIR model with the application in Excel/VBA.

- Calculate the forward prices.

- Calculate the mean and the variance of the short rate.

- Plot the rates distribution formula

Introduction to the Vasicek and the CIR model

Vasicek Model

The model was proposed by Vasicek in 1979. It is a yield-based one-factor equilibrium model given by the interest rate process:

$$dr = (b - ar)dt + \sigma dz$$

Where: a is a measurement of how the fast the short rate will reach the long-term mean value.
b is the long term equilibrium rate

This model assumes that the short rate is normal distributed and has a so-called "mean reverting process" (under Q)

The Vasicek model possesses an affine term.
Affine Term Structure

$$P(t,T) = e^{A(t,T) - B(t,T)r_0}$$

In some literature the Affine bond prices are written as

$$P(t,T) = A(t,T)e^{-B(t,T)r_0}$$

b/a: long-term equilibrium of mean reverting spot rate process
In the Vasicek model we have A(t, T); B(t, T); like this.

$$A(t,T) = \exp\left[(B(t,T) - T + t)\left(\frac{ba - \sigma^2/2}{a^2}\right) - \frac{\sigma^2 B(t,T)^2}{4a}\right]$$

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

Value of zero coupon bond price can also be written as:

$$p(0,T) = exp\left\{\frac{1}{a}\left[1 - e^{-a(T-t)}\right]\left(\left(\frac{b}{a} - \frac{1}{2}\left(\frac{\sigma}{a}\right)^2\right) - r(0)\right) - (T - t)\left[\frac{b}{a} - \frac{1}{2}\left(\frac{\sigma}{a}\right)^2\right] - \frac{\sigma^2}{4a^3}\left[1 - e^{-a(T-t))}\right]^2\right\}$$

Graph Vasicek model:

Steady state probability density function for spot-rate r.
P is normally distributed with

$$P \sim N\left(\frac{b}{a}, \frac{\sigma}{\sqrt{2a}}\right)$$

We can also write as follows

$$p = \sqrt{\frac{a}{\pi}}\frac{1}{\sigma}e^{-\frac{a(r_0 - b)^2}{\sigma^2}}$$

Vasicek volatility of zero rate:

$$\sigma_{Y(t,T)} = \sigma \frac{1 - e^{-a(T-t)}}{a(T-t)} = \sigma \frac{B(t,T)}{(T-t)}$$

Vasicek Discount function

$$P(t,T) = A(t,T)e^{-B(t,T)r_0}$$

Vasicek Term Structure Interest

Infinitely-long Rate:

$$Y = b - \frac{\sigma^2}{2a^2}$$

Vasicek zero rate:

$$r = \frac{ln \frac{1}{A(t,T)e^{-B(t,T)r_0}}}{(T-t)}$$

The Vasicek model is popular in the academic community because of its analytic tractability. However, since the model is not necessarily arbitrage-free with respect to the actual underlying securities in the marketplace, the model is not used much.

CIR Term Structure Model

CIR model prevents the negative interest rates.
According to interest rate process:

$$dr = a(b-r)dt + \sigma\sqrt{r}dz$$

First we need to set the values of a, b and σ which are variable, and dz is standard Wiener process.
As we know the value of coupon bond:
$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$

In this function we need to know A and B, so we have

$$A(t,T) = \left[\frac{2\gamma e^{(a+\gamma)(T-t)/2}}{(\gamma+a)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{2ab/\sigma^2}$$

$$B = \frac{2\left(e^{\gamma(T-t)}-1\right)}{(\gamma+a)(e^{\gamma(T-t)}-1)+2\gamma}$$

Then we also know $\gamma = \sqrt{a^2 + 2\sigma^2}$

Infinitely-long Rate (Y_∞): $Y_\infty = {}^{2ab}/_{(a+\gamma)}$

Discount factor:
$P(t,T) = A(t,T)e^{-B(t,T)r_0}$
Then we use discount factor to get CIR Zero Rate.

