Option Pricing With Dividends

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Chapter 1

Introduction

Chash dividends issued by stocks have a big impact on their option prices. This is because the underlying stock price is expected to drop by the said dividend on the ex-dividend date. Options are valued taking into account the projected dividends receivable in the coming weeks and months up to the option expiration date.

Chapter 2

Solution One: Include Any Dividends After Expiration

2.1 Expiry before the dividend

$$S(t) = D_0 e^{rt} + (S_0 - D_0) e^{(r-0.5\sigma^2)t+\sigma w(t)} \qquad t < T_1 < t_d \tag{2.1}$$

Let

- $\tau = T - t$
- $\sigma(W(T) - W(t)) = \sigma\sqrt{T-t}Z$

At maturity, the stock price is given by

$$S(T) = D_t e^{r\tau} + (S_t - D_t) e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} \tag{2.2}$$

The payoff for a call option is $Max(S_T - K, 0)$. Therefore to calculate the call price we need to solve the following expression

$$e^{-r\tau} E^Q \left[Max(S_T - K, 0) | F_t \right] \tag{2.3}$$

Putting the expression for S_T in 2.2 into 2.3 we have (without the discounting factor)

$$E^Q \left[Max(D_t e^{r\tau} + (S_t - D_t) e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} - K, 0) | F_t \right] \tag{2.4}$$

Note: From here on, we assume the expect ion is in risk-neutral world and is with respect to the filtration at time t Recall that the payoff is always equal to zero when $S_T \leq K$ and this further implies that the expectation will also be zero. With this result we have that

$$\begin{aligned}
D_t e^{r\tau} + (S_t - D_t) e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} &< K \\
(S_t - D_t) e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} &< K - D_t e^{r\tau} \\
e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} &< \frac{K - D_t e^{r\tau}}{(S_t - D_t)}
\end{aligned}$$

Taking ln on both sides, we have

$$\begin{aligned}
(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z &< ln(\tfrac{K-D_te^{r\tau}}{(S_t-D_t)}) \\
\sigma\sqrt{\tau}Z &< ln(\tfrac{K-D_te^{r\tau}}{(S_t-D_t)})-(r-0.5\sigma^2)\tau \\
Z &< \tfrac{ln(\frac{K-D_te^{r\tau}}{(S_t-D_t)})-(r-0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}
\end{aligned}$$

From the last expression, we let $Z<-d_2^*$ and considering expectations for the positive part of the payoff, we have

$$E\left[Max(S_T-K,0)\right]=E\left[Z*I_{Z>-d_2^*}\right]$$

$$=\int_{-d_2^*}^{\infty}(D_te^{r\tau}+(S_t-D_t)e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z}-K)\phi(z)dz$$

$$=\int_{-d_2^*}^{\infty}(D_te^{r\tau}-K)\phi(z)dz+\int_{-d_2^*}^{\infty}(S_t-D_t)e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z}\phi(z)dz$$

- Solving the first integral we have that

$$(D_te^{r\tau}-K)\int_{-d_2^*}^{\infty}\phi(z)dz$$

But $\phi(z)$ is the standard normal distribution and we know by symmetry of the distribution

$$\int_{-d_2^*}^{\infty}\phi(z)dz=P\{Z>-d_2^*\}=P\{Z<d_2^*\}=\Phi(d_2^*)$$

This implies

$$(D_te^{r\tau}-K)\int_{-d_2^*}^{\infty}\phi(z)dz=(D_te^{r\tau}-K)\Phi(d_2^*) \tag{2.5}$$

- Solving the second integral: Let $y=z-\sigma\sqrt{\tau}$, therefore

$$\int_{-d_2^*-\sigma\sqrt{\tau}}^{\infty}(S_t-D_t)e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}(y+\sigma\sqrt{\tau})}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y+\sigma\sqrt{\tau})^2}{2}}dy$$

$$(S_t-D_t)e^{r\tau}\int_{-d_2^*-\sigma\sqrt{\tau}}^{\infty}e^{-0.5\sigma^2\tau+\sigma\sqrt{\tau}(y+\sigma\sqrt{\tau})}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y+\sigma\sqrt{\tau})^2}{2}}dy$$

