American and European Put Option

Analytical Finance I

Kinda Sumlaji
Table of Contents:

1. Introduction .................................................................................................................. 3

2. Option Style .................................................................................................................. 4

3. Put Option ..................................................................................................................... 4
   3.1 Definition .................................................................................................................. 4
   3.2 Payoff at Maturity .................................................................................................... 4
   3.3 Example of a put option on a stock .......................................................................... 5
   3.4 Option’s parameters ................................................................................................. 6

4. Option Pricing ................................................................................................................ 7
   4.1 Black-Scholes Model ............................................................................................... 7
   4.2 Binomial options pricing model ............................................................................... 8
   4.3 More on Binomial models ....................................................................................... 10

5. Example ........................................................................................................................ 12

6. References ..................................................................................................................... 14

Table of graphs:

Graph1: American put option ........................................................................................... 15
Graph2: European put option ........................................................................................... 16
Graph3: Difference between American and European put option .................................. 17
Graph4: comparing put option value at t=0.5 ................................................................. 18
1. Introduction

European and American options are most popular options in finance, European options are typically valued using the Black–Scholes or Black model formula. This is a simple equation with a closed-form solution that has become standard in the financial community. There are no general formulae for American options, but a choice of models to approximate the price are available (for example Roll-Geske-Whaley, Barone-Adesi and Whaley, binomial options model by Cox-Ross-Rubinstein, Black's approximation and others; there is no consensus on which is preferable).

Thus, an American option is a European option with the additional right to exercise it any time prior to expiration. This feature naturally raises the interesting question of whether this right is worth something and, if so, how much,

In this paper I am going to study the difference between the American and European put option for different maturity.
2. Option Style

In finance, the style of an option usually defined by the dates on which the option may be exercised. The most popular options are either European or American (style) options.

The difference between American and European options relates to when the options can be exercised:

- A European option may be exercised only at the expiration date of the option.
- An American option may be exercised at any time before the expiry date.

3. Put Option

3.1. Definition

A put or put option is a contract between two parties to exchange an asset, at a specified price (the strike \( K \)), by a predetermined date (the expiry or maturity \( T \)). One party, the buyer of the put, has the right, but not an obligation, to sell the asset at the strike price by the future date, while the other party, the seller of the put, has the obligation to buy the asset at the strike price if the buyer exercises the option.

3.2. Payoff at Maturity

When the buyer exercises the put at a time \( t \), he can expect to receive a payout of \( K - S(t) \), if the price of the underlying \( S(t) \) at that time is less than \( K \).

The exercise must occur by time \( T \), and what exact times are allowed is specified by the type of put option. An American option can be exercised at any time before or equal to \( T \), a European option can be exercised only at time \( T \).

\[
P_T = \begin{cases} 
K - S_T & \text{if } S_T \leq K \\
0 & \text{if } S_T > K
\end{cases}
\]

Or more simply \( P_T = \max\{K - S_T, 0\} \).

- If \( S_T < K \), then the put is said to finish in-the-money and the option will be exercised, the holder of this option will buy the underlying stock at a price of \( S_T \) and exercise his right to sell it to the writer at the strike price of \( K \), to make a profit of \( K - S_T \).
• If $S_T > K$, the option is said to finish out-of-the-money and exercising the right to sell the underlying asset would result in a loss.

• If the option is not exercised by maturity, it expires worthless.

### 3.3. Example of a put option on a stock

#### Buying a put

A buyer thinks the price of a stock will decrease. He pays a premium which he will never get back, unless it is sold before it expires. The buyer has the right to sell the stock at the strike price.

![Long Put Diagram]

#### Writing a put

The writer receives a premium from the buyer. If the buyer exercises his option, the writer will buy the stock at the strike price. If the buyer does not exercise his option, the writer's profit is the premium.

![Short Put Diagram]
3.4. Option’s parameters

In their derivation of the Black-Scholes option-pricing model, Black, Scholes, and Merton showed that the value of a call or put option depends on the following parameters:

- Current stock price.
- Option’s exercise price.
- Time (in years) until the option expires (referred to as the option’s duration).
- Interest rate (per year on a compounded basis) on a risk-free investment throughout the duration of the investment. This rate is called the risk-free rate. Compound interest simply means that at every instant, you are earning interest on your interest.
- Annual rate (as a percentage of the stock price) at which dividends are paid.
- Stock volatility (measured on an annual basis).

