MÄLARDALEN UNIVERSITY
SWEDEN

Division of Applied Mathematics
School of Education, Culture and Communication
Box 833, SE-721 23 Västerås
Sweden.

MMA 708 - Analytical Finance II
Teacher: Jan Roman

Calculation of Swaptions Using BDT Trees

Ikechukwu Chinaka
Henry Obeng Tawiah

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Abstract

In this Report, we study Swaptions and use a spot rate market data to build a Black-Derman-Toy Tree to calculate the values of the Swaptions. A comparison with the Black'76 model is also made.

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INTRODUCTION

A Swap can be described as a contractual agreement between two parties in which they agree to make periodic payments to each other according to different indices. A swaption can therefore be described as the option on a swap. It gives the holder the right to enter into the swap, without any obligation to enter into the underlying swap. As the underlying asset is a swap, which is itself a derivative, a swaption is a derivative of a derivative.

The holder of a swaption may have the right to exchange a fixed interest rate for a floating rate (payer swaption) or to exchange a floating rate for a fixed rate (receiver option). Although options can be traded on a variety of swaps, the term swaption typically refers to options on interest rate swaps. Swaptions like vanilla options can be European, American or Bermudian.

In this seminar we are going to show the calculation of swaptions using BDT trees and compare the results to Black (1976)

1.0 The Black-Derman-Toy Model (BDT)

The standard BDT model is constructed algorithmically to be consistent with both the existing term structure of zero-coupon yields, and (optionally) the term structure of yield volatilities. That is, the main aim of the tree building procedure is to derive a binomial representation for the level of the short-term interest rate such that zero-coupon bond prices computed from the tree are exactly equal to the set of zero coupon prices that are directly observable in the market. If desired, the model can also be constructed so that the implied interest rate distribution at each time step matches an observed interest rate volatility curve.

The short-rate volatility is potentially time dependent, and the continuous process of the short-term interest rate is given by:

$$dln(r) = \left\{\theta(t) + \frac{\dot{\sigma}(t)}{\sigma(t)}\ln(r)\right\}dt + \sigma(t)dv$$

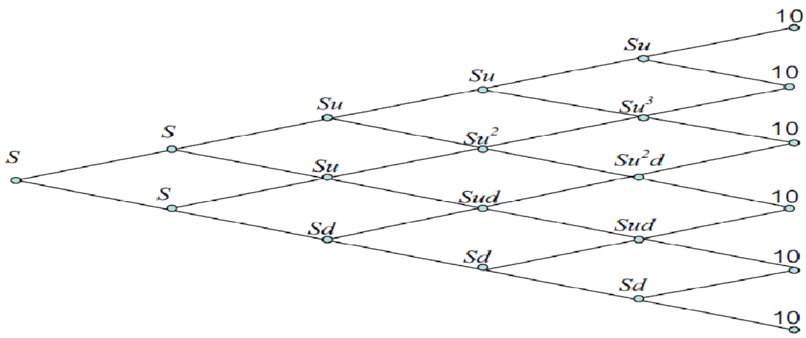
Where the factor in front of $ln(r)$ is the speed of mean reversion ("gravity"), and $\sigma(t)$divided by the speed of mean reversion is a time-dependent mean-reversion level. The BDT model incorporates as we see, two independent functions of time, $\theta(t)$ and $\sigma(t)$, chosen so that the model fits the term structure of spot interest rates and the term structure of spot rate volatilities. The changes in the short rate are log normally distributed, with the resulting advantage that interest rates cannot become negative. Once $\theta(t)$ and $\sigma(t)$ are chosen, the future short-rate volatility, by definition, is entirely determined.

This model does not have an affine term structure since the volatility term is proportional to the level of the short rate. Many practitioners choose to fit the rate structure only, holding the future short-rate volatility constant. The convergent limit therefore reduces to the following:

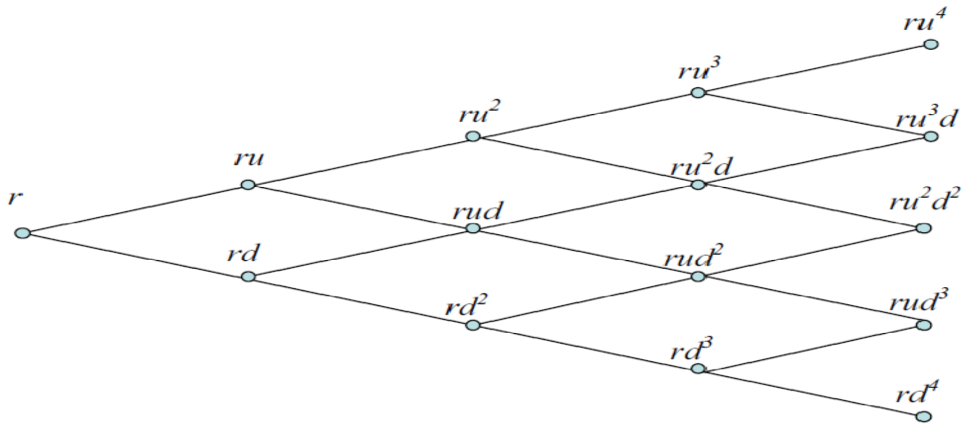
$$dln(r) = \theta(t)dt + \sigma(t)dv$$

1.2 Construction of the BDT model

The first step in the construction of the BDT interest rate tree is the computation of the discount factors for each maturity according to the yield curve structure.
Secondly, we build a zero-coupon bond tree that pays a face value of 100 in 10 years. Each node is the average of the two resulting prices (up and down) in the subsequent period discounted back at the corresponding interest rate (that would be retrieved from the BDT interest rate tree) adjusted for the time interval. This is done starting in the 10 years moment and computing backwards until the present moment. At this point the entire tree shows the value "100" at each node (except for the first), as the interest rate tree is not yet built.