Solving the exponents inside the integral, many terms will cancel out and thus we have

$$(S_t-D_t)e^{r\tau}\int_{-d_2^*-\sigma\sqrt{\tau}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy$$

This results in

$$(S_t-D_t)e^{r\tau}\Phi(d_1^*) \qquad d_1^*=d_2^*+\sigma\sqrt{\tau} \tag{2.6}$$

Therefore, the expectation becomes

$$E\left[Max(S_T - K, 0)\right] = (S_t - D_t)e^{r\tau} + (D_t e^{r\tau} - K)\Phi(d_2^*) \tag{2.7}$$

But the call price is given by (2.3). Therefore replacing (2.7) into (2.3) we have the call price as

$$C = (S_t - D_t)\Phi(d_1^*) + (D_t - Ke^{-r\tau})\Phi(d_2^*) \tag{2.8}$$

Where

$$d_2^* = \frac{ln(\frac{S_t - D_t}{K - D_t e^{r\tau}}) + (r - 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$d_1^* = d_2^* + \sigma\sqrt{\tau}$$

2.1.1 Examples

1. Consider a case where the stock price is 100, dividend is 20, strike price is 100, maturity time is six months, risk-free interest rate is 10%, volatility is 40%. Implementing equation (2.8), the European call price is given by

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[1] 11.37461
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2. Considering the same example but pricing in the binomial model. The European call option price is given by

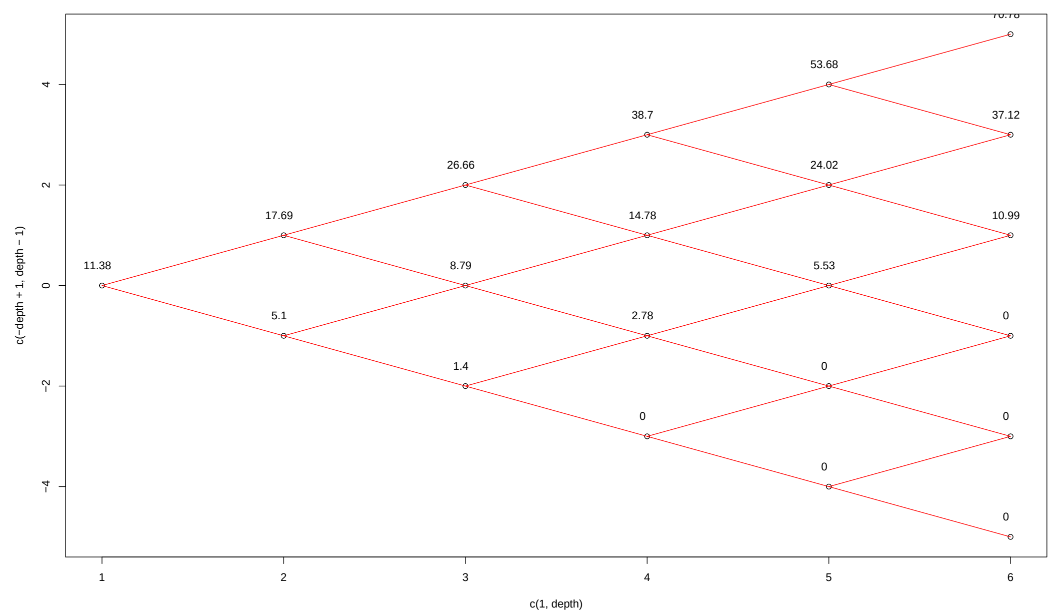

Figure 2.1: Binomial Lattice for stocks with dividend payment after expiry of the option

2.2 Expiry After Dividends

2.2.1 Case 1

$$S(t) = D_0 e^{rt} + (S_0 - D_0)e^{(r-0.5\sigma^2)t+\sigma w(t)} \tag{2.9}$$

The solution is the same as solution (2.8) but the times have changed in the implementation.

