MMA707

Black-Scholes formula with dividends: some solutions

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Content

Introduction and example from Jan Roman's Problems with binominal

The standard Black-Scholes formula does not work properly if we consider stock dividends.

In this report, firstly we show this problem, and its reasons. In the second part we propose two different solutions to solve the problem and we will see if they are appropriate to solve it.

We are only going to consider the case with one dividend but it can be easily generalized to the multi-dividend case.

The problem with Black-Scholes formula when we have stock dividends can be observed easily, just showing to American call options with the same underling stock and strike price. The only difference is that one of them expiry just before a dividend, and the other one expiry just after this dividend. Namely:

We have this stock, dividend, time to dividend, volatility and interest rate:

$S = 100.00$ (Stock price)

$d = 10.00$ (Dividend)

$t_d = 0.50$ (Time to dividend)

$\sigma = 0.30$ (Volatility)

$r = 0.03$ (Interest rate)

Now we have two European put options A and B:

$X_A = 100.00$ (Strike price of A)

$T_A = 0.50 - \varepsilon$ (Option A expires just before the dividend)

$X_B = 90.00$ (Strike price of B, which is $X_A - d$)

$T_B = 0.50 + \varepsilon$ (Option B expires just after the dividend)

Using standard Black-Scholes formula used with dividends, we get that

$P_A = bs(S, X_A, T_A, \sigma, r) = bs(100, 100, 0.5, 0.3, 0.03) = 7.661$

$P_B = bs(S - de^{-r\,t_d}, X_B, T_B, \sigma, r) = bs(90.149, 90, 0.5, 0.3, 0.03) = 6.831$

If we assume that the market is trading the two options at volatility 30%, we can sell in short one Put A and buy in long one Put B, getting 7.661 – 6.831 = 0.83 in cash.

After 6 months (t = 0.5) we have two scenarios:

If the stock price S is higher than X_A = 100 at expiry, Put A expires worthless just before the dividend, and also Put B, because the stock price just after the dividend will be higher than 100 – 10 = 90, the strike price of B.

If the stock price is lower than X_A = 100 at expiry, we have to buy the stock at 100 at expiry of Put A just before the dividend, however, we can also sell the stock it at 90 just after the dividend when we exercise Put B, and we will keep the dividend d = 10.

It has been showed that in both cases the sum is zero. But we have 0.83 from t = 0.

We have made arbitrage, which is not possible in Black-Scholes world.

However, not only American options suffer the problem, also European options deal with it.

SOLUTIONS

There are many solutions to solve our initial problem, some of them are better, and other ones are not very acceptable, according to the last part of this report.

Here we show two of these solutions:

Solution 1: Include any dividends after expiration.

The stock price diffusion process expiring before the dividend is

$$S(t) = D_0\, e^{rt} + (S_0 - D_0)\, e^{(r - \frac{1}{2}\sigma^2)\, t + \sigma w(t)} \qquad t < T_1 < t_d$$

Note that it is not the same as the standard Black-Scholes.

The stock price diffusion process expiring after the dividend is

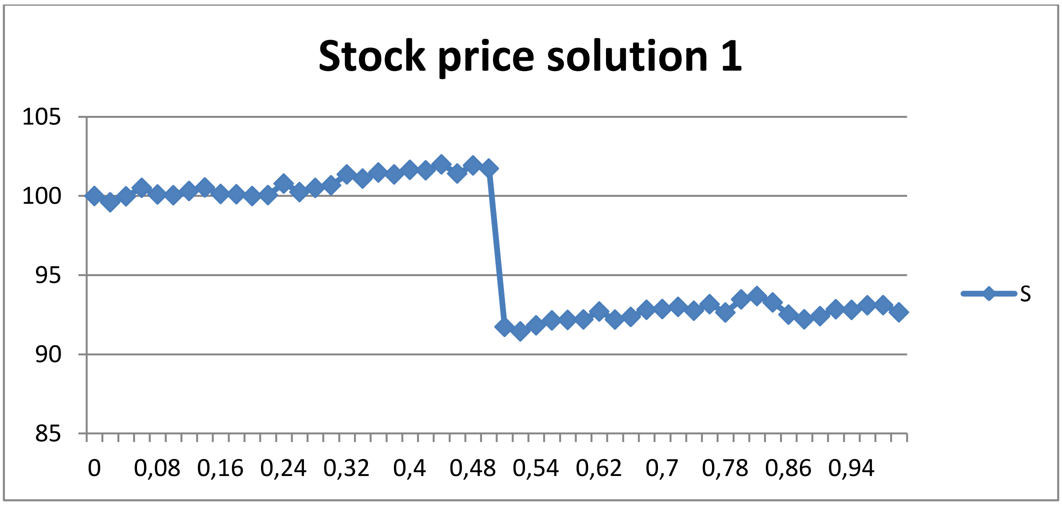
$$S(t) = D_0\, e^{rt} + (S_0 - D_0)\, e^{(r - \frac{1}{2}\sigma^2)\, t + \sigma w(t)} \qquad t < t_d < T_2$$

