MÄLARDALEN UNIVERSITY
SWEDEN
School of Education, Culture and Communication
Division of Applied Mathematics

Analytical Finance 1, MMA707
Autumn 2013
Mälardalens högskola

THE BLACK-SCHOLES MODEL
When the underlying pays dividends

Jonsson, Robin
Karlén, Anne
Weigardh, Anton

Instructor: Jan Röman

Abstract

In this paper, we evaluate the famous Black-Scholes model (1973) for options with dividends in underlying. We use various extensions of the Black-Scholes framework to see how the theoretical price of plain vanilla European options is affected when dividends are assumed. We also test for early exercises of American Options. We infer from our results that the size of a discrete dividend or the size of a dividend yield has a greater impact than when a discrete dividend occurs, at least for options with short time to maturity.

Contents

1 Introduction **3**

2 Theory around the Black-Scholes model and dividends **4**
2.1 What is a dividend and how does it influence the underlying asset? 4
2.2 Assumptions in the Black-Scholes World 5
2.3 The price in a world without dividends 5
2.4 Numerical Example 6
2.5 Introducing Dividends to the Black-Scholes World 7
2.6 Derivation of Black-Scholes with Continuous Dividends 8
2.7 Parts of the Solution Affected by Dividend. 10

3 Application and results **12**
3.1 European case with dividend yield 12
3.2 American Options 14
3.3 Black's approximation 15
3.4 User version 16

4 Discussion and conclusion **18**
4.1 Comments 18
4.2 Appendix 20

Chapter 1

Introduction

During the early 70s, Black and Scholes (1973) and Merton (1973) laid the foundations for a framework that came to change the way in which options were priced. Using only five parameters to price a plain vanilla European option (price of underlying asset, strike price, risk-free interest rate, volatility of the underlying asset and time to maturity), the Black-Scholes-Merton model spoke to a vast range of traders which earlier used complex multi-factor models. The model gained immense popularity and has since then become the benchmark. In order to calculate the theoretical price of a call- or a put option using the Black-Scholes formulas, several assumptions have to be taken (see section 2.3). One keen assumption is that the underlying asset (usually a stock) pays no dividends during the life of the option. The postulation of no dividends in the underlying asset only holds for options with short maturities. In reality, a vast bulk of traded companies pay dividends on an annual, semi-annual, quarterly or even monthly basis. Problems then arise when one tries to price an option that spans over one or several dividends. A large concern is whether one can assume the dividends to be recurring or if they are just some random events. Furthermore, the size of the dividend and its occurrence in time will also have an impact on the theoretical price of a derivative. Does there exist one or several good solutions to cope with the fact that we live in a world where dividends are common? In this report, we investigate the Black-Scholes world with dividends and simulate some different scenarios. By using MATLAB, we develop an application of financial model where the user can see how the theoretical price of a plain vanilla European option changes while altering the parameters.

Chapter 2

Theory around the Black-Scholes model and dividends

In this part, we recap the concept surrounding dividends and the Black-Scholes framework. We present alternative results using different parameters.

2.1 What is a dividend and how does it influence the underlying asset?

Dividends are earnings that a company distributes to its shareholders. There are different types of dividend payouts but for the sake of this report we will assume that dividends are paid in cash. If a person wants to buy a stock, she will be willing to pay a certain amount of money. If this stock is expected to pay a dividend within a short future, the stock is very attractive since it will generate an extra income. Therefore it seems reasonable to pay some extra money in order to become the owner. The price of this stock will be higher prior to a dividend compared to the price of a stock that already had a dividend, which is called *ex dividend*. At ex-dividend date the price of the stock will drop approximately by the amount of the dividend size. A stock can therefore differ in price over time, despite the fact that it represents the exact same size of ownership. [1]

The consequence of this is that option prices on these stocks will differ as well. The more the stock increases its value, the more valuable its call option will be. After the dividend payout the stock price will drop and thus increasing the value of put options. [2]

2.2 Assumptions in the Black-Scholes World

In order for the Black-Scholes-Merton differential equation in section 2.3 to hold, we take on the following assumptions.

1. There is no opportunity to perform arbitrage. (Arbitrage means to earn money through buying cheap and selling expensive.)
2. There is market efficiency which means that the future development of the stock price and the trend of the market are unpredictable. Instead they follow an Ito Process (which depends on the present value) and are continuously guessed.
3. The expected return μ and volatility σ (risk) of the stock (that underlies the option) is constant during the option's life time.
4. The stock pays no dividends.
5. Tax, transaction costs, fees and commissions are excluded from the calculations.
6. During the option's life time, the risk free interest rate is known and constant.
7. Small changes in the stock price S with respect to time $\triangle t$ follow a normal distribution $\frac{\triangle S}{S} \sim \phi(\mu \triangle t, \sigma\sqrt{\triangle t})$ [3]

2.3 The price in a world without dividends

We will use the following notation for the different parameters: $S(t)$ = the price of the underlying
$f(S,t)$ = price of the option depending on the underlying and time
$C(S,t)$ = price of a Call
$P(S,t)$ = price of a Put
K = strike price
r = risk free interest rate, continuously compounded
σ = volatility
t = time

The standard Black- Scholes PDE

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

where $\frac{\partial f}{\partial t}$ represents the change in the option value with respect to time, called Θ
$\frac{\partial f}{\partial S}$ represents the change in the option value with respect to the underlying, called Δ
$\frac{\partial^2 f}{\partial S^2}$ represents the change in the option value with respect to the volatility of the underlying,

called Γ.

The standard formulae for European Call and Put

In order to use the formulae we need to know the current price of the underlying stock S_0, the strike price K (the contracted price of the stock at the date of maturity), the risk free interest rate r and the time to maturity T of the option. Discounting back the strike price with continuous compounding calculating Ke^{-rT} gives us the value that the strike price would have today, i.e. the present value of the strike price. We also need information about the volatility σ which is a measurement of the risk due the uncertainty of the return of the stock.

Furthermore we need to determine what the change of the option price will be with respect to a change of the underlying stock $N(d_1)$ and we need to estimate the probability that the option will be exercised $N(d_2)$. The formulae for d_1 and d_2 are

$$d_1 = \frac{ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

and

$$d_2 = \frac{ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

.

Plugging d_1 and d_2 into N(x) gives the cumulative probability distribution of the values, i.e the probability that variables with standard normal distributions $\phi(0,1)$ will be smaller than d_1 and d_2 respectively.