2.3 Examples

1. Consider a case where the stock price is 100, dividend is 20, strike price is 100, maturity time is six months, risk-free interest rate is 10%, volatility is 40%. Implementing equation (2.8), the dividend is paid 3 months before maturity and thus European call price is given by

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[1] 11.42881
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2. The binomial equivalent to the above example is given by

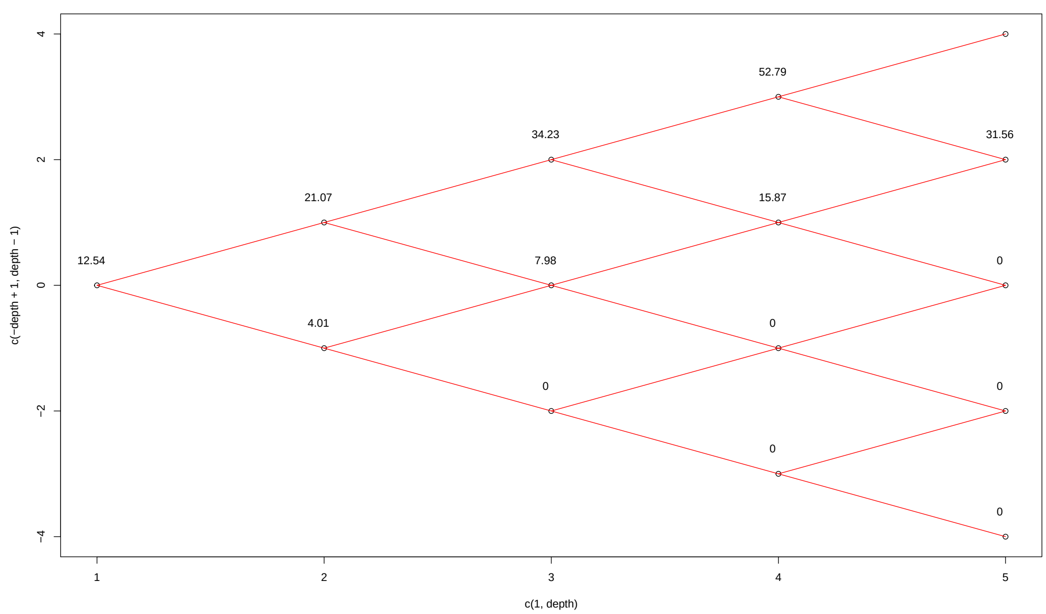

Figure 2.2: Binomial Lattice for stocks with dividend payment before expiry of the option

2.3.1 Case 2

$$(S_0 - D_0)e^{(r-0.5\sigma^2)t+\sigma w(t)} \tag{2.10}$$

$$S(T) = (S_t - D_t)e^{(r-0.5\sigma^2)(T-t)+\sigma(w(T)-w(t))}$$

Let

- $\tau = T - t$
- $\sigma(W(T) - W(t)) = \sigma\sqrt{T-t}Z$

At maturity, the stock price is given by

$$S(T) = (S_t - D_t)e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} \tag{2.11}$$

The payoff for a call option is $Max(S_T - K, 0)$. Therefore to calculate the call price we need to solve the following expression

$$e^{-r\tau}E^Q\left[Max(S_T - K, 0)|F_t\right] \tag{2.12}$$

Putting the expression for S_T in (2.11) into (2.12) we have (without the discounting factor)

$$E^Q\left[Max((S_t - D_t)e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} - K, 0)|F_t\right]$$

Note: From here on, we assume the expection is in risk-neutral world and is with respect to the filtration at time t Recall that the payoff is always equal to zero when $S_T \leq K$ and this further implies that the expectation will also be zero. With this result we have that

$$\begin{aligned}
(S_t - D_t)e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} &< K \\
(S_t - D_t)e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} &< K \\
e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} &< \frac{K}{(S_t - D_t)}
\end{aligned}$$