$$S(t) = (S_0 - D_0)\, e^{(r - \frac{1}{2}\sigma^2)\, t + \sigma w(t)} \qquad t_d < t < T_2$$

Here, we use the same stock price diffusion process for all options, i.e. we include dividends after expiration as well.

Let's check if solution 1 is "acceptable".

After stock price simulation:



We consider two European put options:

S(0)=100

A) Stock price just before the dividend=97,65, strike=100

B) Stock price just after the dividend=87,65, strike=90

Time to maturity=0,5

Volatility=0,3

Interest rate=0,03

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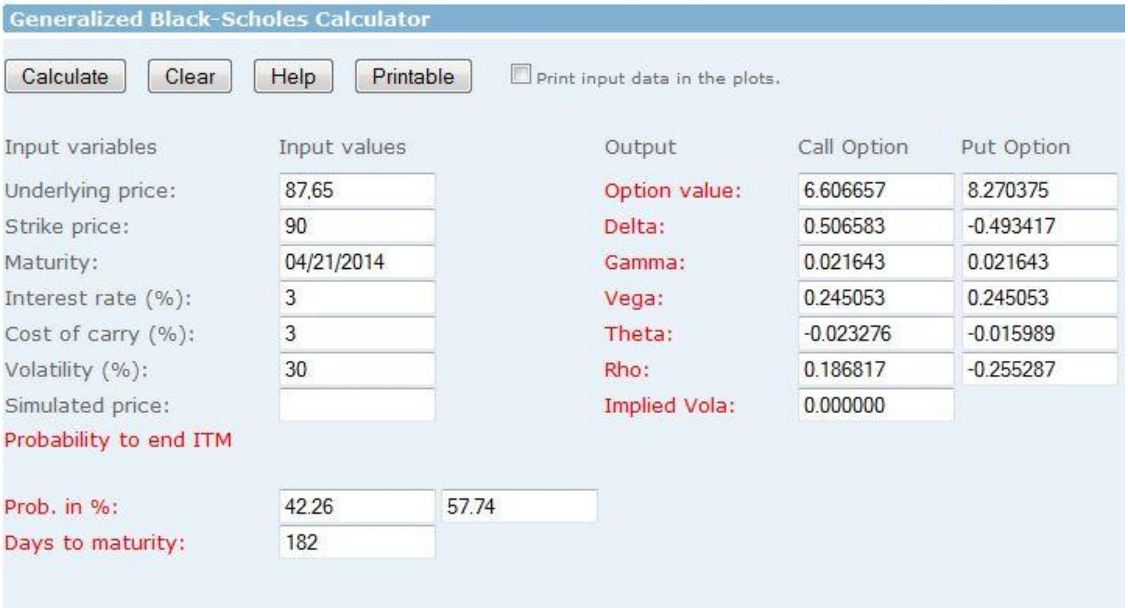


Generalized Black-Scholes Calculator

Calculate | Clear | Help | Printable | Print input data in the plots.

Input variables	Input values		Output	Call Option	Put Option
Underlying price:	87,65		Option value:	6.606657	8.270375
Strike price:	90		Delta:	0.506583	-0.493417
Maturity:	04/21/2014		Gamma:	0.021643	0.021643
Interest rate (%):	3		Vega:	0.245053	0.245053
Cost of carry (%):	3		Theta:	-0.023276	-0.015989
Volatility (%):	30		Rho:	0.186817	-0.255287
Simulated price:			Implied Vola:	0.000000	
Probability to end ITM					
Prob. in %:	42.26	57.74			
Days to maturity:	182				

Now we see a difference between option which expires before dividend and option which expires after dividend. The difference is 9,025914 – 8,270375 = 0,755539. The problem is that this case is not arbitrage free because of higher option A price.