Current stock price and strike price are expressed in the value of its currency (e.g. 100$), volatility and interest rate are expressed in decimals (e.g. 50% = 0.5). Time to maturity is expressed in years (e.g. $200days = \frac{200}{365} \approx 0.55$) .

Now we have everything we need to show the actual formulae for calls c and puts p

$$c = S_0N(d_1) - Ke^{-rT}N(d_2)$$

and

$$p = Ke^{-rT}N(-d_2) - S_0N(-d_1)$$

2.4 Numerical Example

The inputs for this example are for the stock of Eniro AB. At october 27 2012 it was traded for SEK 9.05. It's volatility σ was according to its business report (2012-06-30) 41.18%, [4].

The risk-free interest rate r (STIBOR rate, 2012-10-26, [5]) was 1.57%. The calculations in this example are made for both call and put options with a strike price $K = 15$ and a maturity date in $6 months$ ($\approx 182 days$) from now.

$$c = 9.05N(d_1) - 15e^{-0.0157*0.5}N(d_2)$$

$$d_1 = \frac{ln(\frac{9.05}{15}) + (0.0157 + \frac{0.4118^2}{2})0.5}{0.04118\sqrt{0.5}} = -1.5627$$

We can use the table for N(x) in Hulls literature [6] to obtain the value

$$N(d_1) = 0.059076$$

$$d_2 = \frac{ln(\frac{9.05}{15}) + (0.0157 - \frac{0.04118^2}{2})0.5}{0.4118\sqrt{0.5}} = -1.8539 \Longrightarrow N(d_2) = 0.03192136$$

.

Now we plug the values into our formula for c which yields

$$c \approx 0.06$$

We can also calculate a put:

$$p = Ke^{-rT}N(-d_2) - S_0N(-d_1) = 15e^{-0.0157*0.5}N(1.8539) - 9.05N(1.5627) \approx 5.89$$

2.5 Introducing Dividends to the Black-Scholes World

When we relax assumption nr 4, we face the problem how to correctly value an option. One alternative is to assume that the underlying asset pays a continuous fraction in dividend known as dividend yield. A second alternative arises when the underlying asset pays a finite amount of dividend at one or several certain dates, the discrete case.

For the case of American Options where the holder of the option has the possibility to exercise early, there might be certain conditions that, if they are fulfilled, make it beneficiary to exercise early.

2.6 Derivation of Black-Scholes with Continuous Dividends

The stock price $S(t)$ at time t can be expressed as a function of two deterministic constants, i.e a drift $\mu(t)$ and a diffusion $\sigma(t)$ as well as stochastic component which is governed by a Brownian motion $W(t)$. The function is given by

$$\begin{aligned} S(t) &= \mu(t) + \sigma(t)W(t), \\ S(0) &= S_0. \end{aligned} \tag{2.1}$$

Now, the change in (2.1) between two time periods can be expressed as a ratio given by $\frac{dS(t)}{S(t)}$ where $dS(t)$ is the increment at time t. It follows that

$$dS(t) = S(t)\mu(t)dt + S(t)\sigma(t)dW(t). \tag{2.2}$$

For a stock yielding a continuous dividend at time t, the price of the stock decreases by the rate of the yield. This means that if we express the dividend by $\delta(t)$, the stock decreases by $\delta(t)S(t)dt$, which is equivalent with subtracting the yield from the drift. We can rewrite (2.2) as

$$dS(t) = S(t)(\mu(t) - \delta(t))dt + S(t)\sigma(t)dW(t). \tag{2.3}$$

Further, we introduce a money market account $B(t) = e^{rt}$ which grows continuously by a constant rate r. The characteristics of the increment is given by

$$\begin{aligned} dB(t) &= rB(t)dt, \\ B(0) &= 1. \end{aligned} \tag{2.4}$$

To derive the Black-Scholes PDE we need to express the stock and the money market account as a value process $V(t)$. The value process is weighted by a stochastic function $f(x,y)$ where x is the portion of stocks held and y is the portion of money held at the money market market account. $f(x,y)$ rebalances in proportion to the assets to maintain the value. The process is given by

$$V(t) = x(t)S(t) + y(t)B(t),$$

and is self-financing if

$$\begin{aligned} dV(t) &= x(t)dS(t) + y(t)dB(t) \\ &= x(t)\Big[S(t)(\mu(t) - \delta(t))dt + S(t)\sigma(t)dW(t)\Big] + y(t)rB(t)dt \\ &= \Big[x(t)S(t)(\mu(t) - \delta(t)) + y(t)rB(t)\Big]dt + x(t)S(t)\sigma(t)dW(t). \end{aligned} \tag{2.5}$$

Now, we must introduce a relative portfolio $g(u,w)$ such that the portion of respective asset is expressed as a ratio of the total value of $dV(t)$. This is done by letting

$$u(t) = \frac{x(t)S(t)\left(1 - \frac{\delta(t)}{\mu(t)}\right)}{V(t)} \quad \text{and} \quad w(t) = \frac{y(t)B(t)}{V(t)}, \quad u(t) + w(t) = 1. \tag{2.6}$$

By putting the expressions from (2.6) into (2.8) we get

$$dV(t) = V(t)\Big[u(t)\mu(t) + rw(t)\Big]dt + V(t)w(t)\sigma(t)dW(t). \tag{2.7}$$

By assuming that $V(t) = V(t, S(t))$ we can use ItÃť Lemma on (2.7) and get

$$\begin{aligned} dV(t) &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 \\ &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\Big[S(t)(\mu(t) - \delta(t))dt + S(t)\sigma(t)dW(t)\Big] + \frac{S(t)^2\sigma^2}{2}\frac{\partial^2 V}{\partial S^2}dt \\ &= \left[\frac{\partial V}{\partial t} + S(t)(\mu(t) - \delta(t))\frac{\partial V}{\partial S} + \frac{S(t)^2\sigma^2}{2}\frac{\partial^2 V}{\partial S^2}\right]dt + S(t)\sigma(t)dW(t). \end{aligned}$$

