MÄLARDALEN UNIVERSITY
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American and European Put Option

Analytical Finance I

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1. Introduction

European and American options are most popular options in finance, European options are typically valued using the Black–Scholes or Black model formula. This is a simple equation with a closed-form solution that has become standard in the financial community. There are no general formulae for American options, but a choice of models to approximate the price are available (for example Roll-Geske-Whaley, Barone-Adesi and Whaley, binomial options model by Cox-Ross-Rubinstein, Black's approximation and others; there is no consensus on which is preferable).

Thus, an American option is a European option with the additional right to exercise it any time prior to expiration. This feature naturally raises the interesting question of whether this right is worth something and, if so, how much,

In this paper I am going to study the difference between the American and European put option for different maturity.

2. Option Style

In finance, the style of an option usually defined by the dates on which the option may be exercised. The most popular options are either European or American (style) options.

The difference between American and European options relates to when the options can be exercised:

- A European option may be exercised only at the expiration date of the option.
- An American option may be exercised at any time before the expiry date.

3. Put Option

3.1. Definition

A put or put option is a contract between two parties to exchange an asset, at a specified price (*the strike K),* by a predetermined date (*the expiry* or *maturity T*). One party, the buyer of the put, has the right, but not an obligation, to sell the asset at the strike price by the future date, while the other party, the seller of the put, has the obligation to buy the asset at the strike price if the buyer exercises the option.

3.2. Payoff at Maturity

When the buyer exercises the put at a time *t*, he can expect to receive a payout of *K-S(t)*, if the price of the underlying *S(t)* at that time is less than *K*.

The exercise must occur by time *T*, and what exact times are allowed is specified by the type of put option. An *American option* can be exercised at any time before or equal to *T*, a *European option* can be exercised only at time *T*.

$$P_T = \begin{cases} K - S_T & if\ S_T \le K \\ 0 & if\ S_T > K \end{cases}$$

Or more simply $P_T = \max\{K - S_T\ , 0\}$.

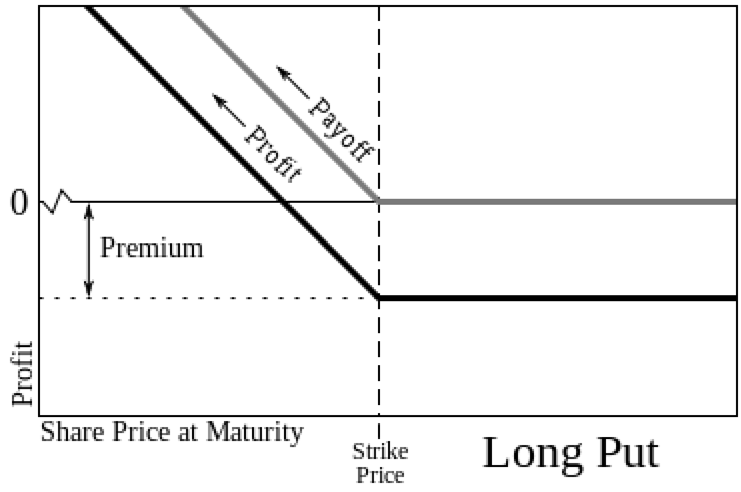
- If $S_T < K$, then the put is said to finish in-the-money and the option will be exercised, the holder of this option will buy the underlying stock at a price of S_T and exercise his right to sell it to the writer at the strike price of K, to make a profit of $K - S_T$.

- If $S_T > K$, the option is said to finish out-of-the-money and exercising the right to sell the underlying asset would result in a loss.
- If the option is not exercised by maturity, it expires worthless.

3.3. Example of a put option on a stock

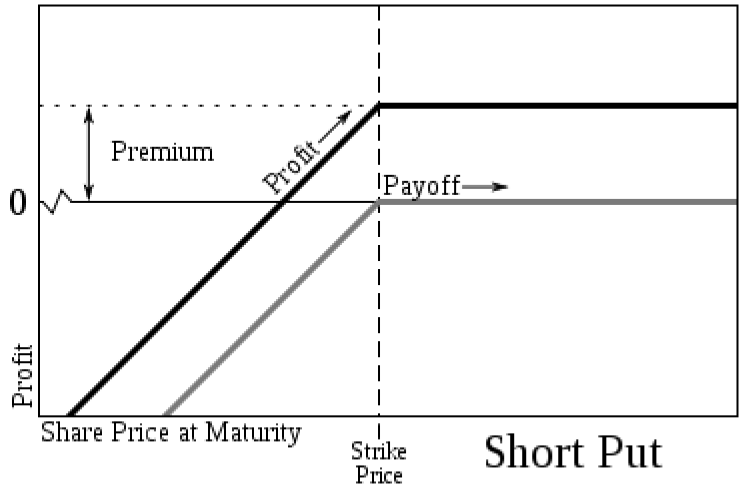
- **Buying a put**

A buyer thinks the price of a stock will decrease. He pays a premium which he will never get back, unless it is sold before it expires. The buyer has the right to sell the stock at the strike price.



- **Writing a put**

The writer receives a premium from the buyer. If the buyer exercises his option, the writer will buy the stock at the strike price. If the buyer does not exercise his option, the writer's profit is the premium.



