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School of Education, Culture and Communication
Division of Applied Mathematics
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"Asset-or-nothing digitals"

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Abstract

An asset-or-nothing digital is an option where the buyer either gets the underlying at a certain date (maturity) or gets nothing, depending on whether the underlying price reaches a certain level or not. In this assignment we calculate the price of such options, as well as the partial (numerical) derivatives, called the greeks. Results are shown in graphs using Excel/VBA.

Contents

Introduction

An asset-or-nothing binary option is an exotic option, meaning it is not a regular call or put option, often referred to as vanilla options. The asset-or-nothing binary is simply a bet on a price of an asset at a certain date in the future. If you own an option and win the bet you receive the asset, so there are only two possible outcomes, hence it is called binary. This makes of course the price of the option higher than the price of a vanilla option but at the same time the pay-off is higher. The simple conclusion is that the asset-or-nothing is just a way to gamble but for financial engineers they often play an important role in constructing more complex products in the derivatives market.

As well as stocks and vanilla options, binary options are traded at the stock exchange market. The asset-or-nothing and cash-or-nothing are the simplest binary options. There exist more advanced binary options (i.e. American types of digitals) but for this assignment we limit the calculations to the asset-or-nothing with corresponding partial differential equations (PDEs).

Asset-or-nothing digital

Definition

At expiration, the asset-or-nothing call option pays 0 if $S \leq K$ and S if $S > K$. Similarly, a put option pays 0 if $S > K$ and S if $S < K$. In other words, the asset-or-nothing call pays one unit of asset if the spot is above the strike at maturity while the put pays one unit of asset if the spot is below the strike at maturity.

Asset-or-nothing options are usually European-style. Options are in general considered high-risk-investments and that goes for binary options as well. One advantage with these kinds of options though, is that the maximum possible loss is known, as is not the case for vanilla options.