By multiplying the right hand side by $\frac{V(t)}{V(t)}$

$$= V(t)\left[\frac{\frac{\partial V}{\partial t} + S(t)(\mu(t) - \delta(t))\frac{\partial V}{\partial S} + \frac{S(t)^2\sigma^2}{2}\frac{\partial^2 V}{\partial S^2}}{V(t)}\right]dt + V(t)\frac{S(t)\sigma(t)dW(t)}{V(t)},$$

and comparing with experssion (2.7) we have that

$$u(t) = \frac{\big(1 - \frac{\delta(t)}{\mu(t)}\big)S(t)\frac{\partial V}{\partial S}}{V(t)}, \tag{2.8}$$

and

$$w(t) = \frac{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S(t^2)\frac{\partial^2 V}{\partial S^2}}{V(t)r}. \tag{2.9}$$

By now we should substantially reduce the partial derivative notations. We use $\frac{\partial V}{\partial t} = V_t$, $\frac{\partial V}{\partial S} = V_S$ and $\frac{\partial^2 V}{\partial S^2} = V_{SS}$. By going back and plug (8) and (9) into (2.6) and let $r = \mu(t)$, the result will become

$$\begin{aligned} u(t) + w(t) &= 1 \\ \frac{\big(1 - \frac{\delta(t)}{r}\big)S(t)V_S}{V(t)} + \frac{V_t + \frac{1}{2}\sigma^2 S(t)^2 V_S S}{V(t)r} &= 1 \\ \big(r - \delta(t)\big)S(t)V_S + V_t + \frac{1}{2}\sigma^2 S(t)^2 V_{SS} &= rV(t), \end{aligned} \tag{2.10}$$

which is the Black-Scholes PDE modified for a continuous yield.

2.7 Parts of the Solution Affected by Dividend.

The pay-off for an European call option at maturity is

$$V_T = \max S(T) - K, 0, \tag{2.11}$$

and suppose that V(t,S(t)) is a solution to (2.10). We recall the incremental stock price and it characteristics given in (2.1) and (2.2), and use Itô Lemma to write (2.10) as

$$\begin{aligned} dV &= V_t dt + V_S dS + \frac{1}{2} V_{SS} (dS)^2 \\ &= V_t dt + V_S \Big[S(t)(\mu(t) - \delta(t))dt + S(t)\sigma(t) dW(t) \Big] + \frac{1}{2} S(t)^2 \sigma^2 V_{SS} dt \\ &= \Big[V_t + V_S S(\mu(t) - \delta(t)) + \frac{1}{2} S(t)^2 \sigma^2 V_{SS} \Big] dt + S(t)\sigma(t) dW(t) \\ &= rV dt + S(t)\sigma(t) dW(t). \end{aligned} \tag{2.12}$$

We also show that the price diffusion process will be slightly different when introducing a dividend. Recall equation (2.2), to solve this by Itô Lemma we must introduce a new variable $Z = ln(S(t))$ solve for Z. We have

$$\begin{aligned} dZ &= Z_t dt + Z_S dS + \frac{1}{2} Z_{SS} (dS)^2 \\ &= \frac{1}{S} \Big[(r - \delta(t)) S dt + \sigma(t) S dW(t) \Big] - \frac{1}{2S^2} \sigma^2 S^2 dt \\ &= \big(r - \delta(t) - \frac{1}{2} \sigma^2 \big) dt + \sigma(t) dW(t). \end{aligned}$$

If we integrate and exponentiate both hand sides of the expression above it gives

$$\begin{aligned} \int_t^T dZ &= \int_t^T \big(r - \delta(t) - \frac{1}{2} \sigma^2 \big) dt + \int_t^T \sigma(t) dW(t) \\ ln(S(T)) - ln(S(t)) &= \big(r - \delta(t) - \frac{1}{2} \sigma^2 \big)(T - t) + \sigma(t)(W(T) - W(t)) \\ e^{ln(S(T))} &= e^{S(t) + \left(r - \delta(t) - \frac{1}{2}\sigma^2 \right)(T-t) + \sigma(t)\left(W(T) - W(t) \right)} \\ S(T) &= S(t) e^{\left(r - \delta(t) - \frac{1}{2}\sigma^2 \right)(T-t) + \sigma(t)\left(W(T) - W(t) \right)} \\ &= S(t) e^{y}. \end{aligned} \tag{2.13}$$

Let $(T - t) = \tau$, then the probability distribution of Z is normal with mean $\big(r - \delta(t) - \frac{1}{2}\sigma^2 \big)\tau$ and variance $\sigma^2 \tau$ and the probability density function is given by

$$g(S) = \frac{1}{\sigma S(t) \sqrt{2\pi\tau}} \exp \left\{ \frac{\big((r - \delta(t) - \frac{1}{2}\sigma^2)\tau \big)^2}{2\sigma^2 \tau} \right\}$$

Before we can continue and finish the solution, we must clarify two things. First of all, we need to find the value of the pay-off when y equals zero. That is when

$$S(t)e^y - K = 0$$
$$y_0 = ln\left(\frac{K}{S(t)}\right),$$

From that solution we can introduce a new variable z and rewrite $g(S)$ into

$$g(S) = \frac{1}{\sigma S(t)\sqrt{2\pi\tau}} \exp\left\{\frac{-z^2}{2}\right\}$$

by showing that

$$z = \frac{y-(r-\delta)\tau}{\sqrt{\tau}} = \frac{ln\left(\frac{K}{S(t)}\right) - (r-\delta)\tau}{\sqrt{\tau}} = z_0 = -d_1$$

The rest of the solution to Black-Scholes follows the same procedure as for the non-dividend case. Note that $N(-z_0) = N(d_1)$ and that z_0 is slightly modified by the dividend.

Chapter 3

Application and results

We developed a MATLAB program where the user can choose several properties of the underlying asset and the dividends. We start of by investigating the theoretical European Call and Put price for different dividend yields. Later we turn to the discrete case and investigate how the size of dividends influences the price. Additionally we consider different times for dividends keeping the other parameters untouched.

In the second part we turn our attention to American Options. Using the Black's Approximation, we evaluate if early exercise is optimal under certain criteria, such as size of and time to dividend. Black's Appr. can also be used if there are more than one dividend payout within the lifetime of an American Option.

At last we present a simple MATLAB program where the user can change parameters, with or without dividends, in order to see theoretical prices and if early exercise would be optimal.

3.1 European case with dividend yield

Here we show how the option price is influenced by a changing yield.

As anticipated we see that the theoretical price of a Call is negatively correlated to the size of the yield while the Put shows the reverse pattern. Obviously one could not expect yields of magnitude 25 %.

In fig. 3.2 we have calculated the present value of the dividend using the simple formula $De^{-t_n r}$. Where D is the size of the dividend, t_n is time to dividend and r is the risk free interest rate.