3.4. Option's parameters

In their derivation of the Black-Scholes option-pricing model, Black, Scholes, and Merton showed that the value of a call or put option depends on the following parameters:

- Current stock price.
- Option's exercise price.
- Time (in years) until the option expires (referred to as the option's duration).
- Interest rate (per year on a compounded basis) on a risk-free investment throughout the duration of the investment. This rate is called the risk-free rate. Compound interest simply means that at every instant, you are earning interest on your interest.
- Annual rate (as a percentage of the stock price) at which dividends are paid.
- Stock volatility (measured on an annual basis).

In general, the effect of changing an input parameter on the value of a put is given in the following table:

PARAMETER	EUROPEAN PUT	AMERICAN PUT
Stock price	-	_
Exercise price	+	+
Time to expiration	?	+
Volatility	+	+
Risk-free rate	-	-
Dividends	+	+

- An increase in today's stock price always decreases the value of a put.
- An increase in the exercise price always increases the value of a put.
- An increase in the duration of an option always increases the value of an American option. In the presence of dividends, an increase in the duration of an option can either increase or decrease the value of a European option.
- An increase in volatility always increases option value.

- An increase in the risk-free rate always decreases the value of a put because the higher growth rate of the stock tends to hurt the put, as does the fact that future payoffs from the put are worth less.
- Dividends tend to reduce the growth rate of a stock price, so increased dividends increase the value of a put.

4. Option Pricing

4.1. Black-Scholes Model

- **Definition**

The Black-Scholes formula is a mathematical formula designed to price an option as a function of certain variables, generally stock price, striking price, volatility, time to expiration, dividends to paid, and the current risk-free interest rate.

- **Black-Scholes Formula**

The payoff to a European put option with strike price K at the maturity date T is

$$P(T) = max\{K - S(T), 0\}$$

as the put option gives the right to sell underlying asset at the strike price of K.

The Black-Scholes formula for the price of the put option is given by

$$p(0) = c(0) + e^{-rT}K - S(0) = e^{-rT}K\big(1 - N(d_2)\big) - S(0)(1 - N(d_1))$$

where d1 and d2 are defined by

$$d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

Here σ^2 is the variance of the continuously compounded rate of return on the underlying asset.

By the symmetry of the standard normal distribution *N(-d) = (1-N(d))* so the formula for the put option is usually written as

$$p(0) = e^{-rT}KN(-d_2) - S(0)N(-d_1)$$

- **Black-Scholes Model Assumptions**

There are several assumptions underlying the Black-Scholes model of calculating options pricing. The most significant is that volatility, a measure of how much a stock can be expected to move in the near-term, is a constant over time. The Black-Scholes model also assumes stocks move in a manner referred to as a random walk; at any given moment, they are as likely to move up as they are to move down. These assumptions are combined with the principle that options pricing should provide no immediate gain to either seller or buyer.

4.2. Binomial options pricing model

One simple and powerful model for stock price evolution is the binomial tree model. It is useful for pricing and hedging options. In the binomial tree model, stock can take 1 of 2 possible values at each node in the tree.

It is not a bad approximation when the time intervals are very small. The model is called a tree (or sometimes lattice) because each branch splits into more branches and so on.

- **Binomial Method**

The binomial pricing model traces the evolution of the option's key underlying variables in discrete-time. This is done by means of a binomial lattice (tree), for a number of time steps between the valuation and expiration dates. Each node in the lattice represents a possible price of the underlying at a given point in time.

Valuation is performed iteratively, starting at each of the final nodes (those that may be reached at the time of expiration), and then working backwards through the tree towards the first node (valuation date). The value computed at each stage is the value of the option at that point in time.

Option valuation using this method is, as described, a three-step process:

1. Price tree generation.
2. Calculation of option value at each final node
3. Sequential calculation of the option value at each preceding node.

STEP 1: Create the binomial price tree

The tree of prices is produced by working forward from valuation date to expiration.

At each step, it is assumed that the underlying instrument will move up or down by a specific factor (u or d) per step of the tree (where, by definition, $u \geq 1$, and $0 < d \leq 1$).

If S is the current price, then in the next period the price will either be

$$S_{up} = S.u \quad or \quad S_{down} = S.d$$

The up and down factors are calculated using the underlying volatility,σ, and the time duration of a step, Δt, measured in years, we have:

$$u = e^{\sigma\sqrt{t}}$$

$$d = e^{-\sigma\sqrt{t}} = \frac{1}{u}$$

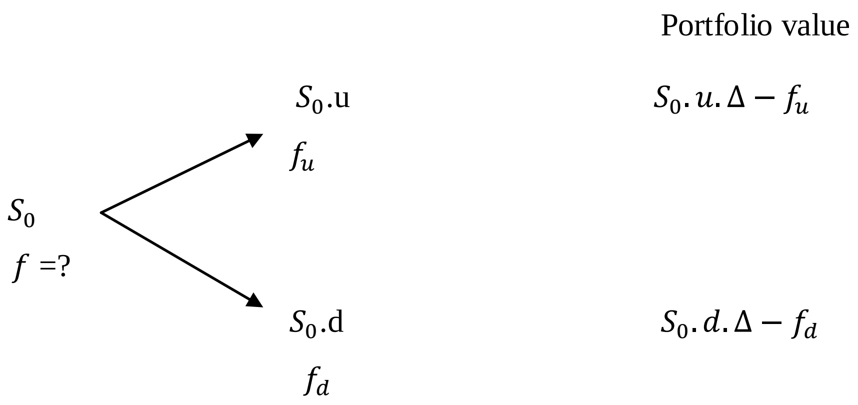
STEP 2: Find Option value at each final node

At each final node of the tree, the option value is equal to:

$max\{K - S_n, 0\}$, for a put option.

