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Tasks

- Given the Swedish Bond prices to create a zero coupon curve fitted to the Vasicek and the CIR model with the application in Excel/VBA.
- Calculate the forward prices.
- Calculate the mean and the variance of the short rate.
- Plot the rates distribution formula
Introduction to the Vasicek and the CIR model

**Vasicek Model**
The model was proposed by Vasicek in 1979. It is a yield-based one-factor equilibrium model given by the interest rate process:

\[ dr = (b - ar)dt + \sigma dz \]

Where: 
- \( a \) is a measurement of how fast the short rate will reach the long-term mean value.
- \( b \) is the long-term equilibrium rate

This model assumes that the short rate is normally distributed and has a so-called “mean reversion” process (under \( Q \)).

The Vasicek model possesses an affine term.

**Affine Term Structure**

\[ P(t, T) = e^{A(t, T) - B(t, T)r_0} \]

In some literature, the Affine bond prices are written as

\[ P(t, T) = A(t, T)e^{-B(t, T)r_0} \]

\( b/a \): long-term equilibrium of mean reversion spot rate process

In the Vasicek model, we have \( A(t, T) \); \( B(t, T) \); like this.

\[ A(t, T) = \exp \left[ (B(t, T) - T + t) \left( \frac{b - \sigma^2/2}{a^2} \right) - \frac{\sigma^2 B(t, T)^2}{4a} \right] \]

\[ B(t, T) = \frac{1 - e^{-a(T-t)}}{a} \]

Value of zero coupon bond price can also be written as:

\[ p(0, T) = \exp \left\{ \frac{1}{a} \left[ 1 - e^{-a(T-t)} \right] \left( \frac{b}{a} - \frac{1}{2} \left( \frac{\sigma}{a} \right)^2 \right) - r(0) \right\} - (T - t) \left[ \frac{b}{a} - \frac{1}{2} \left( \frac{\sigma}{a} \right)^2 \right] - \frac{\sigma^2}{4a^2} \left[ 1 - e^{-a(T-t)} \right]^2 \]

**Graph Vasicek model:**

Steady state probability density function for spot-rate \( r \).

\( P \) is normally distributed with

\[ P \sim N \left( \frac{b}{a}, \frac{\sigma}{\sqrt{2a}} \right) \]

We can also write as follows

\[ p = \frac{1}{\sqrt{\pi \sigma}} e^{-\frac{(r_0 - b)^2}{\sigma^2}} \]

Vasicek volatility of zero rate:
The Vasicek model is popular in the academic community because of its analytic tractability. However, since the model is not necessarily arbitrage-free with respect to the actual underlying securities in the marketplace, the model is not used much.

**CIR Term Structure Model**

CIR model prevents the negative interest rates. According to interest rate process:

\[ dr = a(b - r)dt + \sigma \sqrt{r}dz \]

First we need to set the values of \( a, b \) and \( \sigma \) which are variable, and \( dz \) is standard Wiener process.

As we know the value of coupon bond:

\[ P(t, T) = A(t, T)e^{-B(t,T)r(t)} \]

In this function we need to know \( A \) and \( B \), so we have

\[
A(t, T) = \left[ \frac{2\gamma \left( e^{\gamma(T-t)} - 1 \right)}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma} \right]^{2ab/\sigma^2}
\]

\[
B = \frac{2\left( e^{\gamma(T-t)} - 1 \right)}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}
\]

Then we also know \( \gamma = \sqrt{\alpha^2 + 2\sigma^2} \)
Infinitely-long Rate \( Y_r \): \( Y_r = \frac{2ab}{a + \gamma} \)

Discount factor:
\[ P(t, T) = A(t, T) e^{-B(t,T)r_0} \]
Then we use discount factor to get CIR Zero Rate.

CIR volatility of zero rate \( \sigma_{\gamma(t,T)} \)
\[ \sigma_{\gamma(t,T)} = \sigma \sqrt{\frac{B(t,T)}{(T - t)}} \]

Long-term distribution of \( r \) (Steady State Probability Density Function):
\[ p_r = \left( \frac{2a}{\sigma^2} \right)^k r^{k-1} e^{-2a r / \sigma^2} = \left( \frac{2a}{\sigma^2} \right)^k r^{k-1} e^{-2a r / \sigma^2} - \ln \left( \Gamma(k) \right) \]

With \( k = \frac{2ab}{\sigma^2} \)

This is gamma distributed.
\( \Gamma(.) \) is Gamma Function. Mean & standard deviation gamma distribution:
\[ \Gamma(Mean) = k \frac{\sigma^2}{2a} = b \]
\[ \Gamma(Stdev) = \sqrt{k} \frac{\sigma^2}{2a} = \sqrt{\frac{b}{2a}} \sigma \]

Gamma distribution for probability, we use in excel notation
\[ f(x, \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha - 1} e^{-x / \beta} \]
\[ \alpha = K, \beta = \frac{\sigma}{2a} \]
We downloaded the data from the NASDAQ.OMX.

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Total vol SEKm: 17.037

By using the VBA in Excel, we get the followings:

**Vasicek Model:**

![vasicek discount function](image)
If $r_0$ is 1.25%, the simulated term structure of interest rates is shown in the figure below. It is obvious to see that the Vasicek zero rates always approach to the infinitely long rate.

PDF for Vasicek:

A disadvantage of this model is that the short rates have a positive probability to become negative. A simulation of the distribution of the short rates with $a = 0.15$, $b/a = 1.67 \times 10^{-2}$ and $\sigma = 1.33\%$ is shown as the above one.
CIR Model:

For the graph of the PDF, what we need to do firstly is to draw the probability density function. The X-axis is the spot rate, the Y-axis is the probability.

A simulation of the short rates with the same parameter as in the Vasicek Model is shown in the figure below. As we can see, the probability of negative interest rate is zero.