As expected the price of a Call/Put decreases/increases when the size of the dividend increases. We see that for a sufficiently a large dividend, the Call price approaches zero while the Put approaches K.

Figure 3.1: **European Options with changing size of continuous dividend yield**

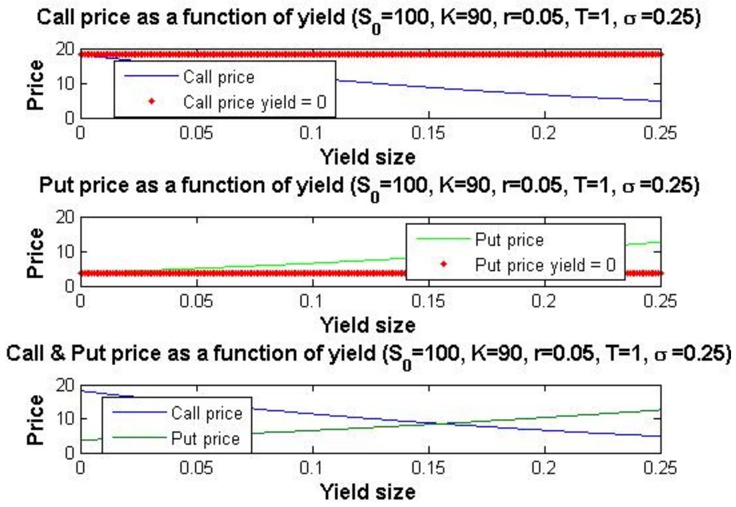
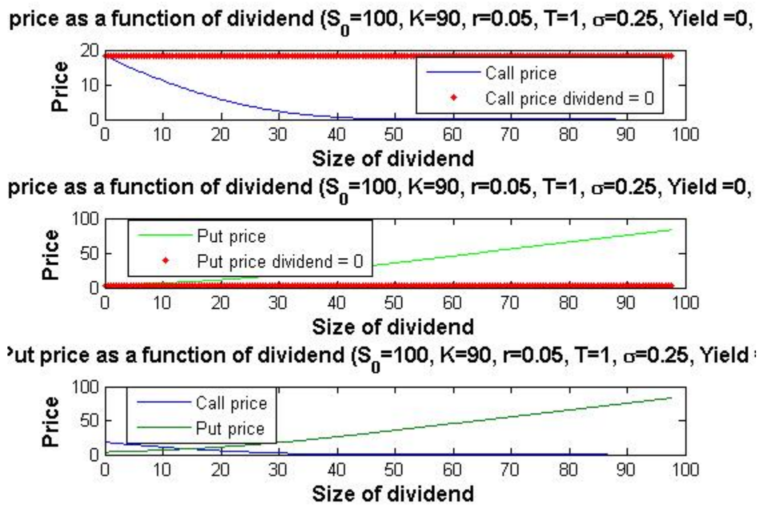


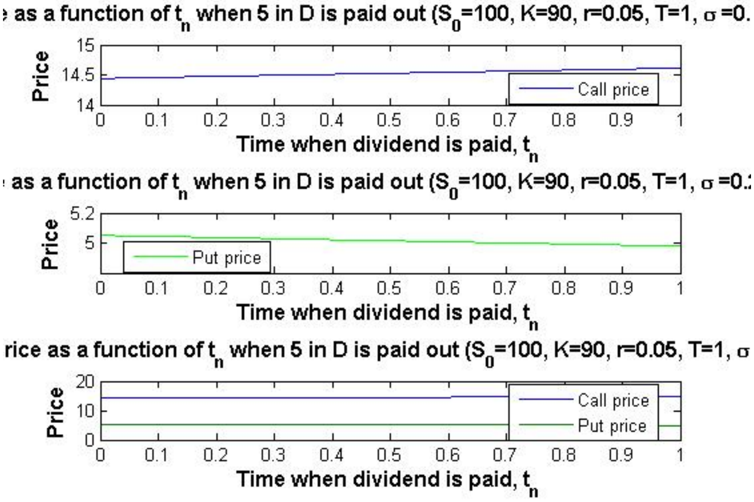
Figure 3.2: **European Options with different discrete dividend values**



Using the previous method to calculate the present value of the dividend, we see in Figure 3.3 how the value for Calls and Puts changes depending on when the dividend is paid. For Calls the value seems to increase with the decrease of time left to next dividend. Despite this, the changes in the theoretical prices are very small, approximately 0.10.

We can conclude that the size of the dividend or the yield has a much larger impact on the price than when a dividend is paid out for instruments with one year to maturity.

Figure 3.3: **European Options with different times when dividends are paid**



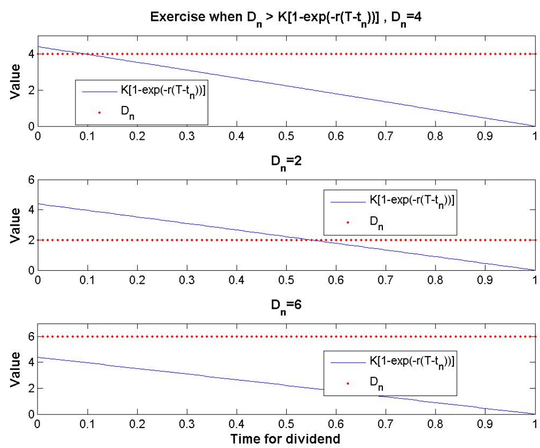
3.2 American Options

It can be shown that it is always optimal to exercise at time t_n for a sufficiently high value of the stock price $S(t_n)$ if the following inequality is valid.[8]

$$D_n > K[1 - e^{-r(T-t_n)}]$$

where D_n is the size of the dividend, K is the strike price, r is the risk free interest rate and t_n is time to dividend.

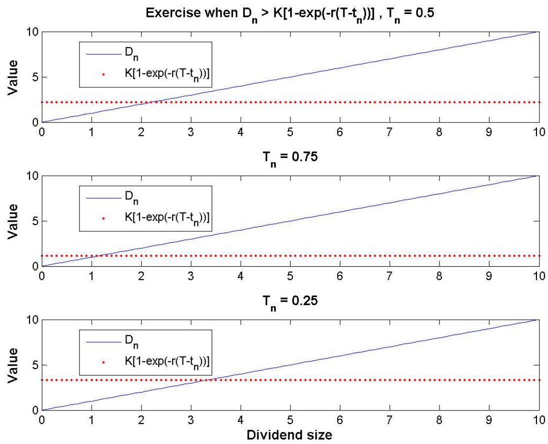
Figure 3.4: **American Options with discrete dividend of constant size**



We can see that the inequality above will tend to be satisfied when the dividend is large and/or the dividend date is fairly close to maturity. In the third panel we can see that the option will always be exercised for sufficiently high stock prices regardless of when the dividend is paid out.