Taking ln on both sides, we have

$$\begin{aligned}
(r-0.5\sigma^2)\tau + \sigma\sqrt{\tau}Z &< ln(\frac{K}{(S_t - D_t)}) \\
\sigma\sqrt{\tau}Z &< ln(\frac{K}{(S_t - D_t)}) - (r-0.5\sigma^2)\tau \\
Z &< \frac{ln(\frac{K}{(S_t - D_t)}) - (r-0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}
\end{aligned}$$

From the last expression, we let $Z < -d_2^*$ and considering expectations for the positive part of the payoff, we have

$$E\left[Max(S_T - K, 0)\right] = E\left[Z * I_{Z>-d_2^*}\right]$$

$$= \int_{-d_2^*}^{\infty} ((S_t - D_t)e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} - K)\phi(z)dz$$

$$= \int_{-d_2^*}^{\infty} (S_t - D_t)e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z}\phi(z)dz - \int_{-d_2^*}^{\infty} K\phi(z)dz$$

- Solving the second integral we have that

$$K \int_{-d_2^*}^{\infty} \phi(z)dz \tag{2.13}$$

But $\phi(z)$ is the standard normal distribution and we know by symmetry of the distribution

$$\int_{-d_2^*}^{\infty} \phi(z)dz = P\{Z > -d_2^*\} = P\{Z < d_2^*\} = \Phi(d_2^*)$$

This implies

$$K \int_{-d_2^*}^{\infty} \phi(z)dz = K\Phi(d_2^*) \tag{2.14}$$

- Solving the first integral: Let $y = z - \sigma\sqrt{\tau}$, therefore

$$\int_{-d_2^*-\sigma\sqrt{\tau}}^{\infty} (S_t - D_t)e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}(y+\sigma\sqrt{\tau})} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+\sigma\sqrt{\tau})^2}{2}} dy \tag{2.15}$$

$$(S_t - D_t)e^{r\tau} \int_{-d_2^*-\sigma\sqrt{\tau}}^{\infty} e^{-0.5\sigma^2\tau+\sigma\sqrt{\tau}(y+\sigma\sqrt{\tau})} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+\sigma\sqrt{\tau})^2}{2}} dy$$

Solving the exponents inside the integral, many terms will cancel out and thus we have

$$(S_t - D_t)e^{r\tau} \int_{-d_2^*-\sigma\sqrt{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

This results in

$$(S_t - D_t)e^{r\tau}\Phi(d_1^*) \qquad d_1^* = d_2^* + \sigma\sqrt{\tau} \tag{2.16}$$

Therefore, the expectation becomes

$$E\left[Max(S_T - K, 0)\right] = (S_t - D_t)e^{r\tau} + K\Phi(d_2^*) \tag{2.17}$$

But the call price is given by (2.12). Therefore replacing (2.17) into (2.12) we have the call price as

$$C = (S_t - D_t)\Phi(d_1^*) + Ke^{-r\tau}\Phi(d_2^*) \tag{2.18}$$

Where

$$d_2^* = \frac{ln(\frac{S_t - D_t}{K}) + (r - 0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$d_1^* = d_2^* + \sigma\sqrt{\tau}$$

2.3.2 Examples

1. Consider a case where the stock price is 100, dividend is 20, strike price is 100, maturity time is six months, risk-free interest rate is 10%, volatility is 40%. Implementing equation (2.18), the European call price is given by

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[1] 4.058085
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2. The equivalent binomial option price is given by

