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MMA 707 – Analytical Finance 1

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Barrier Option Valuation with Binomial Model

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Abstract

Our task is to calculate barrier option price. In theoretical part of this paper we present the analytical formulas for this calculation. However, barrier option price calculations we do with Cox, Ross and Rubinstein (1979) method. These authors use binomial model to calculate plain vanilla option price. In this paper we give the logic of calculation of barrier option price, however the code is not in the text. As a programming language we used MATLAB.

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1. Introduction

We are using binomial model to find the value of the barrier options. There exits two general classes of barrier options; in-options and out-options. With in-options the buyer get an option that becomes active if and when the underlying hits a given barrier value. If the underlying never reaches this value, the option will expire without a value.

An out-option is an option, which is active from the beginning, but becomes inactive, i.e., expires immediately if the underlying hits the barrier value.

It is possible to combine both types. If we have a down- and-out-call option and a down-and-in-call option and the underlying hit the barrier, the down-and-out becomes inactive while the down-and-in becomes active. Therefore this combination is an exact replication of a plain vanilla European call option. It is possible to create many kinds of barrier options on all kind of markets. All such barrier options can be either call or put options.

2. Theory

Barrier options are one of the most widely-traded exotics on some markets (OTC). When we compare barrier options with the plan vanilla options, we find that they have only one additional key term: a barrier imposed on price of the underlying. The barrier might be below the strike or above the strike. When barrier is hit during the life of the option, there are two possibilities of outcome: the contract might be cancelled or the contract might become effective. Hence there are four basic types of barrier options:

Down-and-out: the barrier is below the strike price; once it is hit, the option dies or become useless and there is no payout for the option holder at maturity, although the option might be in-the-money at maturity;

Down-and-in: the barrier is also below the strike price; once it is hit, the option becomes active or alive and the holder of the option gets usual payout if and only the barrier is hit during the life of the option.

Up-and-out: the barrier is above the strike price; once it is hit, the option becomes worthless or of zero value and there is no payoff for the holder of the option at maturity, no matter if the option is in-the-money at the maturity;

Up-and-in: the barrier is above the strike price; once it is hit, the option becomes valid or alive and has payout if and only if the stock price hits the barrier during the life of the option.

These four basic features of barrier options apply to both call and put options, for European and American type of options. The underlying of a barrier option might be a stock, stock index, commodity, foreign currency or interest rate. So barrier options are path-dependent and the payout at maturity depends not only on the price of the underlying at maturity but also on the path of the underlying during the life of the option.

2.1 Analytical formulas

Some barrier options have analytically solutions. An "in" barrier options becomes a plain vanilla option if the price has been below the barrier level H for a down-and-in option or if the asset price has been above H for an up-and-in option. An "out" barrier option is an option that equals a plain vanilla option as long as the asset price has always been above H for an down-and-out option or below H for and up-and-out option. When some barrier options are knocked, out they pay a rebate K at maturity.

The payoff of an in-barrier in combination with an out-barrier of the same type is equivalent to a plain vanilla option and a cash pay-out equal to the rebate, K.

$$\eta = \begin{cases} 1 & \text{if Down} \\ -1 & \text{if Up} \end{cases} \qquad \phi = \begin{cases} 1 & \text{if Call} \\ -1 & \text{if Put} \end{cases}$$

$$x_1 = \frac{\ln(S/X)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T}, \qquad x_2 = \frac{\ln(S/H)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T},$$

$$x_3 = \frac{\ln\left(H^2/(SX)\right)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T}, \qquad x_4 = \frac{\ln(H/S)}{\sigma\sqrt{T}} + (1+\mu)\sigma\sqrt{T},$$

$$z = \frac{\ln(H/S)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

$$\mu = \frac{b-\sigma^2/2}{\sigma^2}, \qquad \lambda = \sqrt{\mu^2 + \frac{2r}{\sigma^2}}$$

$$C = \phi S e^{(b-r)}\left(\frac{H}{S}\right)^{2(\mu+1)} N(\eta y_1) - \phi X e^{-rT}\left(\frac{H}{S}\right)^{2\mu} N(\eta y_1 - \eta\sigma\sqrt{T})$$