Letting the time for dividend payout be constant, we can see below how the size of the dividend influences whether early exercise might be optimal.

Figure 3.5: **American options with discrete dividends of different size and the same t_n**



All the panels in Figure 3.5 show that there exist boundary values that , if they are exceeded, lead to early exercise at dividend date. The longer away the dividend date lies from today, the lower the dividend needs to be for early exercise. This can be seen in panel 2.

3.3 Black's approximation

The following inequality can be used for the case of several dividends.

$$D_i \leq K[1 - e^{-r(t_{i+1} - t_i)}]$$

If the inequality is fulfilled, it is not optimal to exercise immediately prior to time t_{n-1}.

Assume there are two dividends, both of size 2. The first dividend occurs in one month and the second in seven months. By using the Black's approximation we check whether the inequality above is fulfilled at any time.

$$90[1 - e^{-0.05*(7/12-1/12)}] = 2.2221$$

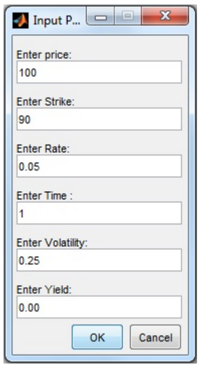
$$90[1 - e^{-0.05*(1-7/12)}] = 1.8556$$

As we can see, 2.21 is greater than 2.00. Thus the option should never be exercised just before the first ex-dividend date. On the contrary, since 1.86 is smaller than 2.00. There are sufficiently high stock prices where early exercise is optimal just prior to the second ex-dividend date.

3.4 User version

A simple MATLAB program is created where the user can change the values of the Black-Scholes parameters in order to get the theoretical option prices. Later, the user can state whether the underlying asset pays zero, one or two dividends during the lifetime of the contract. Updated European prices will then be given. In addition, the Black's Approximation, which was explained earlier in this paper, will test if there might be any optimal times to exercise the call-option prior to expiry. Below we guide the reader through one conceivable procedure. When starting the program the following box will show up

Figure 3.6: **Starting box for user version**



Running the program with the predetermined values gives the following output

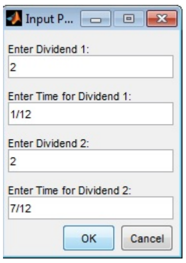
For a European call/put
Price of the call is 18.140763
Price of the put is 3.751411

Do underlying pay dividend? y/n

The second step is to indicate whether the underlying asset pays dividends. Answering 'y' and '2' in the MATLAB command prompt will initiate the following box

Now the program recalculates the theoretical option prices for a European call and put. In addition, it uses the Black's approximation to see if there are any dividend times when early exercise is optimal (given a sufficiently high stock price). Changing the values to use the same parameters as earlier (i.e. dividends of $D_i = [2, 2]$ at time $t_n = [1/12, 7/12]$), yields the following outcome

Figure 3.7: **Box for telling the program if asset pays dividends**



For a European call/put
Price of the call is 15.200774
Price of the put is 4.745616

For a American call
Do not exercise (call) at first dividend
Exercise (call) at second dividend

We see that the call/put price decrease/increase when the two dividends are accounted for. Furthermore, we get the expected result as we did in the previous section using the Black's approximation.

Chapter 4

Discussion and conclusion

In this report, we briefly investigate the effect that dividends have when using the Black-Scholes pricing formulae. In order to calculate the theoretical prices of options we employ several (sometimes unrealistic) assumptions about the market and the underlying asset. Despite that we investigate scenarios that are implausible in the real market (e.g. very high yield or high discrete dividends), it is quite instructive to see how the theoretical option prices are affected when dividends are accounted for. For short maturities (in our case 1 year), the option value is greater influenced by the size of the dividend compared to *when* it occurs. For American options there is always an opportunity where an early exercise might be optimal given that. By using the Black's approximation, we see in general, that if early exercise is optimal, it will happen just prior to the latest dividend. This is seen from our numerical examples above. For instruments where the underlying just pays one dividend, early exercise is depending on the size of the dividend and when it is paid out.

4.1 Comments

When one derives the Black-Scholes PDE for continuous dividends and uses the approach involving a money market account as above, one of the weight equations ($x(t)$) changes from the non-dividend case. This change is compulsory if one doesn't want δ to fall out of the final PDE along with μ. For a non-dividend case, r is what is left in the PDE and $r = \mu(t)$ in a risk neutral world, it is thereby not a valid solution if one first lets $(\mu(t) - \delta(t))$ fall out and then just replace r in the final PDE by $(\mu(t) - \delta(t))$, since $r \neq (\mu(t) - \delta(t))$. Other sources of literature suggests using the delta hedge approach where one creates a value process of Δ shares and a call option instead of using the money market account.

Bibliography

[1] Hillier, D. (2010). Corporate Finance (1 ed.). McGraw Hill,Berkshire, p. 488/489

[2] Hull, John C. (2008). Options, Futures and Other Derivatives (7 ed.). Prentice Hall, p. 202

[3] Hull, John C. (2008). Options, Futures and Other Derivatives (7 ed.). Prentice Hall, p. 286/287

[4] https://www.avanza.se/aza/aktieroptioner/kurslistor/aktie.jsp?orderbookId=176688

[5] https://www.riksgalden.se/sv/omriksgalden/statsskulden/aktuella-siffror/statslanerantan/?year=2012

[6] Hull, John C. (2008). Options, Futures and Other Derivatives (7 ed.). Prentice Hall, p.800

[7] Röman, J *Absolute Dividends and the Inherent Diffusion Model Problem of the Black-Scholes Universe*