Where K is the strike price and S_n is the spot price of the underlying asset at the n^{th} period.

STEP 3: Find Option value at earlier nodes



At time T, the portfolio has value $S_0.u.\Delta - f_u$ at the upper node and $S_0.d.\Delta - f_d$ at the lower node, to make the portfolio riskless, these two values should be set equal:

$$S_0.u.\Delta - f_u = S_0.d.\Delta - f_d$$

$$\Rightarrow \Delta = \frac{f_u - f_d}{S_0.u - S_0.d} \qquad (1)$$

Δ Are the units of stock, $\Delta > 0$ if we have call option and $\Delta < 0$ if it is put option.

The portfolio's value at time 0 is $S_0.\Delta - f$

Since riskless portfolio must earn risk-free rate of interest, we have:

$$S_0.\Delta - f = (S_0.u.\Delta - f_u)e^{-rT}$$

$$\Rightarrow f = S_0.\Delta(1 - u.e^{-rT}) + f_u e^{-rT} \qquad (2)$$

We abstract (1) in (2):

$$f = e^{-rT}(\frac{e^{rT}-d}{u-d}f_u + \frac{u-e^{rT}}{u-d}f_d)$$

And this equition can be written as:

$$f = e^{-rT}(\,p.f_u + (1-p)f_d)$$

Where: $p = \frac{e^{rT}-d}{u-d}$ is called the "risk-neutral probability".