CIR volatility of zero rate $\sigma_{Y(t,T)}$

$$\sigma_{Y(t,T)} = \sigma\sqrt{r_0}\frac{B(t,T)}{(T-t)}$$

Long-term distribution of r (Steady State Probability Density Function):

$$P_\infty = \frac{\left(\frac{2a}{\sigma^2}\right)^k}{\Gamma(k)} r^{k-1} e^{{}^{-2ar}/_{\sigma^2}} = \left(\frac{2a}{\sigma^2}\right)^k r^{k-1} e^{{}^{-2ar}/_{\sigma^2}} - ln(\Gamma(k))$$

With $k = \frac{2ab}{\sigma^2}$

This is gamma distributed.
$\Gamma(.)$is Gamma Function. Mean & standard deviation gamma distribution:

$$\Gamma(Mean) = k\frac{\sigma^2}{2a} = b$$

$$\Gamma(Stdev) = \sqrt{k}\frac{\sigma^2}{2a} = \sqrt{\frac{b}{2a}}\sigma$$

Gamma distribution for probability, we use in excel notation

$$f(x,\alpha,\beta) = \frac{1}{\beta^\alpha\Gamma(\alpha)} x^{\alpha-1} e^{{}^{-x}/_{\beta}}$$

$$\alpha = K, \beta = {}^{\sigma}/_{2a}$$

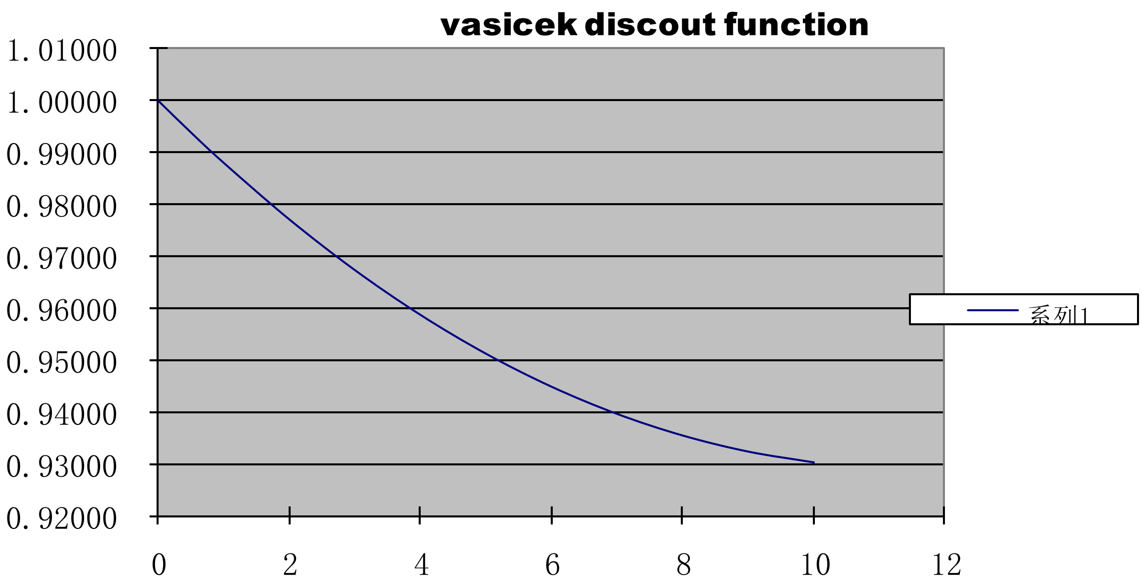
Application

We downloaded the data from the NASDAQ.OMX

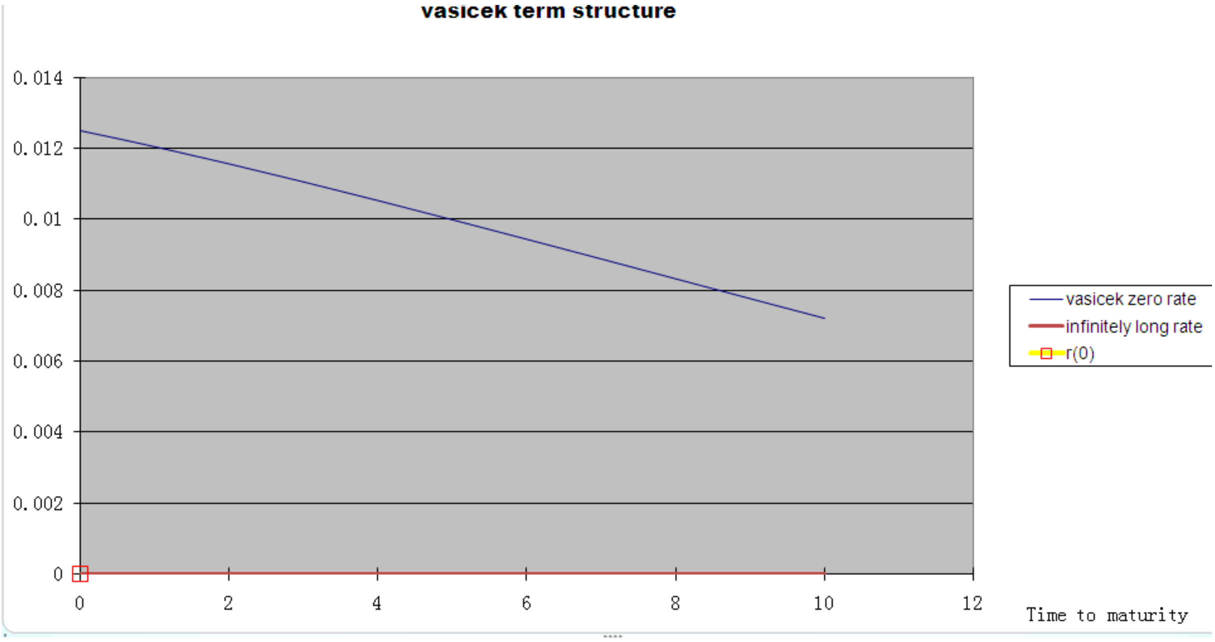