Payoff graphs

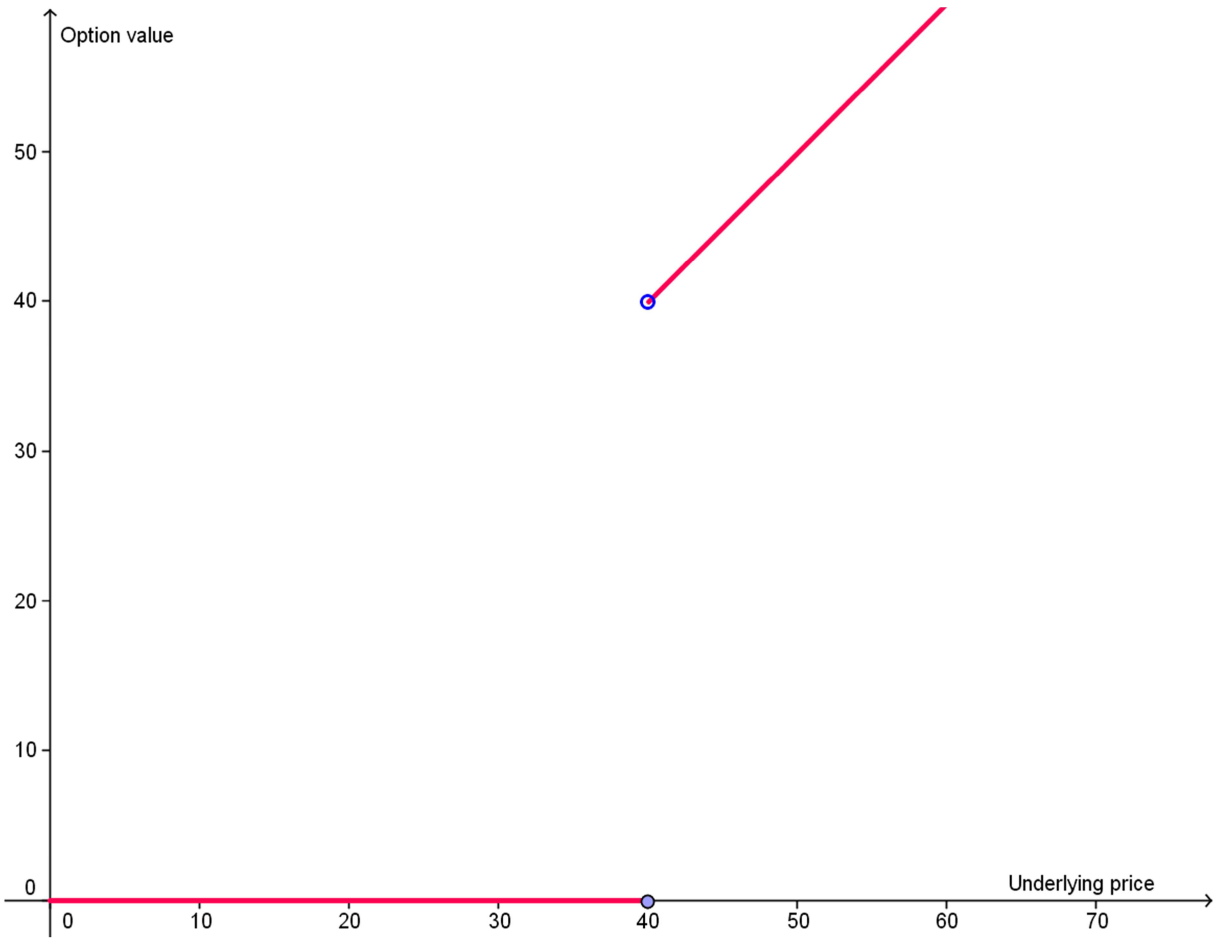

Figure 1 Payoff of binary asset-or-nothing call option with strike price 40 and current stock price 30.