4.3. More on Binomial models

- **The Cox-Ross-Rubenstein model**

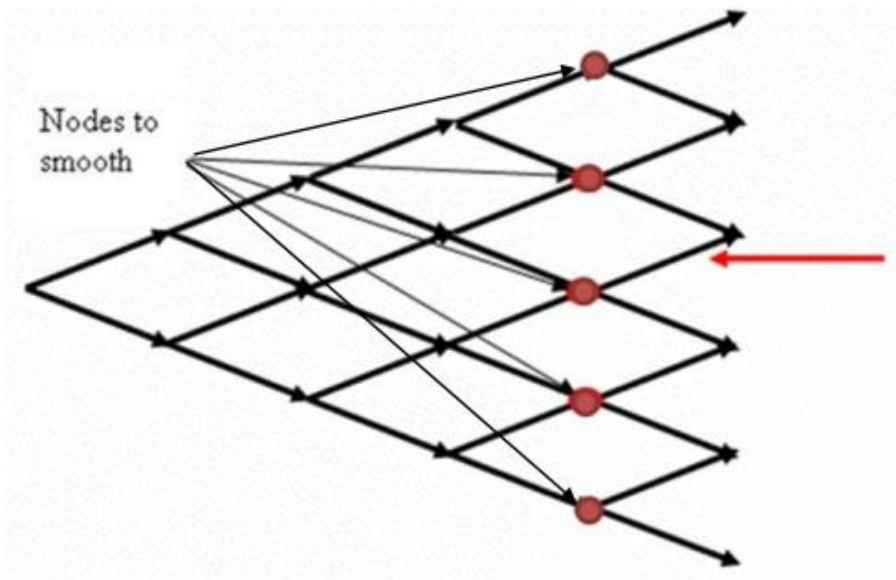
$$u = e^{\sigma\sqrt{t}}$$

$$d = e^{-\sigma\sqrt{t}} = \frac{1}{u}$$

$$p = \frac{e^{rT}-d}{u-d}$$

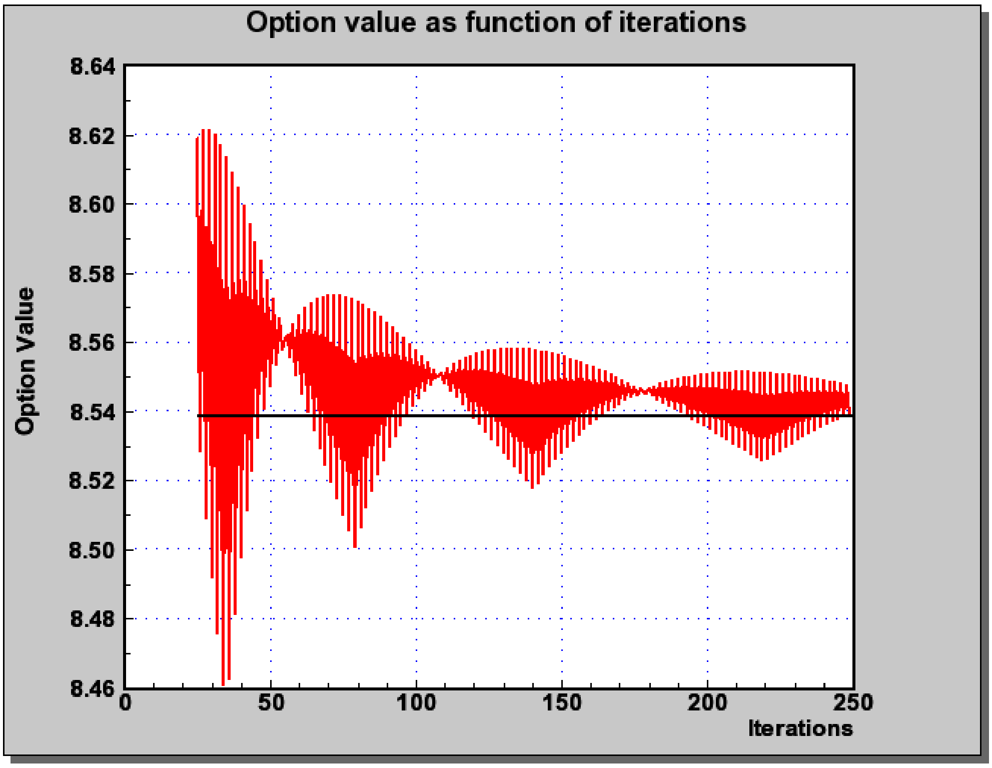
- **Black-Scholes smoothing**

  We use CRR method to calculate price of options but we use black-scholes value For the node closest to strike price. Actually, we can use Black-Scholes value only three node close to strike price but in order to get less oscillation, we apply Black-Scholes value to all node closest to strike price.

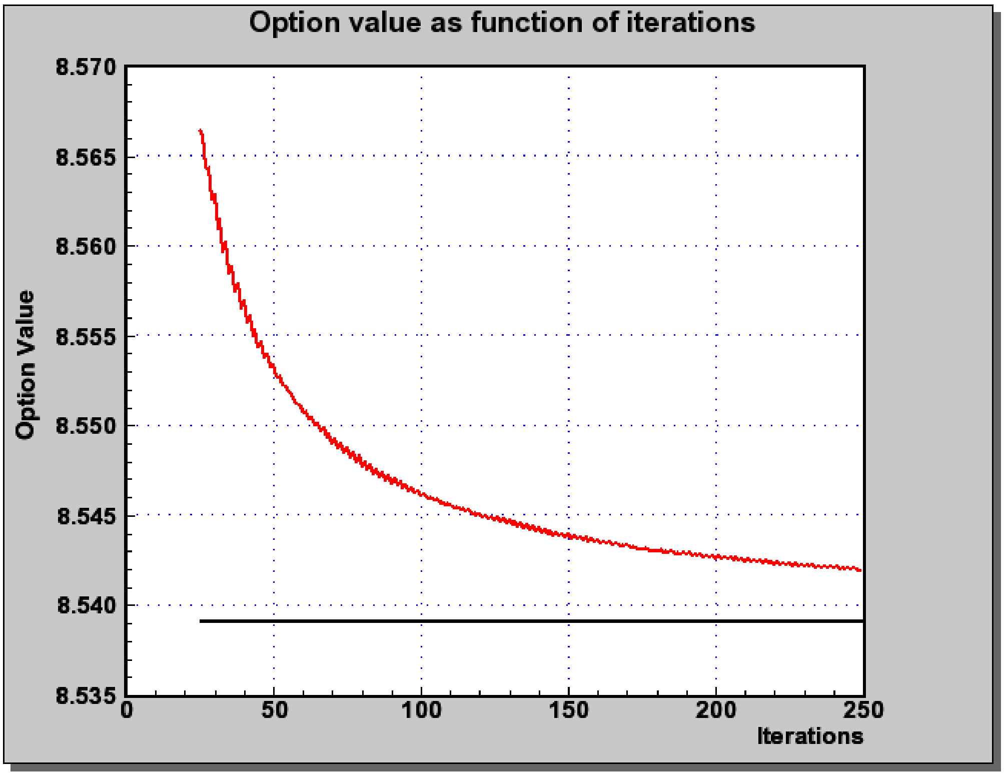


The reason that the Black-Scholes smoothing (also called mollification for dealing with ill-posed problems) minimize the oscillations is that we get a much smoother distribution one step from maturity.

CCR gives the following result:



With Black-Scholes smoothing we get



4. Example:

Input		
Stock price	S =	100
Strike price	K =	105
Maturity	T =	0…3
intrest rate	r =	3%
volatility	σ =	35.00%
number of steps	n =	12
One step	Δt=	0.25

We are going to calculate the put option for different maturity (0.25,0.5,…,3)

The minimum stock price: $S_{min} = S_0 e^{-n\sigma\sqrt{\Delta t}} = 12.24564$

The maximum stock price: $S_{max} = S_0 e^{n\sigma\sqrt{\Delta t}} = 816.617$

Up factor $u = e^{\sigma\sqrt{t}} = 1.191246$

Down factor $d = e^{-\sigma\sqrt{t}} = 0.839457$

By using binomial formula with Black-Scholes smoothing we can calculate the American put and the European.