Thirdly we build the interest rate tree. We determine all our unknowns using the lowest interest rates for each period and the corresponding volatilities. After, we introduce a column denominated as" Destination cells". This column gives for each period the difference between the product of 100 by the discount factor determined through the yield curve, and the present value of the zero coupon bond calculated in the bond tree.



1.2 Computation of the Bond at each node

We strip the bond, making the entire payments (coupons and principal) equivalent to zero coupon bonds. Subsequently, we build a tree for each of these bonds to implement the BDT model. In each tree we know that independently of the state of nature, the bond will pay a certain value (coupon, or coupon plus face value at its maturity). Then, we discount these values to the present moment by applying the risk-neutral probability of 0.5 and the corresponding interest rate. If we add all the corresponding nodes of these trees we obtain the tree of "dirty prices". By removing the respective period coupon (included in the "dirty price"), we get to the "clean prices" tree, which will be the basis for valuing the option.

$$Pt = \frac{0.5\frac{100}{1+r_d} + 0.5\frac{100}{1+r_u}}{1+r_t}$$

Pt : The price today
r_d : Downward rate
r_u : Upward rate
r_t : Zero coupon rate

In a standard binomial tree, we have

$u = e^{\sigma\sqrt{\Delta t}}$ $\quad\quad$ $d = e^{-\sigma\sqrt{\Delta t}}$

$\frac{u}{d} = e^{2\sigma\sqrt{\Delta t}} \quad \Rightarrow \quad ln\left(\frac{u}{d}\right) = 2\sigma\sqrt{\Delta t}$

$$\sigma = \frac{1}{2\sqrt{\Delta t}} ln\left(\frac{u}{d}\right)$$

$where$

$s_u = \frac{100}{1+r_u}$ $\quad\quad$ $s_u = \frac{100}{1+r_u}$

Where Δt is the change in time from the present to the maturity date. The Black-Derman-Toy tree rate are assumed to be log-normally distributed which implies that

$$\sigma_n = \frac{1}{2\sqrt{\Delta t}} ln\left(\frac{r_u}{r_d}\right)$$

In Option Adjusted Spreads model volatility factor is defined by

$$Z_n = e^{2\sigma_n\sqrt{\Delta t}}$$

We are then left with two unknowns r_d and r_n and two sources of information.

1.3 Calculation of the call option's value for a selected strike price

A call option's value at maturity is the maximum of zero and the difference between: the value of the bond (clean price) and the strike price. We discount these values (using the risk-neutral probability of 0.5 and the corresponding interest rate taken from the BDT tree) until we reach the present moment, and hence the option's value.

In what refers to the call option on the bond, some adjustments must be made. Firstly, we do not need the accrual as the underlying asset, but a bond's price. Therefore, we do not need to correct it for the period. Secondly, we estimate the underlying asset's forward price and volatility in a somewhat different ways. The forward price of the bond was estimated by discounting all the subsequent certain cash flows (Coupons and principal) back to the exercise date through the forward rates (between each exercise date and the cash-flow's moment).

1.4 Pricing Swaptions Using "Black 76"

To price European − style swaptions using "Black1976" option pricing model, the following steps have to be taken

1) *Derive an interest rate − swap spot rate curve*
2) *Calculate the forward rate of the swaption from the curve*
3) *Using the forward rate as the underlying price in a "Black 76" option model to calculate the option price*
4) *Annuitize the "Black 76" option model price for the term structure*

1.5 Calculation of the Option Price

Using the semiannual spot rates given by this method, we can calculate the semiannual forward rates from the formula

$$1 + nfj = \frac{\left(1 + R_{n+j}\right)^{n+j}}{(1 + R_n)^n}$$

where f = *the forward rate*

n = *the point in time at which the rate is effective*

j = *the period of time over which it applies*

the values for a call price C *and put price* P *are*

$$C = e^{-rt}[f\phi(d1) - x\phi(d2)]$$

$$P = e^{-rt}[x\phi(-d2) - f\phi(-d1)]$$

$$d1 = \frac{\log\left(\frac{f}{x}\right) + (\sigma^2/2)t}{\sigma\sqrt{t}}$$

$$d2 = d1 - \sigma\sqrt{t}$$

f = *the current underlying forward price*
r = *the continosly compounded risk free interest rate*
x = *the strike price*

$t = the\ time\ in\ years\ until\ the\ expiration\ of\ the\ option$

$\sigma = the\ implied\ volatility\ for\ the\ underlying\ forward\ price$

$\phi = the\ standard\ normal\ cummulative\ distribution\ function$

1.6 BDT Compared with Black'76 Model

The value of the option prices depends on the model used. The reasons can be the ff

- The Black 76 model uses a central estimate of the discount rate while the BDT uses a whole set of rates
- The Difference may also be due to the assumption of volatilities. The Black Model has a constant volatility while the BDT has many varying volatilities in different time periods.

References

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[3] John C. Hull, *Option, Futures and Other Derivatives*, 7th Edition, Prentice Hull New Jersey, 2009