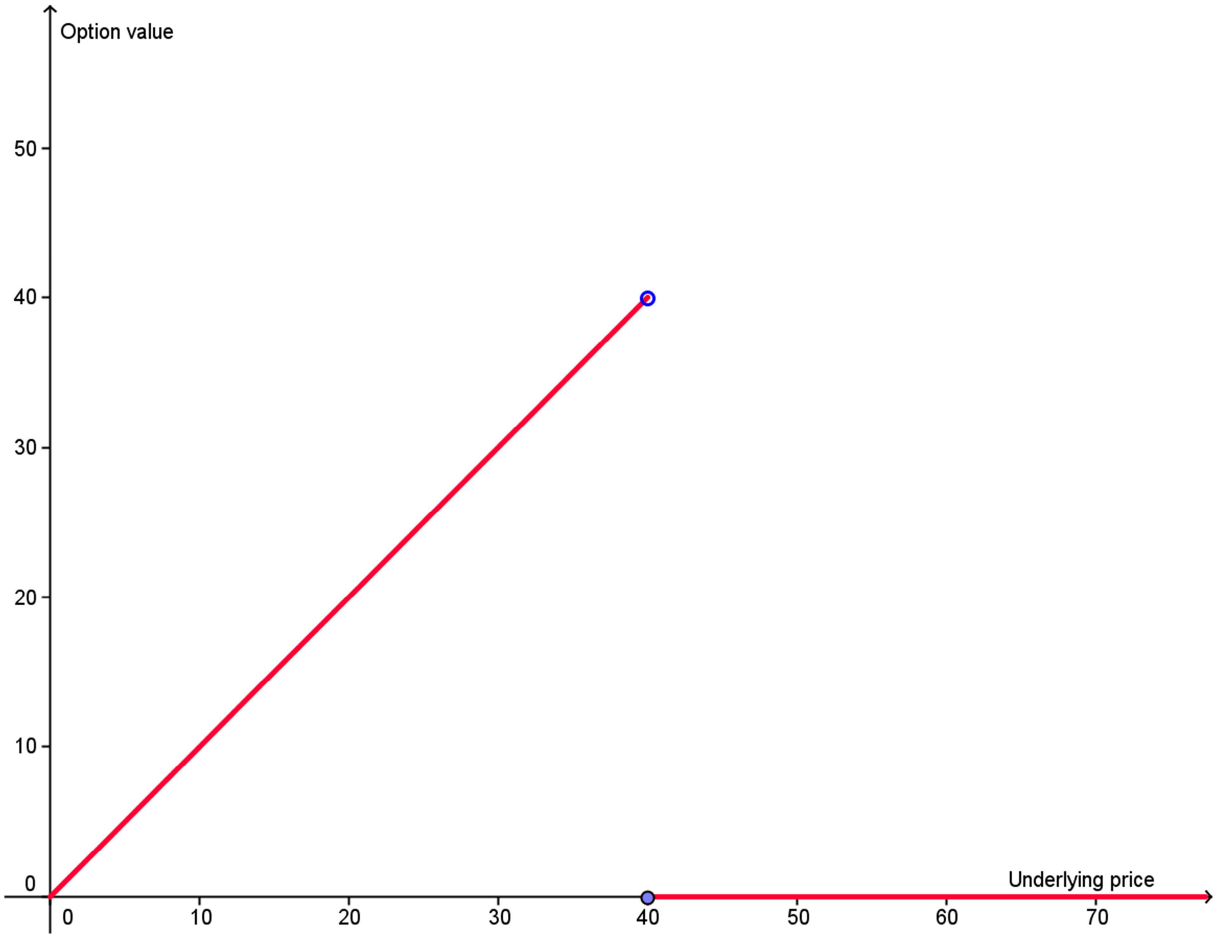

Figure 2 Payoff of binary asset-or-nothing put option with strike price 40 and current stock price 30.

Notation

In this part we use the following notation

t = current time

S = time t stock price

K – strike price

T – maturity

S_T –stock price at maturity

σ – volatility

Calculating the price of the binary asset-or-nothing call option

The price of a binary asset-or-nothing call option is given by

$$\prod_t = e^{-r(T-t)} E^Q\left[S_T I_{\{S_T>K\}} | \mathcal{F}_T\right] = e^{-r(T-t)} \int_{Z_0}^{\infty} se^z \varphi(z) dz$$

Where φ is the density of a $N\left[\left(r-\frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right]$-distribution. Now we use that the density function for a $N\left[\left(r-\frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right]$-distributed and we complete the square in the exponent. This yields

$$\prod_t = s\int_{Z_0}^{\infty} \Psi(z)dz$$

Where Ψ denotes the density for a $N\left[\left(r-\frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right]$-distribution. We thus have that

$$\prod_t [BAC_t] = \mathrm{sQ}(\mathrm{Z} > Z_0)$$
$$= S\left[1 - N\left(\frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left\{\frac{K}{S}\right\} - \left(r+\frac{1}{2}\sigma^2\right)(T-t)\right\}\right)\right]$$
$$= SN\left(\frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left\{\frac{S}{K}\right\} + \left(r+\frac{1}{2}\sigma^2\right)(T-t)\right\}\right)$$

We denote

$$d = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left\{\frac{S}{K}\right\} + \left(r+\frac{1}{2}\sigma^2\right)(T-t)\right\}$$

Which means that we can express the price of a binary asset-or-nothing call option as

$$\prod_t [BAC_t] = \mathrm{SN(d)}$$

Which is equal to the first part of the Black-Scholes formula.

Asset-or-nothing put option

Given the definitions of an asset-or-nothing call and put option and referring to the calculated price of the call option the put-call parity gives us the price of the put option as follows:

$$P + C = S$$
$$P = S - C$$
$$= S - SN(d)$$
$$= S(1 - N(d)$$
$$= SN(-d)$$

$$\prod_t [BAP_t] = \mathrm{SN}(-\mathrm{d})$$

The greeks

The greeks are the partial derivatives (PDEs) of the Black-Scholes formula. They represent the measure of sensitivity of the option price when a parameter in the formula is slightly changed. It is very important to have knowledge about, not only the pricing of an option but also how the price is changed when there are small changes in the parameters of the pricing formula above. That is why we now introduce the greeks, often called the hedge parameters and for traders calculation of the greeks with following changes of the option price is called hedging. Calculating the greeks are not only important initially, but frequently during the contract period, since the hedge parameters changes over time.

The greeks are associated respectively with different type of sensitivity that effects the option price. There exist lots of different greeks, even mixed ones and third-order PDEs for speed measure. For the purpose of this report we don't mention them all but we present some of the most common; delta, gamma, rho, vega and theta. Below we shortly introduce what kind of sensitivity measure each of them represents, followed by the analytical solution to the PDEs respectively.

Before calculating the PDEs for the greeks we present the PDEs of *d* since these are used for substitution when simplifying the PDEs of the greeks. In calculation of the PDEs of *d* we use the following definition:

$$d = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left\{\frac{S}{K}\right\} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}$$