1- American put:
We take the stock price values between the S_{min} and S_{max},

T \ S	12.24564	49.65853	83.9457	100	119.1246	239.8875	816.617
0.25	92.75436	55.34147	21.43423	9.549806	2.521361	2.37E-11	0
0.5	92.75436	55.34147	22.5189	12.0608	4.931877	0.000859	0
0.75	92.75436	55.34147	23.6035	13.92375	6.845147	0.021428	0
1	92.75436	55.34147	24.58995	15.44211	8.442673	0.09727	0
1.25	92.75436	55.34147	25.47363	16.73602	9.820691	0.244159	5.14E-16
1.5	92.75436	55.34147	26.27139	17.86864	11.04752	0.457279	2.09E-11
1.75	92.75436	55.34147	27.00928	18.87797	12.14643	0.72582	1.56E-08
2	92.75436	55.34147	27.69063	19.78895	13.14053	1.039854	1.17E-06
2.25	92.75436	55.34147	28.32029	20.61908	14.05122	1.383454	2.15E-05
2.5	92.75436	55.34147	28.90509	21.38428	14.9028	1.750973	0.000161
2.75	92.75436	55.34147	29.45052	22.09697	15.69809	2.134212	0.000678
3	92.75436	55.34147	29.96104	22.75946	16.43739	2.533219	0.001953

2- European put:

	12.24564283	49.65853	83.9457	100	119.1246217	239.8875	816.617
0.25	91.52821663	54.55692	21.10573	9.462386	2.511549016	2.37E-11	0
0.5	90.82496124	53.78141	22.0456	11.88611	4.887174183	0.000859	0
0.75	90.41823708	53.04699	22.98085	13.65528	6.752188412	0.021428	0
1	89.6511382	52.38721	23.8197	15.07393	8.299238225	0.09727	0
1.25	88.88977104	51.80757	24.56398	16.26453	9.625855277	0.243469	5.14E-16
1.5	88.13409277	51.29585	25.22543	17.29118	10.78815463	0.456298	2.09E-11
1.75	87.38406089	50.84208	25.81672	18.19247	11.82183458	0.72194	1.56E-08
2	86.63963323	50.43468	26.34794	18.99382	12.75126483	1.028932	1.17E-06
2.25	85.90076869	50.06467	26.82722	19.71305	13.59396508	1.368078	2.15E-05
2.5	85.16743369	49.7239	27.26113	20.36323	14.36303797	1.730792	0.000161
2.75	84.43962151	49.40603	27.65499	20.95429	15.0685928	2.109426	0.000678
3	83.7173707	49.10625	28.01324	21.494	15.71862815	2.497676	0.001953

The graph (1) represents the option value when the stock price changes and for different maturities.

After a specific price of stock, the American put option has a fixed value for each price stock smaller than the later price, whatever the maturity is. (graph 1).

The graph (2) represents the option value when the stock price changes and for different maturities.

The European put option value decreases when the maturity increases for all the stock values (graph 2).

Both options give a null value after particular stock price.

The graph (3) represents the difference between the American and European put.

It is clear that when the time of maturity increase, the difference between them increase and we get a bigger value for an American put option.