[8] Hull, p.299

4.2 Appendix

```
close all
clear all
clc
tic;


%% European options, dividend yield
%[Call, Put] = blsprice(Price, Strike, Rate, Time, Volatility, Yield)

% Setting some initial conditions, ITM(OTM) Call(Put)
Price = 100;
Strike = 90;
Rate = 0.05;
Time = 1;
Volatility = 0.25;

% No dividend case
[temp1,temp2] = blsprice(Price,Strike,Rate,Time,Volatility,0);
Call_DivYield0 = temp1;
Put_DivYield0 = temp2;

Yield = 0:0.001:0.25;
% Loop for Yield Ă[0,0.25] with small increments
for i = 1:1:length(Yield)
[Call_EUyield(i), Put_EUyield(i)] = blsprice(Price,Strike,Rate,Time,V
end

% Plot results
figure(1)
subplot(3,1,1)
plot(Yield,Call_EUyield)
hold on
plot(Yield,Call_DivYield0,'.r')
title(['Call␣price␣as␣a␣function␣of␣yield␣(S_0=' num2str(Price) ',␣K=
xlabel('Yield␣size','fontsize',12,'fontweight','bold')
ylabel('Price','fontsize',12,'fontweight','bold')
legend('Call␣price','Call␣price␣yield␣=␣0','Location','Best')
hold off

subplot(3,1,2)
plot(Yield,Put_EUyield,'g')
```

```
hold on
plot(Yield,Put_DivYield0,'.r')
title(['Put␣price␣as␣a␣function␣of␣yield␣(S_0=' num2str(Price) ',␣K=' num2st
xlabel('Yield␣size','fontsize',12,'fontweight','bold')
ylabel('Price','fontsize',12,'fontweight','bold')
legend('Put␣price','Put␣price␣yield␣=␣0','Location','Best')
hold off

subplot(3,1,3)
plot(Yield,Call_EUyield,Yield,Put_EUyield)
hold on
title(['Call␣&␣Put␣price␣as␣a␣function␣of␣yield␣(S_0=' num2str(Price) ',␣K='
xlabel('Yield␣size','fontsize',12,'fontweight','bold')
ylabel('Price','fontsize',12,'fontweight','bold')
legend('Call␣price','Put␣price','Location','Best')

%% Individual plots, uncomment to plot and save
% figure(2)
% plot(Yield,Call_EUyield)
% hold on
% plot(Yield,Call_DivYield0,'.r')
% title(['Call price as a function of yield (S_0=' num2str(Price) ', K=' num
% xlabel('Yield size','fontsize',12,'fontweight','bold')
% ylabel('Price','fontsize',12,'fontweight','bold')
% legend('Call price','Call price yield = 0','Location','Best')
% hold off
%
% figure(3)
% plot(Yield,Put_EUyield,'g')
% hold on
% plot(Yield,Put_DivYield0,'.r')
% title(['Put price as a function of yield (S_0=' num2str(Price) ', K=' num2
% xlabel('Yield size','fontsize',12,'fontweight','bold')
% ylabel('Price','fontsize',12,'fontweight','bold')
% legend('Put price','Put price yield = 0','Location','Best')
% hold off
%
% figure(4)
% plot(Yield,Call_EUyield,Yield,Put_EUyield)
% hold on
% title(['Call & Put price as a function of yield (S_0=' num2str(Price) ', K
% xlabel('Yield size','fontsize',12,'fontweight','bold')
% ylabel('Price','fontsize',12,'fontweight','bold')
% legend('Call price','Put price','Location','Best')
```

```
%
% saveas(figure(2), 'EUFigure2.jpeg','jpeg');
% saveas(figure(3), 'EUFigure3.jpeg','jpeg');
% saveas(figure(4), 'EUFigure4.jpeg','jpeg');

%% European options - discrete dividends

% Setting some initial conditions, ITM(OTM) Call(Put)
Price = 100;
Strike = 90;
Rate = 0.05;
Time = 1;
Volatility = 0.25;
Yield = 0;

% No dividend case
[temp1,temp2] = blsprice(Price, Strike, Rate, Time, Volatility,Yield)
Call_Divdis0 = temp1;
Put_Divdis0 = temp2;

% Letting time for dividend be constant, Dtime
Dtime = 0.5;

Dividend_D =[0:0.5:100]*exp(-Dtime*Rate);
% Looping for DivĂ[0,100] with small increments
for i = 1:1:length(Dividend_D)
[Call_EUdis_D(i), Put_EUdis_D(i)] = blsprice(Price-Dividend_D(i),Strik
end

% Plot results
figure(5)
subplot(3,1,1)
plot(Dividend_D,Call_EUdis_D)
hold on
plot(Dividend_D,Call_Divdis0,'.r')
title(['Call␣price␣as␣a␣function␣of␣dividend␣(S_0=' num2str(Price) ',
xlabel('Size␣of␣dividend','fontsize',12,'fontweight','bold')
ylabel('Price','fontsize',12,'fontweight','bold')
legend('Call␣price','Call␣price␣dividend␣=␣0','Location','Best')
hold off

subplot(3,1,2)
plot(Dividend_D,Put_EUdis_D,'g')
```

```
hold on
plot(Dividend_D,Put_Divdis0,'.r')
title(['Put price as a function of dividend (S_0=' num2str(Price) ', K=' num
xlabel('Size of dividend','fontsize',12,'fontweight','bold')
ylabel('Price','fontsize',12,'fontweight','bold')
legend('Put price','Put price dividend = 0','Location','Best')
hold off

subplot(3,1,3)
plot(Dividend_D,Call_EUdis_D,Dividend_D,Put_EUdis_D)
title(['Call & Put price as a function of dividend (S_0=' num2str(Price) ',
xlabel('Size of dividend','fontsize',12,'fontweight','bold')
ylabel('Price','fontsize',12,'fontweight','bold')
legend('Call price','Put price','Location','Best')

%% Individual plots, uncomment to plot and save
% figure(6)
% plot(Dividend_D,Call_EUdis_D)
% hold on
% plot(Dividend_D,Call_Divdis0,'.r')
% title(['Call price as a function of dividend (S_0=' num2str(Price) ', K='
% xlabel('Size of dividend','fontsize',12,'fontweight','bold')
% ylabel('Price','fontsize',12,'fontweight','bold')
% legend('Call price','Call price dividend = 0','Location','Best')
% hold off
%
% figure(7)
% plot(Dividend_D,Put_EUdis_D,'g')
% hold on
% plot(Dividend_D,Put_Divdis0,'.r')