Name	ISIN code	Coupon	Maturity date	Bid	Ask	Avg rate	High rate	Low rate	Vol SEKm
				Indicative pricing		Reported turnover for trading day: 2010-12-16			
RGKB 1041	SE0000412389	6.75	2014-05-05			2.405	2.460	2.405	2,089
RGKB 1045	SE0000722852	5.25	2011-03-15			1.230	1.230	1.230	500
RGKB 1046	SE0000909640	5.5	2012-10-08			1.825	1.860	1.645	7,696
RGKB 1047	SE0001149311	5.0	2020-12-01			3.268	3.290	3.260	2,068
RGKB 1049	SE0001250135	4.5	2015-08-12			2.721	2.749	2.705	1,080
RGKB 1050	SE0001517699	3.0	2016-07-12			2.930	2.931	2.888	2,653
RGKB 1051	SE0001811399	3.75	2017-08-12			3.058	3.065	3.045	713
RGKB 1052	SE0002241083	4.25	2019-03-12			3.177	3.200	3.165	125
Total vol SEKm									**17,037**

By using the VBA in Excel, we get the followings:

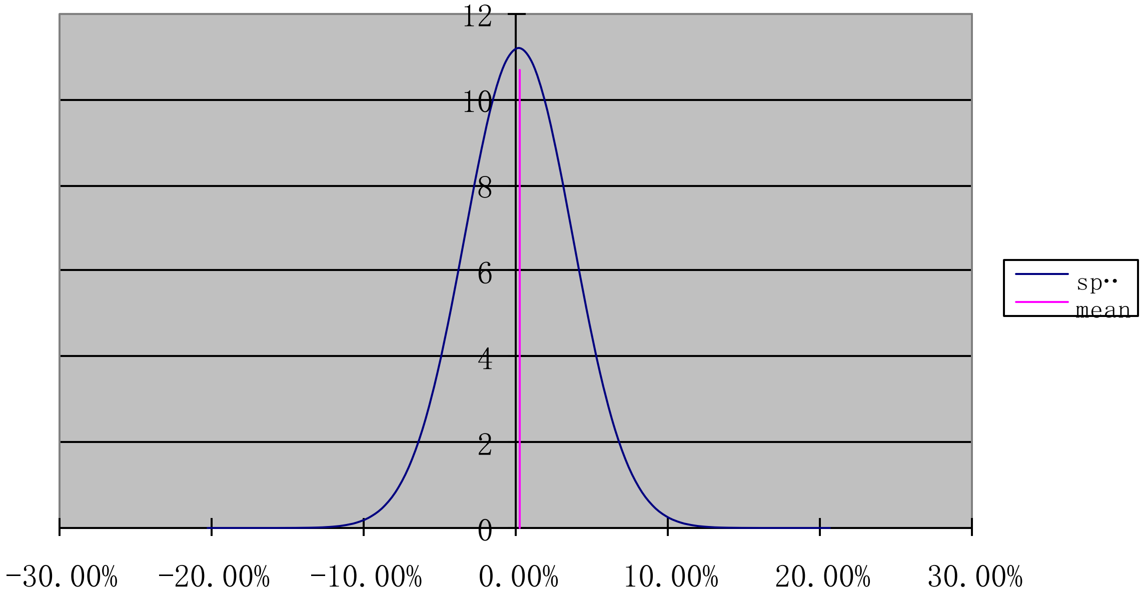
Vasicek Model:



If r_0 is 1.25%, the simulated term structure of interest rates is shown in the figure below. It is obvious to see that the Vasicek zero rates always approach to the infinitely long rate.