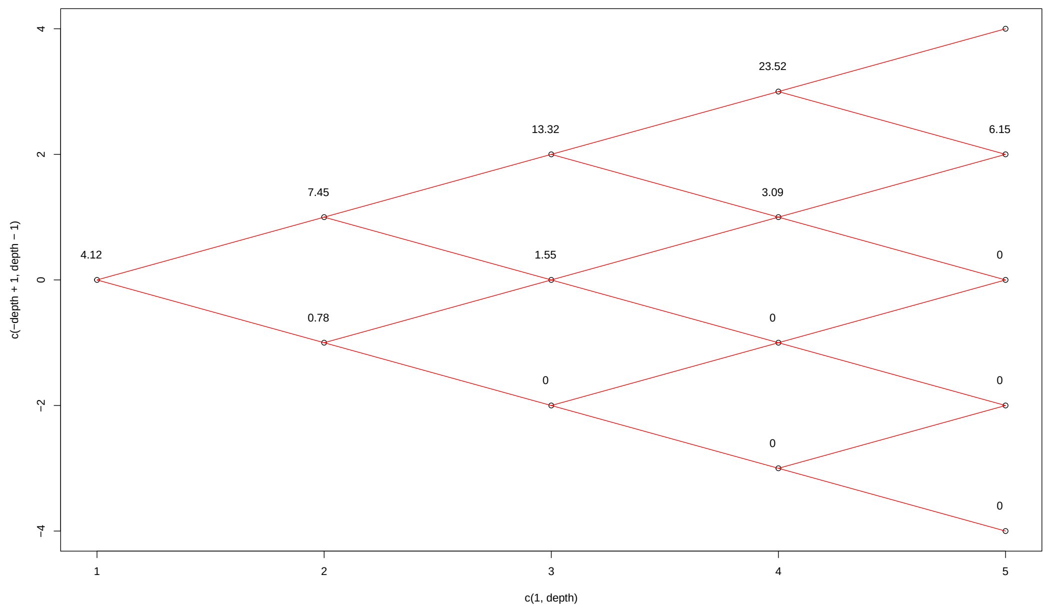

Figure 2.3: Binomial option price for option with dividend before inception

Chapter 3

Treat All Dividends as Proportional

3.1 Expiry before dividend

$$S_0 e^{(r-0.5\sigma^2)t+\sigma w(t)} \tag{3.1}$$

$$S(T) = S_t e^{(r-0.5\sigma^2)(T-t)+\sigma(w(T)-w(t))}$$

Let

- $\tau = T - t$
- $\sigma(W(T) - W(t)) = \sigma\sqrt{T-t}Z$

At maturity, the stock price is given by

$$S(T) = S_t e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} \tag{3.2}$$

The payoff for a call option is $Max(S_T - K, 0)$. Therefore to calculate the call price we need to solve the following expression

$$e^{-r\tau}E^Q\left[Max(S_T - K, 0)|F_t\right] \tag{3.3}$$

Putting the expression for S_T in (3.2) into (3.3) we have (without the discounting factor)

$$E^Q\left[Max(S_t e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} - K, 0)|F_t\right] \tag{3.4}$$

Note: From here on, we assume the expect ion is in risk-neutral world and is with respect to the filtration at time t Recall that the payoff is always equal to zero when $S_T \leq K$ and this further implies that the expectation will also be zero. With this result we have that

$$\begin{aligned} S_t e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} &< K \\ S_t e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} &< K \\ e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z} &< \frac{K}{S_t} \end{aligned}$$

Taking ln on both sides, we have

$$\begin{aligned}(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}Z &< ln(\tfrac{K}{S_t})\\ \sigma\sqrt{\tau}*Z &< ln(\tfrac{K}{S_t})-(r-0.5\sigma^2)\tau\\ Z &< \tfrac{ln(\frac{K}{S_t})-(r-0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}\end{aligned}$$

- Solving the second integral we have that

$$K\int_{-d_2^*}^{\infty}\phi(z)dz \tag{3.5}$$

But $\phi(z)$ is the standard normal distribution and we know by symmetry of the distribution

$$\int_{-d_2^*}^{\infty}\phi(z)dz = P\{Z>-d_2^*\} = P\{Z<d_2^*\} = \Phi(d_2^*)$$

This implies

$$K\int_{-d_2^*}^{\infty}\phi(z)dz = K\Phi(d_2^*) \tag{3.6}$$

- Solving the first integral: Let $y = z-\sigma\sqrt{\tau}$, therefore

$$\int_{-d_2^*-\sigma\sqrt{\tau}}^{\infty} S_t e^{(r-0.5\sigma^2)\tau+\sigma\sqrt{\tau}(y+\sigma\sqrt{\tau})}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y+\sigma\sqrt{\tau})^2}{2}}dy \tag{3.7}$$

$$S_te^{r\tau}\int_{-d_2^*-\sigma\sqrt{\tau}}^{\infty} e^{-0.5\sigma^2\tau+\sigma\sqrt{\tau}(y+\sigma\sqrt{\tau})}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y+\sigma\sqrt{\tau})^2}{2}}dy$$