Solution 2: Treat all dividends as proportional.

The stock price diffusion process expiring before the dividend is

$$S(t) = S_0\, e^{(r - \frac{1}{2}\sigma^2)\, t + \sigma w(t)} \qquad t < T_1 < t_d$$

Note that it is the same as the standard Black-Scholes.

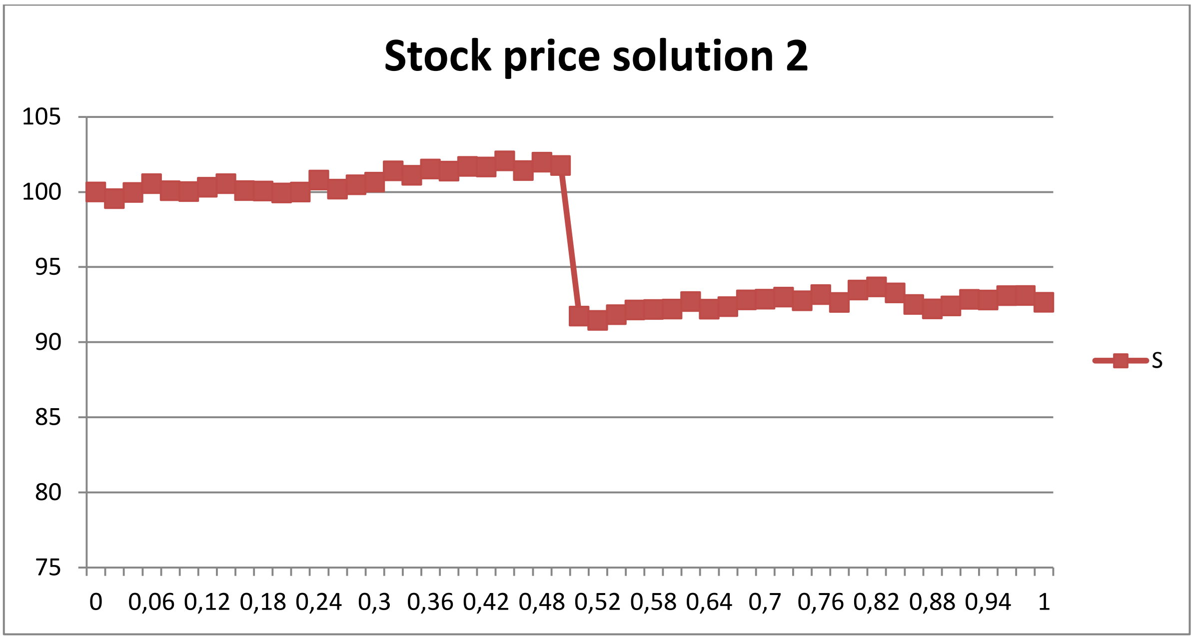
The stock price diffusion process expiring after the dividend is

$$S(t) = S_0\, e^{(r - \frac{1}{2}\sigma^2)\, t + \sigma w(t)} \qquad t < t_d < T_2$$

$$S(t) = (S_0 - D_0)\, e^{(r - \frac{1}{2}\sigma^2)\, t + \sigma w(t)} \qquad t_d < t < T_2$$

Dividends are treated as proportional, so the volatility is both the volatility of the stock and forward price, but there are no differences, since dividends are proportional.

After stock price simulation:



We consider two European put options:

S(0)=100

A) Stock price just before the dividend=101,76, strike=100

B) Stock price just after the dividend=91,76, strike=90

Time to maturity=0,5

Volatility=0,3

Interest rate=0,03

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Generalized Black-Scholes Calculator

Calculate | Clear | Help | Printable | ☐ Print input data in the plots.

Input variables	Input values		Output	Call Option	Put Option
Underlying price:	101,76		Option value:	9.715286	7.230528
Strike price:	100		Delta:	0.588429	-0.411571
Maturity:	04/21/2014		Gamma:	0.018186	0.018186
Interest rate (%):	3		Vega:	0.277507	0.277507
Cost of carry (%):	3		Theta:	-0.026958	-0.018861
Volatility (%):	30		Rho:	0.247899	-0.243328
Simulated price:			Implied Vola:	0.000000	
Probability to end ITM					
Prob. in %:	50.47	49.53			
Days to maturity:	182				

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Generalized Black-Scholes Calculator

Calculate | Clear | Help | Printable | ☐ Print input data in the plots.