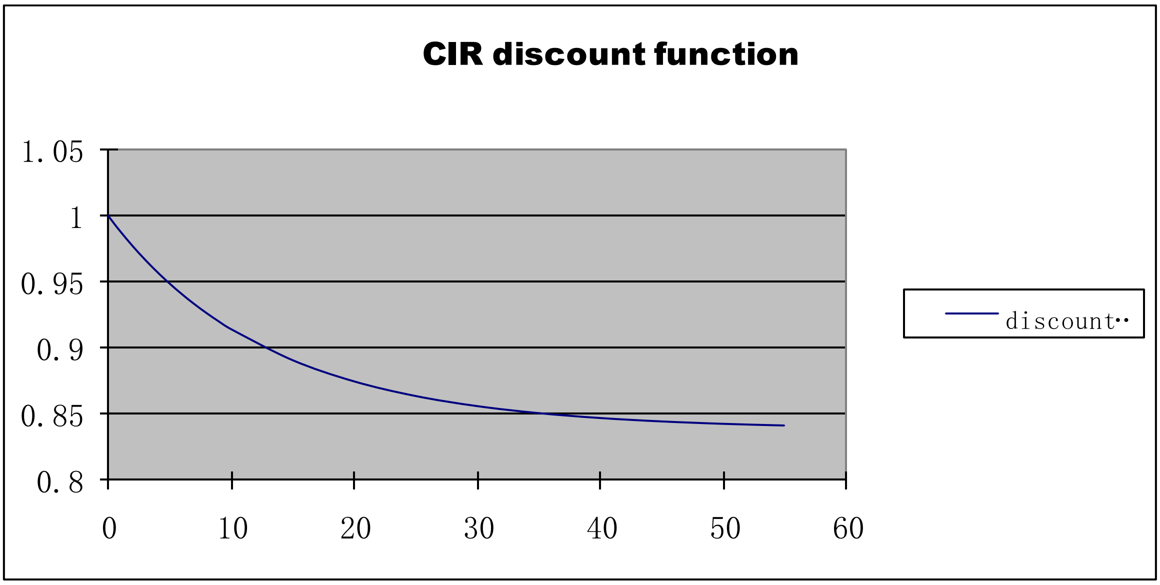


PDF for Vasicek：



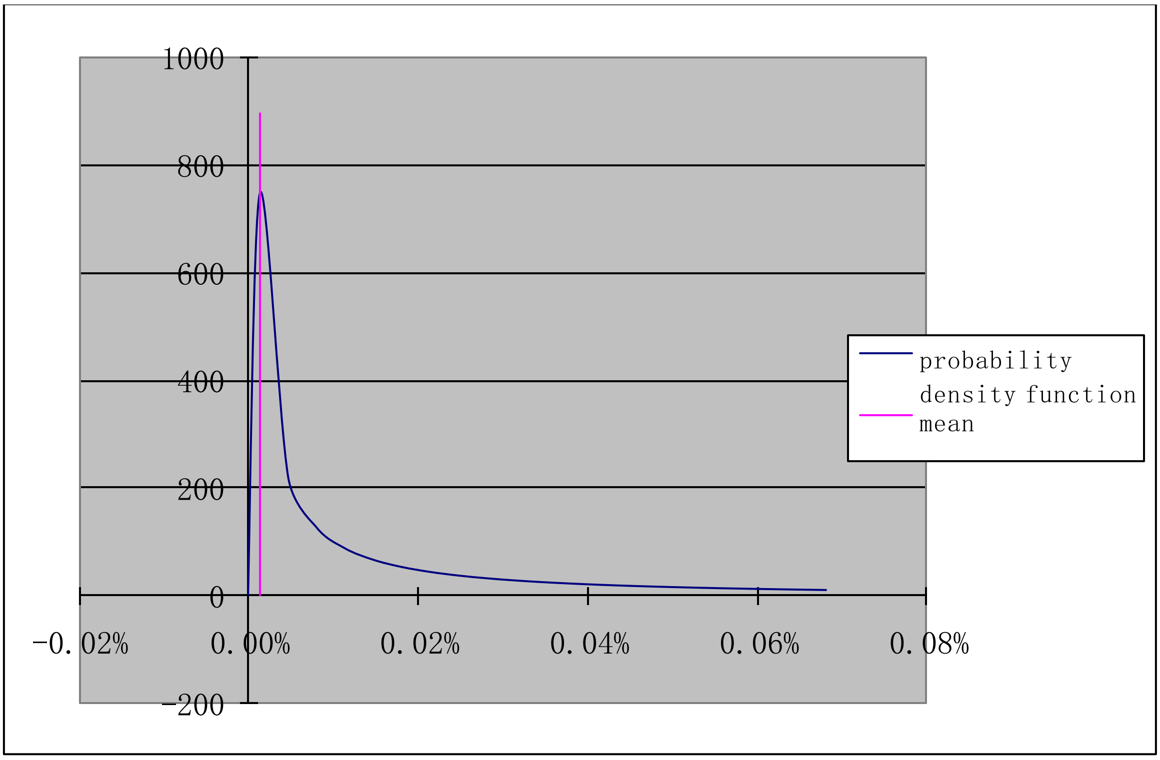
A disadvantage of this model is that the short rates have a positive probability to become negative. A simulation of the distribution of the short rates with $a = 0.15$, $b/a = 1.67 * 10^{-3}$ and $\sigma = 1.33\%$ is shown as the above one.