% title(['Put price as a function of dividend (S_0=' num2str(Price) ', K=' n
% xlabel('Size of dividend','fontsize',12,'fontweight','bold')
% ylabel('Price','fontsize',12,'fontweight','bold')
% legend('Put price','Put price dividend = 0','Location','Best')
% hold off
%
% figure(8)
% plot(Dividend_D,Call_EUdis_D,Dividend_D,Put_EUdis_D)
% title(['Call & Put price as a function of dividend (S_0=' num2str(Price) '
% xlabel('Size of dividend','fontsize',12,'fontweight','bold')
% ylabel('Price','fontsize',12,'fontweight','bold')
% legend('Call price','Put price','Location','Best')
%
% saveas(figure(6), 'EUFigure6.jpeg','jpeg');
```

```
% saveas(figure(7), 'EUFigure7.jpeg','jpeg');
% saveas(figure(8), 'EUFigure8.jpeg','jpeg');


% Letting size of dividend be constant, D
D = 5;

D_time = 0:0.01:1;
% Looping for D_timeĂ[0,1] with small increments
for i = 1:1:length(D_time)
Dividend_T(i) = D*exp(-D_time(i)*Rate);
[Call_EUdis_T(i), Put_EUdis_T(i)] = blsprice(Price-Dividend_T(i),Strik
end

% Plot results
figure(9)
subplot(3,1,1)
plot(D_time,Call_EUdis_T)
title(['Call␣price␣as␣a␣function␣of␣t_n␣when␣',num2str(D),'␣in␣D␣is␣p
xlabel('Time␣when␣dividend␣is␣paid,␣t_n','fontsize',12,'fontweight','
ylabel('Price','fontsize',12,'fontweight','bold')
legend('Call␣price','Location','Best')

subplot(3,1,2)
plot(D_time,Put_EUdis_T,'g')
title(['Put␣price␣as␣a␣function␣of␣t_n␣when␣',num2str(D),'␣in␣D␣is␣pa
xlabel('Time␣when␣dividend␣is␣paid,␣t_n','fontsize',12,'fontweight','
ylabel('Price','fontsize',12,'fontweight','bold')
legend('Put␣price','Location','Best')

subplot(3,1,3)
plot(D_time,Call_EUdis_T,D_time,Put_EUdis_T)
title(['Call␣%␣Put␣price␣as␣a␣function␣of␣t_n␣when␣',num2str(D),'␣in␣
xlabel('Time␣when␣dividend␣is␣paid,␣t_n','fontsize',12,'fontweight','
ylabel('Price','fontsize',12,'fontweight','bold')
legend('Call␣price','Put␣price','Location','Best')

%% Individual plots, uncomment to plot and save
% figure(10)
% plot(D_time,Call_EUdis_T)
% title(['Call price as a function of t_n when ',num2str(D),' in D is
% xlabel('Time when dividend is paid, t_n','fontsize',12,'fontweight'
% ylabel('Price','fontsize',12,'fontweight','bold')
```

```
% legend('Call price','Location','Best')
%
% figure(11)
% plot(D_time,Put_EUdis_T,'g')
% title(['Put price as a function of t_n when ',num2str(D),' in D is paid ou
% xlabel('Time when dividend is paid, t_n','fontsize',12,'fontweight','bold'
% ylabel('Price','fontsize',12,'fontweight','bold')
% legend('Put price','Location','Best')
%
% figure(12)
% plot(D_time,Call_EUdis_T,D_time,Put_EUdis_T)
% title(['Call & Put as a function of t_n when Dividend = ',num2str(D),' (S_
% xlabel('Time when dividend is paid, t_n','fontsize',12,'fontweight','bold'
% ylabel('Price','fontsize',12,'fontweight','bold')
% legend('Call price','Put price','Location','Best')
%
% saveas(figure(10), 'EUFigure10.jpeg','jpeg');
% saveas(figure(11), 'EUFigure11.jpeg','jpeg');
% saveas(figure(12), 'EUFigure12.jpeg','jpeg');

%%

toc;

close all
clear all
clc
tic

%% American options
% Check for early exercise

% Setting some initial conditions, ITM(OTM) Call(Put)
Price = 100;
Strike = 90;
Rate = 0.05;
Time = 1;
Volatility = 0.25;
Yield = 0;
Dn = 4;
Tn = 0:0.01:1;

% Hull 2009
% Dn <= K[1-exp(-r(T-tn)] 13.24
```

```
% Dn > K[1-exp(-r(T-tn)]   13.25


% Checking for which Tn to exercise for some given Dn
% Looping over the range of Tn
for i = 1:1:length(Tn)
RHS(i) = Strike*(1-exp(-Rate*(Time-Tn(i))));
end

figure(1)
subplot(3,1,1)
plot(Tn,RHS)
hold on
plot(Tn,Dn,'.r')
axis([0 1 0 Dn+1])
title(['Exercise␣when␣D_n␣>␣K[1-exp(-r(T-t_n))]␣,␣D_n=' num2str(Dn)],
ylabel('Value','fontsize',12,'fontweight','bold')
legend('K[1-exp(-r(T-t_n))]','D_n','Location','Best')

% Decreasing the dividend
Dn = 2;

% Looping over the range of Tn
for i = 1:1:length(Tn)
RHS(i) = Strike*(1-exp(-Rate*(Time-Tn(i))));
end

subplot(3,1,2)
plot(Tn,RHS)
hold on
plot(Tn,Dn,'.r')
title(['D_n=' num2str(Dn)],'fontsize',12,'fontweight','bold')
ylabel('Value','fontsize',12,'fontweight','bold')
legend('K[1-exp(-r(T-t_n))]','D_n','Location','Best')

% Increasing the dividend
Dn = 6;

for i = 1:1:length(Tn)
RHS(i) = Strike*(1-exp(-Rate*(Time-Tn(i))));
end

subplot(3,1,3)
plot(Tn,RHS)
```

```
hold on
plot(Tn,Dn,'.r')
axis([0 1 0 Dn+1])
title(['D_n=' num2str(Dn)],'fontsize',12,'fontweight','bold')
xlabel('Time for dividend','fontsize',12,'fontweight','bold')
ylabel('Value','fontsize',12,'fontweight','bold')
legend('K[1-exp(-r(T-t_n))]','D_n','Location','Best')

% Uncomment to save
% saveas(figure(1), 'AMFigure1.jpeg','jpeg');

% Checking for which dDn to exercise early for a fixed Tn
% Setting some initial conditions, ITM(OTM) Call(Put)
Price = 100;
Strike = 90;
Rate = 0.05;
Time = 1;
Volatility = 0.25;
Yield = 0;
Tn = 0.5;
D=0:0.1:10;

RHS=Strike*(1-exp(-Rate*(Time-Tn)));

figure(2)
subplot(3,1,1)
plot(D,D)
hold on
plot(D,RHS,'.r')
title(['Exercise when D_n > K[1-exp(-r(T-t_n))] , T_n = ' num2str(Tn)],'font