Which gives us the following PDEs

$$\frac{\partial d}{\partial S} = \frac{\frac{1}{K}\frac{K}{S}}{\sigma\sqrt{T-t}} = \frac{1}{S\sigma\sqrt{T-t}} \qquad (1)$$

$$\frac{\partial d}{\partial t} = \frac{-\left(r + \frac{1}{2}\sigma^2\right)\sigma\sqrt{T-t} + \frac{\sigma}{2\sqrt{T-t}}\left[\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right]}{\sigma^2(T-t)}$$

$$= \frac{-\left(r+\frac{1}{2}\sigma^2\right)2\sigma(T-t)+\sigma\left[\ln\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right]}{2\sigma^2(T-t)\sqrt{T-t}}$$

$$= \frac{1}{2(T-t)}\frac{\left[\ln\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right]}{\sigma\sqrt{T-t}}-\frac{\left(r+\frac{1}{2}\sigma^2\right)2(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{1}{2(T-t)}\left[d_1-\frac{2\sqrt{T-t}\left(r+\frac{1}{2}\sigma^2\right)}{\sigma}\right]$$

$$= \frac{d_1}{2(T-t)}-\frac{\left(r+\frac{1}{2}\sigma^2\right)}{\sigma\sqrt{T-t}} \qquad (2)$$

$$\frac{\partial d}{\partial r}=\frac{T-t}{\sigma\sqrt{T-t}}=\frac{\sqrt{T-t}}{\sigma} \qquad (3)$$

$$\frac{\partial d}{\partial \sigma}=\frac{\sigma^2(T-t)\sqrt{T-t}-\sqrt{T-t}\left[\ln\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right]}{\sigma^2(T-t)}$$

$$= \sqrt{T-t}-\sigma d \qquad (4)$$

Delta (Δ)

Delta is the option's price sensitivity. It represents the rate of change between the option's price and the underlying asset's price.

$$\Delta_{call}=\frac{\partial c}{\partial S}=\frac{\partial}{\partial S}SN(d)$$

$$= N(d)+S\frac{\partial N(d)}{\partial d}\frac{\partial d}{\partial S}$$

$$= N(d)+SN'(d)\frac{\partial d}{\partial S}$$

$$= N(d)+\frac{1}{\sigma\sqrt{T-t}}N'(d)$$

where $N'(d) = \phi(d)$ which is the density function of the standard normal distribution and

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

To calculate Δ_{put} we use

$$P = SN(-d)$$

We denote

$$x = -d$$

$$\Delta_{put} = \frac{\partial p}{\partial S} = N(x) + S\frac{\partial N}{\partial x}\frac{\partial x}{\partial S}$$

$$= N(x) + SN'(x)\left(-\frac{\partial d}{\partial S}\right)$$

$$= N(x) - S\frac{\partial d}{\partial S}N'(x)$$

$$= N(-d) - S\frac{1}{S\sigma\sqrt{T-t}}N^{'}(-d)$$

$$= N(-d) - \frac{1}{\sigma\sqrt{T-t}}N^{'}(-d)$$

Gamma (Γ)

Gamma represents the rate of change between an option portfolio's delta and the underlying asset's price - in other words, it is the second-order time price sensitivity.

$$\Gamma_{call} = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 c}{\partial S^2} = \frac{\partial}{\partial S}[N(d) + \frac{1}{\sigma\sqrt{T-t}}N'(d)$$

$$= \frac{\partial N(d)}{\partial d}\frac{\partial d}{\partial S} + \left[\frac{1}{\sigma S\sqrt{T-t}}\frac{\partial N'}{\partial d}\frac{\partial d}{\partial S}\right]$$

$$= \frac{1}{\sigma S\sqrt{T-t}}N'(d) + \frac{1}{\sigma^2 S(T-t)}(-dN'(d))$$

$$= \frac{N'(d)}{\sigma S\sqrt{T-t}} - \frac{d}{\sigma^2 S(T-t)}N'(d)$$

$$= N'(d)\left[\frac{1}{\sigma S\sqrt{T-t}} - \frac{d}{\sigma^2 S(T-t)}\right]$$

$$= \frac{N'(d)}{\sigma S\sqrt{T-t}}\left[1 - \frac{d}{\sigma\sqrt{T-t}}\right]$$

For the Γ_{put} we get the same result so even though the calculations are slightly different we don't show them here.