References

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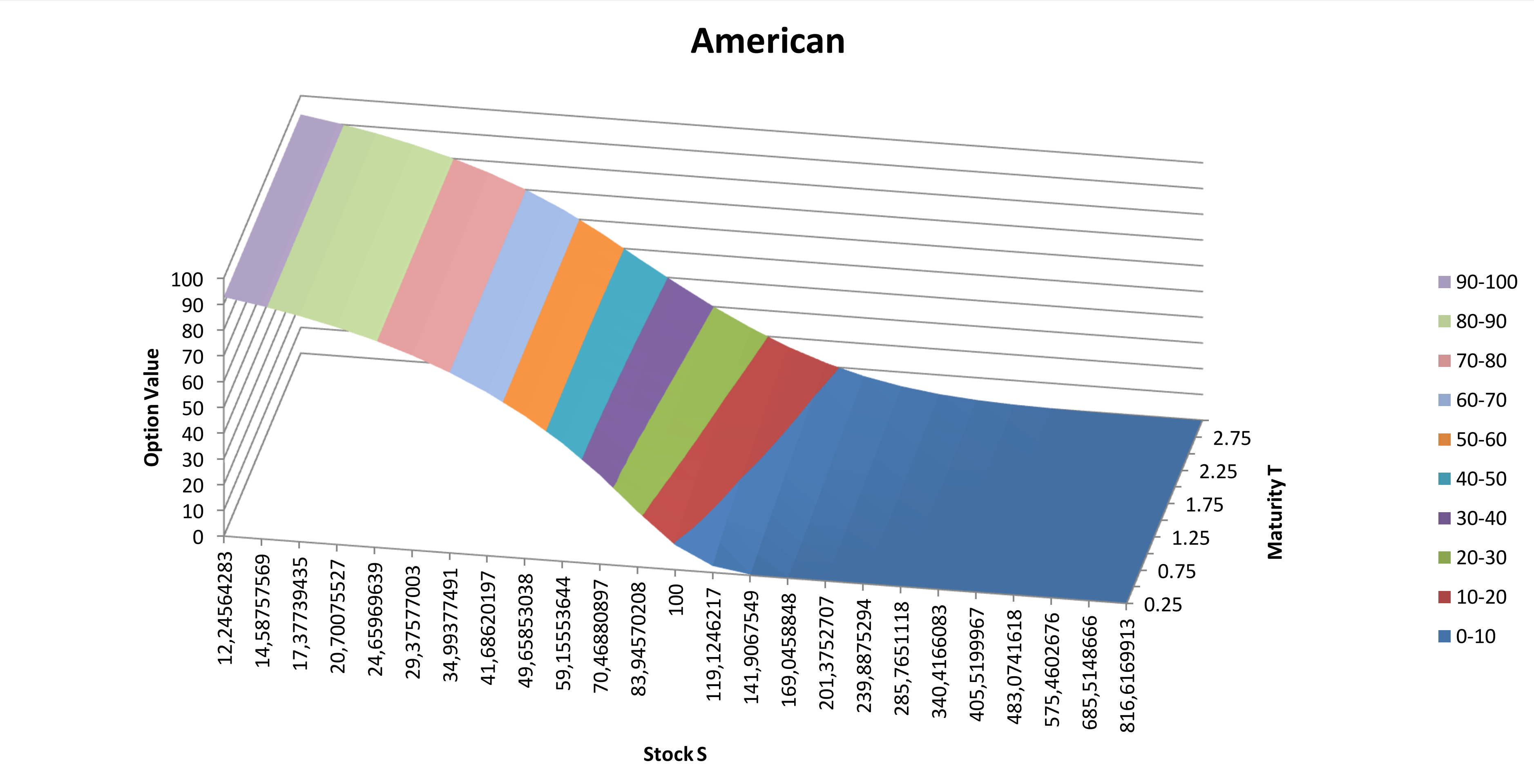
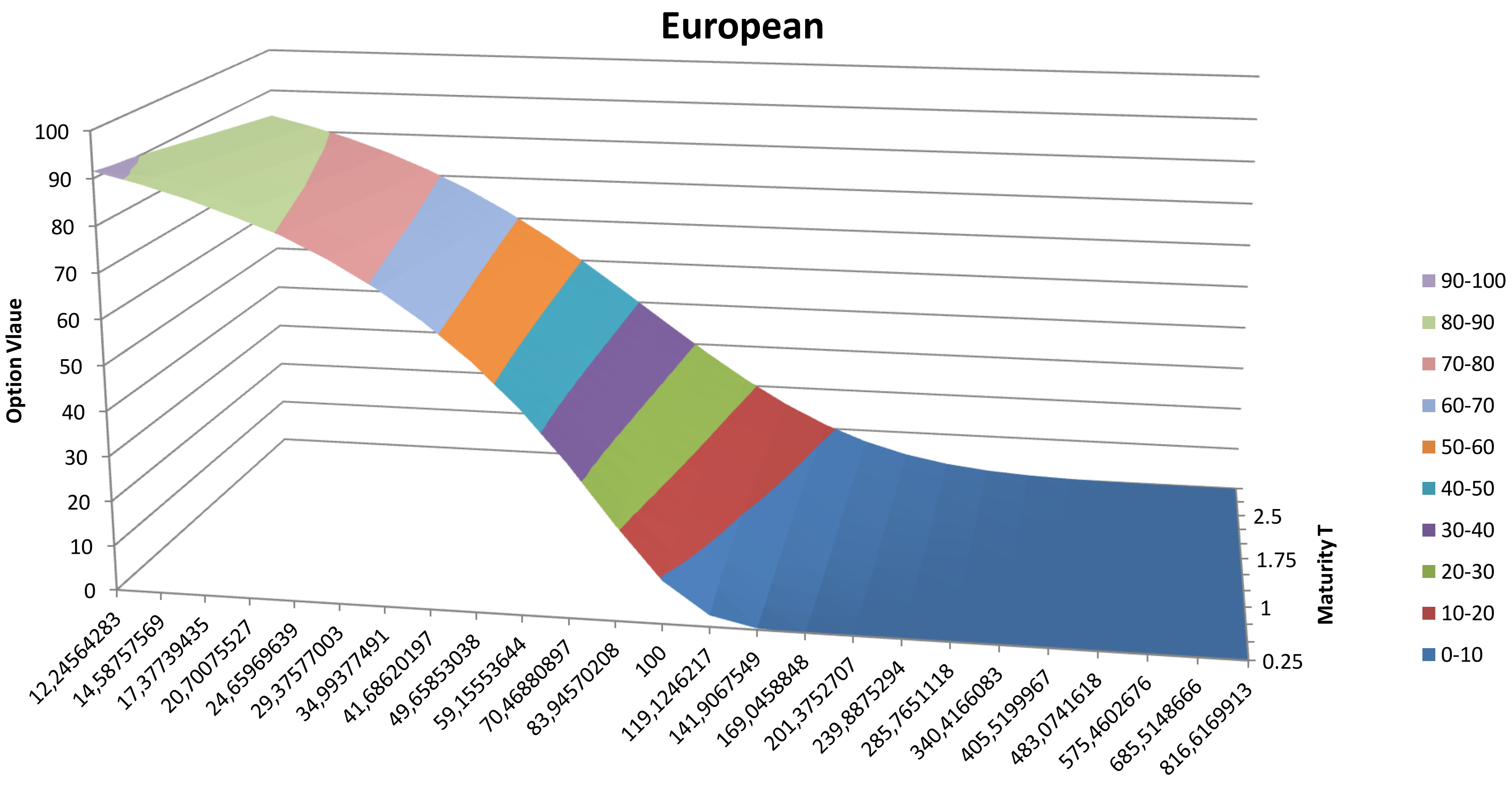