In general, the effect of changing an input parameter on the value of a put is given in the following table:

<table>
<thead>
<tr>
<th>PARAMETER</th>
<th>EUROPEAN PUT</th>
<th>AMERICAN PUT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock price</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Exercise price</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Time to expiration</td>
<td>?</td>
<td>+</td>
</tr>
<tr>
<td>Volatility</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Risk-free rate</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Dividends</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

- An increase in today’s stock price always decreases the value of a put.
- An increase in the exercise price always increases the value of a put.
- An increase in the duration of an option always increases the value of an American option. In the presence of dividends, an increase in the duration of an option can either increase or decrease the value of a European option.
- An increase in volatility always increases option value.
An increase in the risk-free rate always decreases the value of a put because the higher growth rate of the stock tends to hurt the put, as does the fact that future payoffs from the put are worth less.

Dividends tend to reduce the growth rate of a stock price, so increased dividends increase the value of a put.

4. Option Pricing

4.1. Black-Scholes Model

- Definition

The Black-Scholes formula is a mathematical formula designed to price an option as a function of certain variables, generally stock price, striking price, volatility, time to expiration, dividends to paid, and the current risk-free interest rate.

- Black-Scholes Formula

The payoff to a European put option with strike price $K$ at the maturity date $T$ is

$$P(T) = \max\{K - S(T), 0\}$$

as the put option gives the right to sell underlying asset at the strike price of $K$.

The Black-Scholes formula for the price of the put option is given by

$$p(0) = c(0) + e^{-rT}K - S(0) = e^{-rT}K\left(1 - N(d_2)\right) - S(0)(1 - N(d_1))$$

where $d_1$ and $d_2$ are defined by

$$d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

Here $\sigma^2$ is the variance of the continuously compounded rate of return on the underlying asset.

By the symmetry of the standard normal distribution $N(-d) = (1 - N(d))$ so the formula for the put option is usually written as

$$p(0) = e^{-rT}KN(-d_2) - S(0)N(-d_1)$$
- **Black-Scholes Model Assumptions**
  There are several assumptions underlying the Black-Scholes model of calculating options pricing. The most significant is that volatility, a measure of how much a stock can be expected to move in the near-term, is a constant over time. The Black-Scholes model also assumes stocks move in a manner referred to as a random walk; at any given moment, they are as likely to move up as they are to move down. These assumptions are combined with the principle that options pricing should provide no immediate gain to either seller or buyer.

4.2. **Binomial options pricing model**
One simple and powerful model for stock price evolution is the binomial tree model. It is useful for pricing and hedging options. In the binomial tree model, stock can take 1 of 2 possible values at each node in the tree.

It is not a bad approximation when the time intervals are very small. The model is called a tree (or sometimes lattice) because each branch splits into more branches and so on.

- **Binomial Method**

The binomial pricing model traces the evolution of the option's key underlying variables in discrete-time. This is done by means of a binomial lattice (tree), for a number of time steps between the valuation and expiration dates. Each node in the lattice represents a possible price of the underlying at a given point in time.

Valuation is performed iteratively, starting at each of the final nodes (those that may be reached at the time of expiration), and then working backwards through the tree towards the first node (valuation date). The value computed at each stage is the value of the option at that point in time.

Option valuation using this method is, as described, a three-step process:

1. Price tree generation.
2. Calculation of option value at each final node
3. Sequential calculation of the option value at each preceding node.

**STEP 1: Create the binomial price tree**

The tree of prices is produced by working forward from valuation date to expiration.
At each step, it is assumed that the underlying instrument will move up or down by a specific factor \((u\) or \(d\)) per step of the tree (where, by definition, \(u \geq 1\), and \(0 < d \leq 1\)).