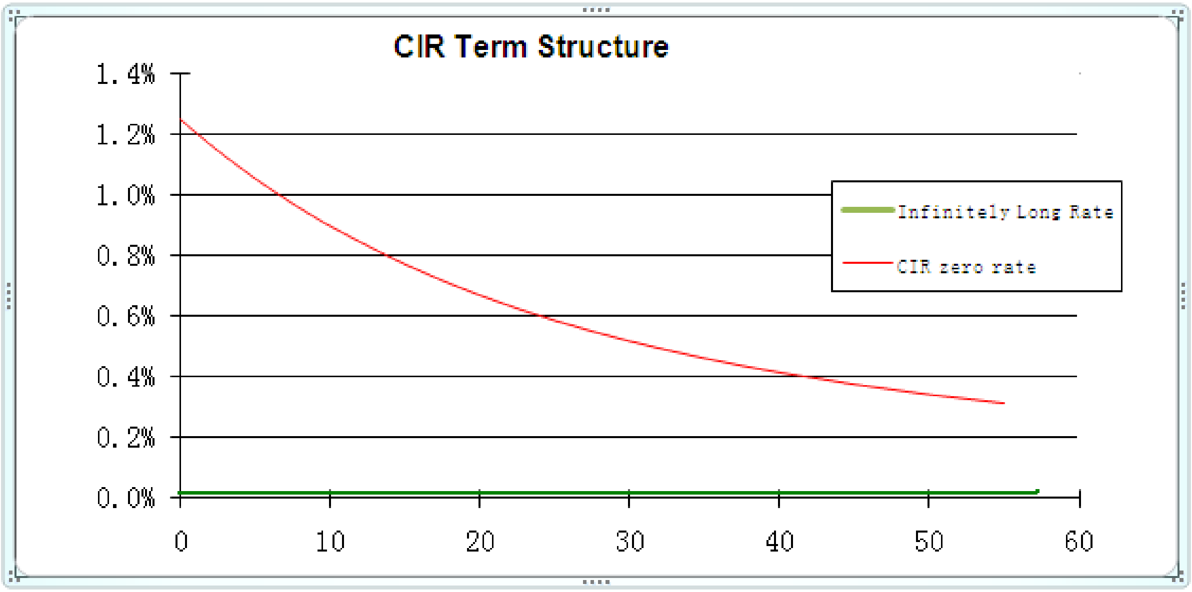
CIR Model:



For the graph of the PDF, what we need to do firstly is to draw the probability density function. The X-axis is the spot rate, the Y-axis is the probability.

A simulation of the short rates with the same parameter as in the Vasicek Model is shown in the figure below. As we can see, the probability of negative interest rate is zero.





CIR Term Structure
1.4%
1.2%
1.0%
0.8%
0.6%
0.4%
0.2%
0.0%
0
10
20
30
40
50
60
Infinitely Long Rate
CIR zero rate