Solving the exponents inside the integral, many terms will cancel out and thus we have

$$S_te^{r\tau}\int_{-d_2^*-\sigma\sqrt{\tau}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy$$

This results in

$$S_te^{r\tau}\Phi(d_1^*) \qquad d_1^* = d_2^*+\sigma\sqrt{\tau} \tag{3.8}$$

Therefore, the expectation becomes

$$E\left[Max(S_T-K,0)\right] = S_te^{r\tau}+K\Phi(d_2^*) \tag{3.9}$$

But the call price is given by (3.3). Therefore replacing (3.9) into (3.3) we have the call price as

$$C = S_t\Phi(d_1^*)+Ke^{-r\tau}\Phi(d_2^*) \tag{3.10}$$

Where

$$d_2^* = \frac{ln(\frac{S_t}{K})+(r-0.5\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$d_1^* = d_2^*+\sigma\sqrt{\tau}$$

3.2 Examples

1. Consider a case where the stock price is 100, dividend is 20, strike price is 100, maturity time is six months, risk-free interest rate is 10%, volatility is 40%.

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   [1] 13.58038
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2. The binomial option for the above example is given by

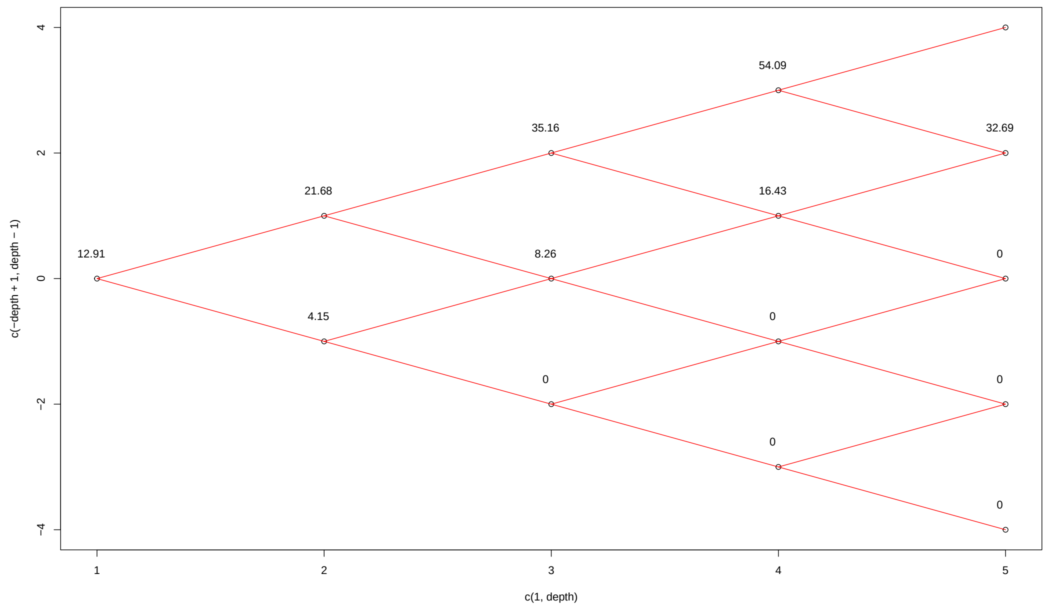

Figure 3.1: Binomial option price for option with proportional dividend

3.3 Expiry After Dividend

$$(S_0 - D_0)e^{(r-0.5\sigma^2)t+\sigma w(t)} \tag{3.11}$$

The solution to the expression (3.11) is the same as the one given in equation (2.18)

Bibliography

[1] Hull, John C. **Options, Futures and Other Derivatives**

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[3] Black, F. and M. Scholes **The Pricing of Options and Corporate Liabilities** *The Journal of Political Economy 8*(1973)