If \(S\) is the current price, then in the next period the price will either be

\[ S_{up} = S \cdot u \quad \text{or} \quad S_{down} = S \cdot d \]

The up and down factors are calculated using the underlying volatility, \(\sigma\), and the time duration of a step, \(\Delta t\), measured in years, we have:

\[ u = e^{\sigma \sqrt{\Delta t}} \]

\[ d = e^{-\sigma \sqrt{\Delta t}} = \frac{1}{u} \]

**STEP 2: Find Option value at each final node**

At each final node of the tree, the option value is equal to:

\[ \max\{K - S_n, 0\} \], for a put option.

Where \(K\) is the strike price and \(S_n\) is the spot price of the underlying asset at the \(n^{th}\) period.

**STEP 3: Find Option value at earlier nodes**

![Diagram](https://via.placeholder.com/150)

Portfolio value

\[ S_0 \cdot u \]

\[ S_0 \cdot d \]

\[ f_u \]

\[ f_d \]

At time \(T\), the portfolio has value \(S_0 \cdot u \cdot \Delta - f_u\) at the upper node and \(S_0 \cdot d \cdot \Delta - f_d\) at the lower node, to make the portfolio riskless, these two values should be set equal:

\[ S_0 \cdot u \cdot \Delta - f_u = S_0 \cdot d \cdot \Delta - f_d \]

\[ \Rightarrow \Delta = \frac{f_u - f_d}{S_0 \cdot u - S_0 \cdot d} \]  

(1)

\(\Delta\) Are the units of stock, \(\Delta > 0\) if we have call option and \(\Delta < 0\) if it is put option.

The portfolio’s value at time 0 is \(S_0 \cdot \Delta - f\)

Since riskless portfolio must earn risk-free rate of interest, we have:

\[ S_0 \cdot \Delta - f = (S_0 \cdot u \cdot \Delta - f_u)e^{-rT} \]

\[ \Rightarrow f = S_0 \cdot \Delta(1 - u \cdot e^{-rT}) + f_u e^{-rT} \]  

(2)
We abstract (1) in (2):

$$f = e^{-rT} \left( \frac{e^{rT-d}}{u-d} f_u + \frac{u^{-e^{rT}}}{u-d} f_d \right)$$

And this equation can be written as:

$$f = e^{-rT} (p \cdot f_u + (1-p) f_d)$$

Where: \( p = \frac{e^{rT-d}}{u-d} \) is called the “risk-neutral probability”.

### 4.3. More on Binomial models

- **The Cox-Ross-Rubenstein model**
  \[
  u = e^{\sigma \sqrt{T}} \\
  d = e^{-\sigma \sqrt{T}} = \frac{1}{u} \\
  p = \frac{e^{rT-d}}{u-d}
  \]

- **Black-Scholes smoothing**
  We use CRR method to calculate price of options but we use Black-Scholes value for the node closest to strike price. Actually, we can use Black-Scholes value only three node close to strike price but in order to get less oscillation, we apply Black-Scholes value to all node closest to strike price.

The reason that the Black-Scholes smoothing (also called mollification for dealing with ill-posed problems) minimize the oscillations is that we get a much smoother distribution one step from maturity.
CCR gives the following result:

With Black-Scholes smoothing we get
4. Example:

<table>
<thead>
<tr>
<th>Input</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock price</td>
<td>$S = 100$</td>
</tr>
<tr>
<td>Strike price</td>
<td>$K = 105$</td>
</tr>
<tr>
<td>Maturity</td>
<td>$T = 0...3$</td>
</tr>
<tr>
<td>Interest rate</td>
<td>$r = 3%$</td>
</tr>
<tr>
<td>Volatility</td>
<td>$\sigma = 35.00%$</td>
</tr>
<tr>
<td>Number of steps</td>
<td>$n = 12$</td>
</tr>
<tr>
<td>One step</td>
<td>$\Delta t = 0.25$</td>
</tr>
</tbody>
</table>

We are going to calculate the put option for different maturity (0.25, 0.5, ..., 3)

The minimum stock price: $S_{min} = S_0 e^{-n\sigma\sqrt{\Delta T}} = 12.24564$

The maximum stock price: $S_{max} = S_0 e^{n\sigma\sqrt{\Delta T}} = 816.617$

Up factor $u = e^{\sigma\sqrt{T}} = 1.191246$

Down factor $d = e^{-\sigma\sqrt{T}} = 0.839457$

By using binomial formula with Black-Scholes smoothing we can calculate the American put and the European.