American
Option Value
100
90
80
70
60
50
40
30
20
10
0
12,24564283
14,58757569
17,37739435
20,70075527
24,65969639
29,37577003
34,99377491
41,68620197
49,65853038
59,15553644
70,46880897
83,94570208
100
119,1246217
141,9067549
169,0458848
201,3752707
239,8875294
285,7651118
340,4166083
405,5199967
483,0741618
575,4602676
685,5148666
816,6169913
Stock S
Maturity T
0.25
0.75
1.25
1.75
2.25
2.75
90-100
80-90
70-80
60-70
50-60
40-50
30-40
20-30
10-20
0-10

Graph 1


European
Option Vlaue
100
90
80
70
60
50
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30
20
10
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12,24564283
14,58757569
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20,70075527
24,65969639
29,37577003
34,99377491
41,68620197
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141,9067549
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239,8875294
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405,5199967
483,0741618
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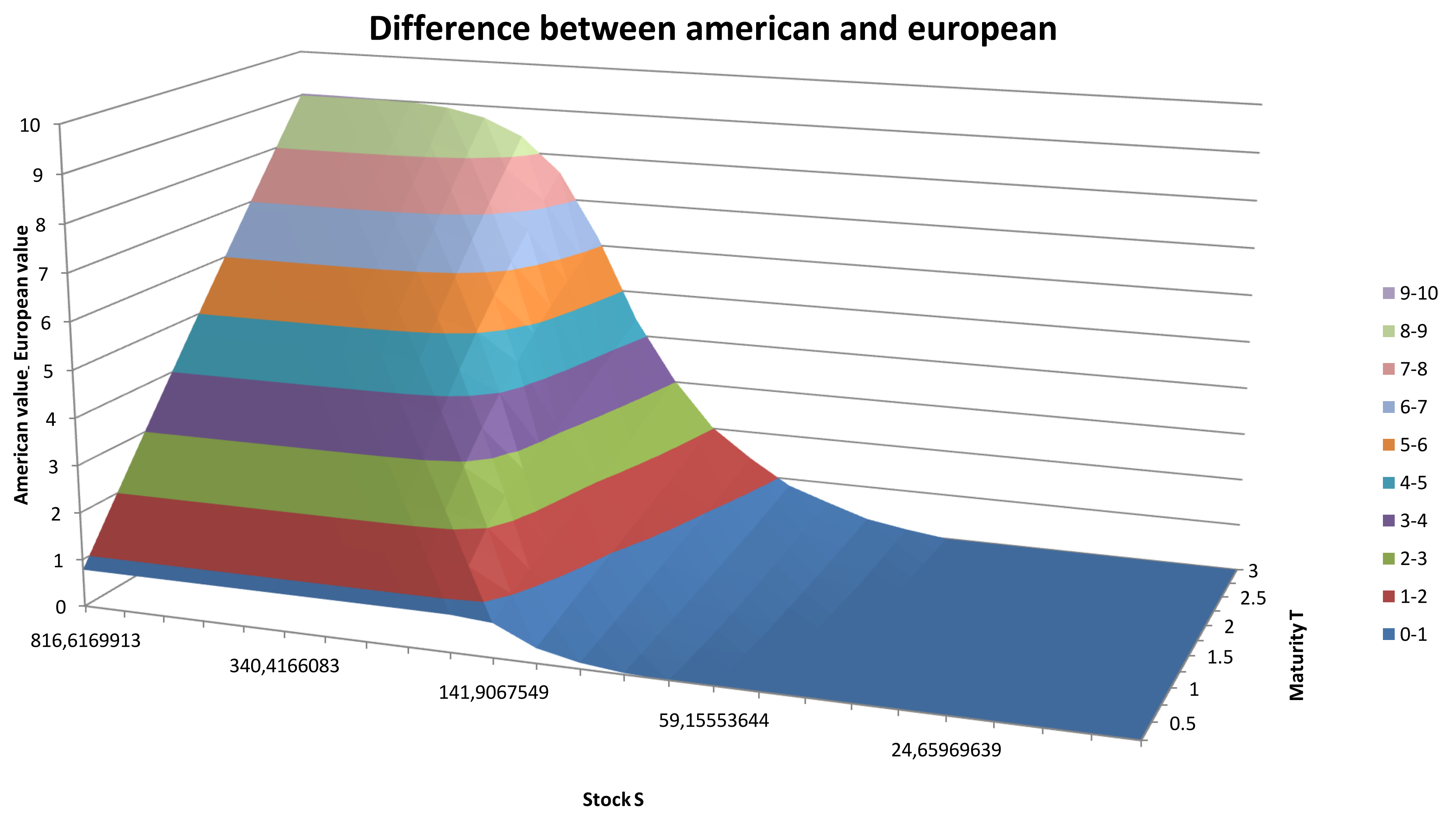
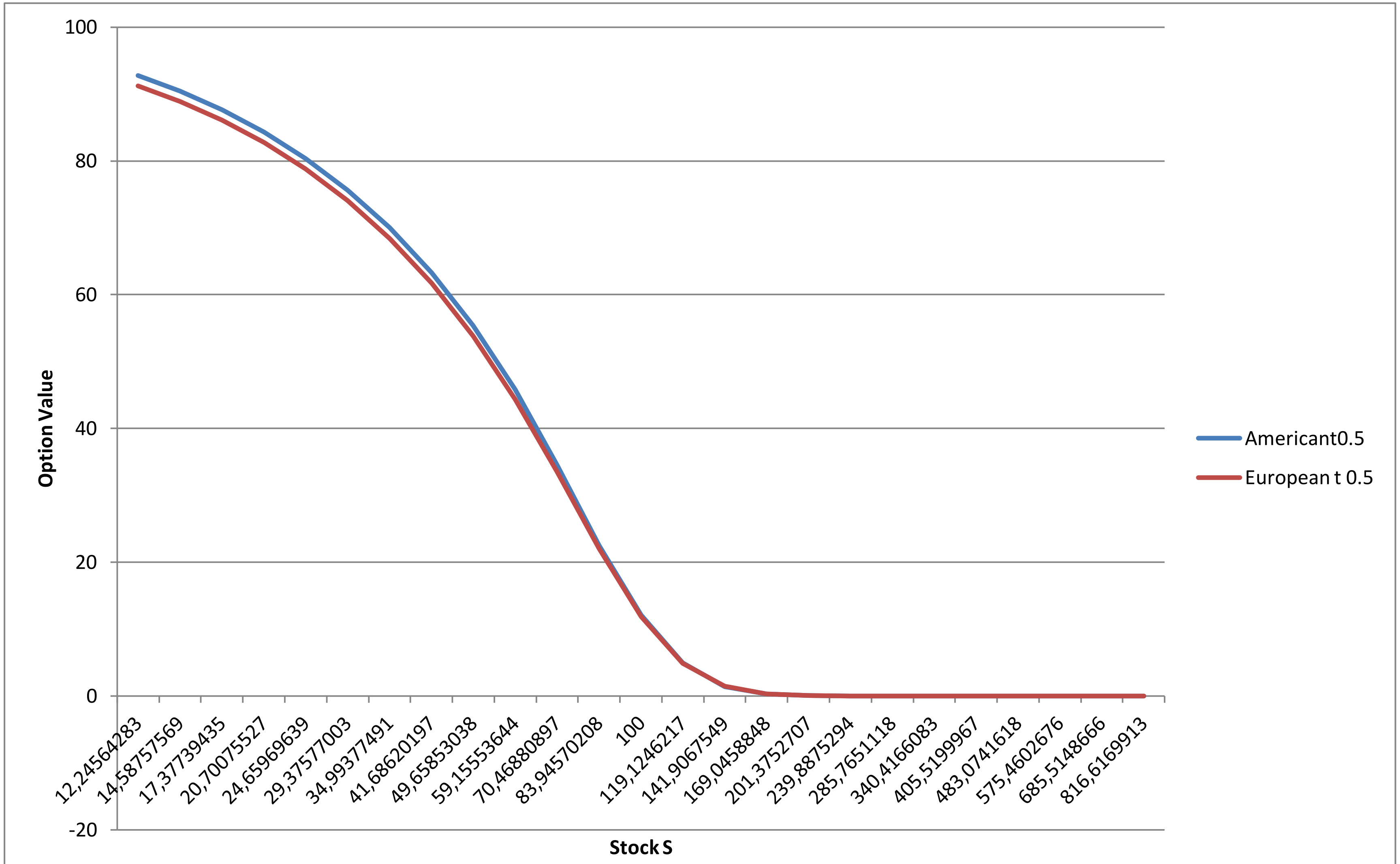
685,5148666
816,6169913
Stock S
2.5
1.75
1
0.25
Maturity T
90-100
80-90
70-80
60-70
50-60
40-50
30-40
20-30
10-20
0-10

Graph 2


Difference between american and european
American value_ European value
10
9
8
7
6
5
4
3
2
1
0
816,6169913
340,4166083
141,9067549
59,15553644
24,65969639
Stock S
3
2.5
2
1.5
1
0.5
Maturity T
9-10
8-9
7-8
6-7
5-6
4-5
3-4
2-3
1-2
0-1
Graph 3


100
80
60
40
20
0
-20
Option Value
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20,70075527
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575,4602676
685,5148666
816,6169913
Americant0.5
European t 0.5
Stock S
Graph 4