Input variables	Input values		Output	Call Option	Put Option
Underlying price:	91,76		Option value:	8.802701	6.466419
Strike price:	90		Delta:	0.590447	-0.409553
Maturity:	04/21/2014		Gamma:	0.020160	0.020160
Interest rate (%):	3		Vega:	0.249737	0.249737
Cost of carry (%):	3		Theta:	-0.024275	-0.016988
Volatility (%):	30		Rho:	0.224024	-0.218080
Simulated price:			Implied Vola:	0.000000	
Probability to end ITM					
Prob. in %:	50.67	49.33			
Days to maturity:	182				

In this solution arbitrage is also available. We make money by selling option A and buying option B. In this case we earn 7,230528 – 6,466419 = 0,764109

Cox-Ross-Rubinstein Model

Now we will see how the option price look like in binominal model. Again we consider European put option.

Let assume:

S (0)=100

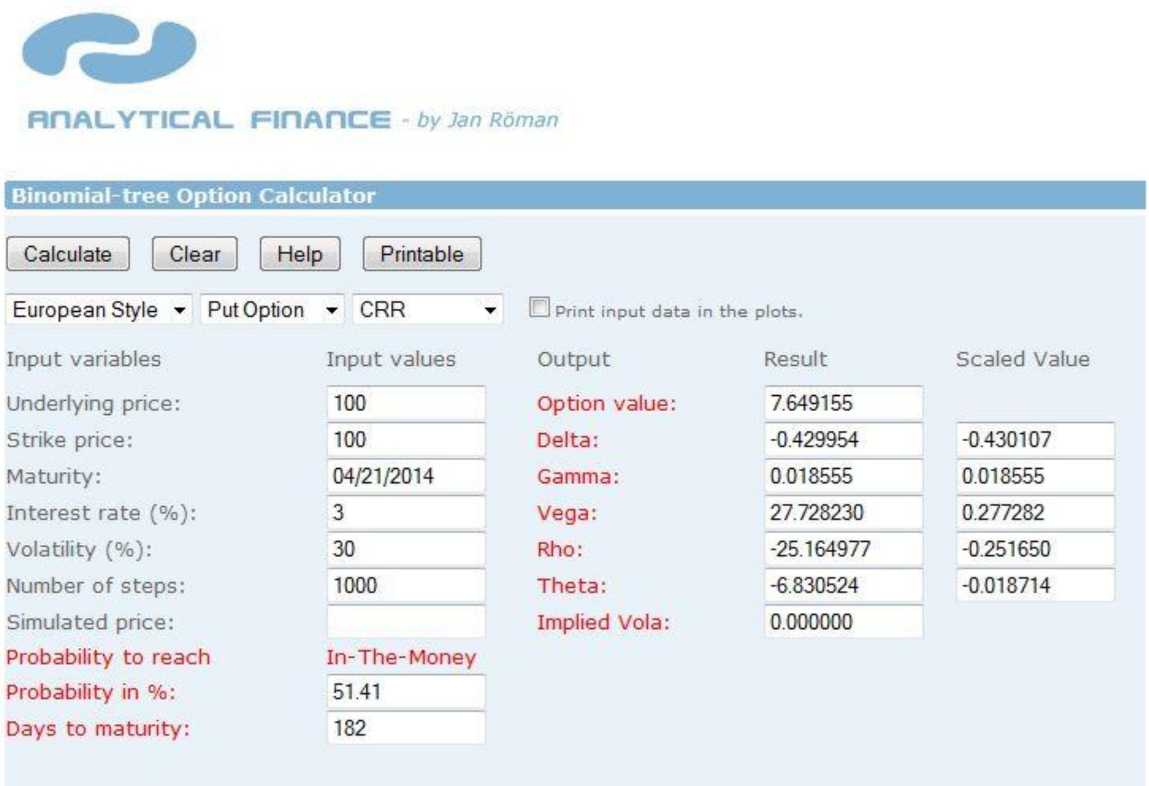
Strike=100

Volatility=0,3

Interest rates=0,03

Time to maturity=0,5

Number of steps=1000



Different approaches to the problem gives us different results. Option price from Cox-Ross-Rubinstein model is not the same as prices from Black-Scholes Universe. Black-Scholes model is continues model so that why the prices are different. Also it is difficult to implement dividend approach to binominal model.