ylabel('Value','fontsize',12,'fontweight','bold')
legend('D_n','K[1-exp(-r(T-t_n))]','Location','Best')

% Increasing Tn
Tn = 0.75;

RHS=Strike*(1-exp(-Rate*(Time-Tn)));

subplot(3,1,2)
plot(D,D)
hold on
plot(D,RHS,'.r')
title(['T_n = ' num2str(Tn)],'fontsize',12,'fontweight','bold')
ylabel('Value','fontsize',12,'fontweight','bold')
```

```
legend('D_n','K[1-exp(-r(T-t_n))]','Location','Best')

% Decreasing Tn
Tn = 0.25;

RHS=Strike*(1-exp(-Rate*(Time-Tn)));

subplot(3,1,3)
plot(D,D)
hold on
plot(D,RHS,'.r')
title(['T_n = ' num2str(Tn)],'fontsize',12,'fontweight','bold')
xlabel('Dividend size','fontsize',12,'fontweight','bold')
ylabel('Value','fontsize',12,'fontweight','bold')
legend('D_n','K[1-exp(-r(T-t_n))]','Location','Best')

% Uncomment to save
% saveas(figure(2), 'AMFigure2.jpeg','jpeg');

%% American options calls, Black's Approximation
% Hull page 300

% Setting some initial conditions, ITM(OTM) Call(Put)
Price = 100;
Strike = 90;
Rate = 0.05;
Time = 1;
Volatility = 0.25;
Yield = 0;

% Setting up a series of dividends Dn that are paid out on some time
Dn = [2,2];
Tn = [1/12,7/12];

first = Strike*(1-exp(-Rate*(Tn(2)-Tn(1))));
second = Strike*(1-exp(-Rate*(Time-Tn(2))));

% Check wheter they should be exercised early
if (first<Dn(1))
    fprintf('Exercise at first dividend\n')
else
    fprintf('Do not exercise at first dividend\n')
end
```

```
if (second<Dn(1))
    fprintf('Exercise at second dividend\n')
else
    fprintf('Do not exercise at second dividend\n')
end

%%

toc;

close all
clear all
clc

% Price = 100;
% Strike = 90;
% Rate = 0.05;
% Time = 1;
% Volatility = 0.25;

prompt={'Enter price:',...
        'Enter Strike:'...
        'Enter Rate:',...
        'Enter Time :'...
        'Enter Volatility:',...
        'Enter Yield:',...
        };
name='Input Parameters';
numlines=1;
defaultanswer={'100','90','0.05','1','0.25','0.00'};
answer=inputdlg(prompt,name,numlines,defaultanswer);

Price = str2num(cell2mat(answer(1)));
Strike = str2num(cell2mat(answer(2)));
Rate = str2num(cell2mat(answer(3)));
Time = str2num(cell2mat(answer(4)));
Volatility = str2num(cell2mat(answer(5)));
Yield = str2num(cell2mat(answer(6)));

[Call,Put] = blsprice(Price,Strike,Rate,Time,Volatility,Yield);
fprintf('For a European call/put\n')
fprintf('Price of the call is %f\n', Call)
fprintf('Price of the put is %f\n', Put)
fprintf('------------------------------------------\n')
```

```
%% Setting up some dividend condition
Dividends = input('Do underlying pay dividend? y/n\n','s');

if (Dividends == 'y')
    Times = input('1 or 2 times? 1/2\n','s');
    fprintf('----------------------------------------\n')

if (Dividends == 'y' && Times == '1')
    prompt={'Enter Dividend 1:',...
        'Enter Time for Dividend 1:'...
        };

    name='Input Parameters';
    numlines=1;
    defaultanswer={'5','0.25'};
    answer=inputdlg(prompt,name,numlines,defaultanswer);
    Dn = str2num(cell2mat(answer(1)));
    Tn = str2num(cell2mat(answer(2)));

    %For European
    Dividend = Dn*exp(-Tn*Rate);
    [Call,Put] = blsprice(Price-Dividend,Strike,Rate,Time,Volatility,
    fprintf('For a European call/put\n')
    fprintf('Price of the call is %f\n', Call)
    fprintf('Price of the put is %f\n\n', Put)

    %For American
    RHS = Strike*(1-exp(-Rate*(Time-Tn)));
    fprintf('For a American put\n')
    if  Dn > RHS
        fprintf('It is optimal to exercise early\n')
    else
        fprintf('It is not optimal at exercise early\n')
    end
        fprintf('----------------------------------------\n')
end

if (Dividends == 'y' && Times == '2')
    prompt={'Enter Dividend 1:',...
        'Enter Time for Dividend 1:'...
        'Enter Dividend 2:',...
        'Enter Time for Dividend 2:'...
        };
```

```
    name='Input Parameters';
    numlines=1;
    defaultanswer={'5','0.25','5','0.75'};
    answer=inputdlg(prompt,name,numlines,defaultanswer);
    Dn = [str2num(cell2mat(answer(1))),str2num(cell2mat(answer(3)))];
    Tn = [str2num(cell2mat(answer(2))),str2num(cell2mat(answer(4)))];


    %For European
    Dividend = Dn(1)*exp(-Tn(1)*Rate) + Dn(2)*exp(-Tn(2)*Rate) ;
    [Call,Put] = blsprice(Price-Dividend,Strike,Rate,Time,Volatility,Yield);
    fprintf('For a European call/put\n')
    fprintf('Price of the call is %f\n', Call)
    fprintf('Price of the put is %f\n\n', Put)

    first = Strike*(1-exp(-Rate*(Tn(2)-Tn(1))));
    second = Strike*(1-exp(-Rate*(Time-Tn(2))));

    % Check wheter they should be exercised early
    fprintf('For a American call\n')
        if (first<Dn(1))
            fprintf('Exercise (call) at first dividend\n')
        else
            fprintf('Do not exercise (call) at first dividend\n')
        end

        if (second<Dn(1))
            fprintf('Exercise (call) at second dividend\n')
        else
            fprintf('Do not exercise (call) at second dividend\n')
        end
        fprintf('--------------------------------------\n')
end

end
```