Rho (ρ)

Rho represents the rate of change between an option portfolio's value and the interest rate, or sensitivity to the interest rate.

$$\rho_{call} = \frac{\partial c}{\partial r} = \frac{\partial}{\partial r}(SN(d) = S\frac{\partial N}{\partial d}\frac{\partial d}{\partial r}$$
$$SN'(d)\frac{\sqrt{T-t}}{\sigma}$$

To calculate ρ_{put} we use

$$P = SN(-d)$$

We denote

$$x = -d$$

$$\rho_{put} = \frac{\partial P}{\partial r} = S\frac{\partial N}{\partial x}\frac{\partial x}{\partial r}$$
$$= SN'(x)\left(-\frac{\partial d}{\partial r}\right)$$
$$= -S\frac{\partial d}{\partial r}N'(-d)$$
$$= -S\frac{\sqrt{T-t}}{\sigma}N'(-d)$$

Vega (ϒ)

Vega represents the rate of change between an option portfolio's value and the underlying asset's volatility - in other words, sensitivity to volatility.

$$\Upsilon_{call} = \frac{\partial c}{\partial \sigma} = S\frac{\partial N(d)}{\partial \sigma}\frac{\partial d}{\partial \sigma}$$
$$= S(\sqrt{T-t} - \sigma d)N'(d)$$

$$
\begin{aligned}
\Upsilon_{put} &= \frac{\partial p}{\partial \sigma} = S\frac{\partial N}{\partial x}\frac{\partial x}{\partial \sigma} \\
&= -S\frac{\partial d}{\partial \sigma}N'(-d) \\
&= -S\left(\sqrt{T-t} - \sigma d\right)N^{'}(-d)
\end{aligned}
$$

Theta (Θ)

Theta represents the rate of change between an option portfolio and time, or time sensitivity.

$$
\begin{aligned}
\Theta_{call} &= -\frac{\partial c}{\partial t} = \frac{\partial}{\partial t}\left(SN(d)\right) = S\frac{\partial N(d)}{\partial d}\frac{\partial d}{\partial t} \\
&= SN'(d)\frac{d}{2(T-t)} - \frac{\left(r + \frac{1}{2}\sigma^2\right)}{\sigma\sqrt{T-t}}
\end{aligned}
$$

$$
\begin{aligned}
\Theta_{put} &= \frac{\partial p}{\partial t} = S\frac{\partial N}{\partial x}\frac{\partial x}{\partial t} \\
&= SN'(x)\left(-\frac{\partial d}{\partial t}\right) \\
&= -S\frac{\partial d}{\partial t}N'(-d) \\
&= -S\frac{d_1}{2(T-t)} - \frac{\left(r + \frac{1}{2}\sigma^2\right)}{\sigma\sqrt{T-t}}N'(-d)
\end{aligned}
$$

Numerical derivatives

Above we showed the analytical partial differential equations for some greeks. In practice, analytical derivatives are not used very often and a better way is to calculate the numerical derivatives. Using the right numerical method, even though it is not exact, it is so closed to exact that it used widely in the financial sector. The reason that numerical methods are more often used than analytical is mainly the fact that when the time to maturity goes to zero, analytical methods no longer work. In the attached Excel/VBA-file we have used numerical derivatives via the following formulas, considering an asset-or-nothing call option:

$$
\Delta = \frac{C(t, S+\delta, \sigma, r, K) - C(t, S-\delta, \sigma, r, K)}{2\delta}
$$

$$
\Gamma = \frac{C(t, S+\delta, \sigma, r, K) - 2C(t, S, \sigma, r, K) + C(t, S-\delta, \sigma, r, K)}{\delta^2}
$$

$$\rho = \frac{C(t,S,\sigma,r+\delta,K) - C(t,S,\sigma,r-\delta,K)}{2\delta}$$

$$\Upsilon = \frac{C(t,S,\sigma+\delta,r,K) - 2C(t,S,\sigma-\delta,r,K)}{2\delta}$$

$$\Theta = \frac{C(t+\delta,S,\sigma,r,K) - C(t,S,\sigma,r,K)}{\delta}$$

Summary

During this project we have got an insight into digital options. Even though we have focused on asset-or-nothing options the idea of cash-or-nothing is the same and we believe that this project has thought us the basics of the ideas with digital options. As stand-alone investments they are very risky, even though the maximum possible loss is known in advance. But options, including binary options are often used to balance a portfolio. In one hand that makes the dynamics more complex and a lot of calculations are needed as we saw when presenting the greeks. Recalculating the values of the greeks and the option price frequently to re-balance is called hedging and that, on the other hand, results in a more risk-neutralized portfolio.

References

Investopedia. (2012). Retrieved 10 16, 2012, from Investopedia: http://www.investopedia.com/terms/g/greeks.asp

Gaarder Haug, E. (2006). *The complete guide to Option Pricing Formulas.* McGraw Hill.

Röman, J. (2012). *Lecture notes in Analytical Finance I.* Mälardalen University.