1- American put:
We take the stock price values between the $S_{min}$ and $S_{max}$.

<table>
<thead>
<tr>
<th>T</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>92.75436</td>
</tr>
<tr>
<td>0.5</td>
<td>92.75436</td>
</tr>
<tr>
<td>0.75</td>
<td>92.75436</td>
</tr>
<tr>
<td>1</td>
<td>92.75436</td>
</tr>
<tr>
<td>1.25</td>
<td>92.75436</td>
</tr>
<tr>
<td>1.5</td>
<td>92.75436</td>
</tr>
<tr>
<td>1.75</td>
<td>92.75436</td>
</tr>
<tr>
<td>2</td>
<td>92.75436</td>
</tr>
<tr>
<td>2.25</td>
<td>92.75436</td>
</tr>
<tr>
<td>2.5</td>
<td>92.75436</td>
</tr>
<tr>
<td>2.75</td>
<td>92.75436</td>
</tr>
<tr>
<td>3</td>
<td>92.75436</td>
</tr>
</tbody>
</table>
2- European put:

<table>
<thead>
<tr>
<th>maturity</th>
<th>American put</th>
<th>European put</th>
<th>difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>91.52821663</td>
<td>21.10573</td>
<td>70.42251</td>
</tr>
<tr>
<td>0.5</td>
<td>90.82496124</td>
<td>11.88611</td>
<td>78.93803</td>
</tr>
<tr>
<td>0.75</td>
<td>90.41823708</td>
<td>22.98085</td>
<td>67.43723</td>
</tr>
<tr>
<td>1</td>
<td>89.6511382</td>
<td>23.8197</td>
<td>65.83154</td>
</tr>
<tr>
<td>1.25</td>
<td>88.88977104</td>
<td>24.56398</td>
<td>64.3258</td>
</tr>
<tr>
<td>1.5</td>
<td>88.13409277</td>
<td>25.22543</td>
<td>62.90434</td>
</tr>
<tr>
<td>1.75</td>
<td>87.38406089</td>
<td>25.81672</td>
<td>61.56832</td>
</tr>
<tr>
<td>2</td>
<td>86.63963323</td>
<td>26.34794</td>
<td>59.29135</td>
</tr>
<tr>
<td>2.25</td>
<td>85.90076869</td>
<td>26.82722</td>
<td>58.08046</td>
</tr>
<tr>
<td>2.5</td>
<td>85.16743369</td>
<td>27.26113</td>
<td>56.90228</td>
</tr>
<tr>
<td>2.75</td>
<td>84.43962151</td>
<td>27.65499</td>
<td>55.78452</td>
</tr>
<tr>
<td>3</td>
<td>83.7173707</td>
<td>28.01324</td>
<td>54.70407</td>
</tr>
</tbody>
</table>

The graph (1) represents the option value when the stock price changes and for different maturities.

After a specific price of stock, the American put option has a fixed value for each price stock smaller than the later price, whatever the maturity is. (graph 1).

The graph (2) represents the option value when the stock price changes and for different maturities.

The European put option value decreases when the maturity increases for all the stock values (graph 2).

Both options give a null value after particular stock price.

The graph (3) represents the difference between the American and European put.

It is clear that when the time of maturity increase, the difference between them increase and we get a bigger value for an American put option.
References

JAN RÖMAN’S Lecture notes in Analytical finance I - http://janroman.net.dhis.org

John HULL, Option Futures and Other Derivatives fifth edition (p167-p184)

http://en.wikipedia.org/wiki/Option_style
American

Graph 1

Stock S

Option Value

Maturity T

- 90-100
- 80-90
- 70-80
- 60-70
- 50-60
- 40-50
- 30-40
- 20-30
- 10-20
- 0-10
Difference between American and European value

Graph 3