$$D = \phi S e^{(b-r)}\left(\frac{H}{S}\right)^{2(\mu+1)} N(\eta y_2) - \phi X e^{-rT}\left(\frac{H}{S}\right)^{2\mu} N(\eta y_2 - \eta\sigma\sqrt{T})$$

$$E = K e^{-rT}\left[N(\eta x_2 - \eta\sigma\sqrt{T}) - \left(\frac{H}{S}\right)^{2\mu} N(\eta y_2 - \eta\sigma\sqrt{T})\right]$$

$$F = K e^{-rT}\left[\left(\frac{H}{S}\right)^{\mu+\lambda} N(\eta z) - \left(\frac{H}{S}\right)^{\mu-\lambda} N(\eta z - 2\eta\lambda\sigma\sqrt{T})\right]$$

The prices of Call barriers are given by

Type		$X < H$	$X > H$
Down-and-In:	$S > H$	A – B + D + E	C + E
Up-and-In:	$S < H$	B – C + D + E	A + E
Down-and-Out:	$S > H$	B – D + F	A – C + F
Up-and-Out:	S < H	A – B + C – D + F	F

The prices of Put barriers are given by

Type		$X < H$	$X > H$
Down-and-In:	$S > H$	A + E	B - C + D + E
Up-and-In:	$S < H$	C + E	A – B + D + E
Down-and-Out:	$S > H$	F	A - B + C - D + F
Up-and-Out:	S < H	A – C + F	B – D + F

There are also many structural variations possible, for example:

- Some contracts have more than one barrier – e.g. a double knock-out knocks out if either a higher or a lower barrier is reached
- Some barrier options knock in or out depending on the performance of a different market. An example of this type is the soft call provision embedded in many Euro

convertible bonds, which gives the issuer the right to call the bond if the underlying shares reach a specified threshold level. Some barrier contracts as we see above, includes a rebate clause.

2.2 Binomial model

The binomial model was first proposed by Cox, Ross and Rubinstein in 1979. In finance, binomial model is generally used for valuation options by numerical method. It is flexible, intuitive and popular approach to option pricing. Binomial model is base on random walks theory. The concept is that, over a single period of time (very short duration) the underlying asset can only move from its current price to two possible levels. It assumes that movement of the price of underlying asset follows a binomial distribution; for many trials, this binomial distribution approach the normal distribution assumed by Black-Scholes-Merton Model.

We will introduce only theory used in our application. It is left to the reader to search more detail, if necessary.

Consider the financial market which consists of bond and stocks. Deterministic bond with process:

$$\begin{cases} B(0) = 1 \\ B(1) = 1 + r \end{cases} \tag{1}$$

Stock with stochastic process:

$$\begin{cases} S(0) = s \\ S(1) = \begin{cases} su \\ sd \end{cases} \quad prob \begin{matrix} p_u \\ p_d \end{matrix} \end{cases} \tag{2}$$

With these prices of stock and bond we can create a replicating portfolio for upward and downward movement of stock price. The replicating portfolio value will be equal to option value in both events. Thus if we write two equations for up and down movements, we can find the amounts to be invested in bond and stock. Thus the option value is given by equation as below

$$C = \frac{C_u - C_d}{u - d} + \frac{uC_d - dC_u}{(u-d)R}. \quad (3)$$

Here C_u and C_d are option values in the stock price up and down movements respectively, u and d are the up and down factors and $R=r+1$is the risk free interest rate.

Let denote the true probabilities as $P=(P_u, P_d)$ and risk-free probabilities as $Q=(Q_u, Q_d)$. If we use continuous compound interest rate, we get the following general pricing formula under risk neutral probability measure Q.

$$S(0) = e^{-rt} E^{Q}[S(1)] \quad (4)$$

We will apply this formula to bino use

$$u = e^{\sigma \cdot \sqrt{\Delta t}}$$

$$d = \frac{1}{u} = e^{-\sigma \cdot \sqrt{\Delta t}}$$

$$p = \frac{e^{r \cdot \Delta t} - d}{u - d} \quad (5)$$

We can evaluate the price of an american plain vanilla option by discounting its expected value under risk neutral probability measure Q and comparing it with the amount that we can get by exercising it, and take the maximum value. For the case of european call option we need to discount the expected value under risk neutral probability measure Q and comparing it with 0, and take the maximum value.

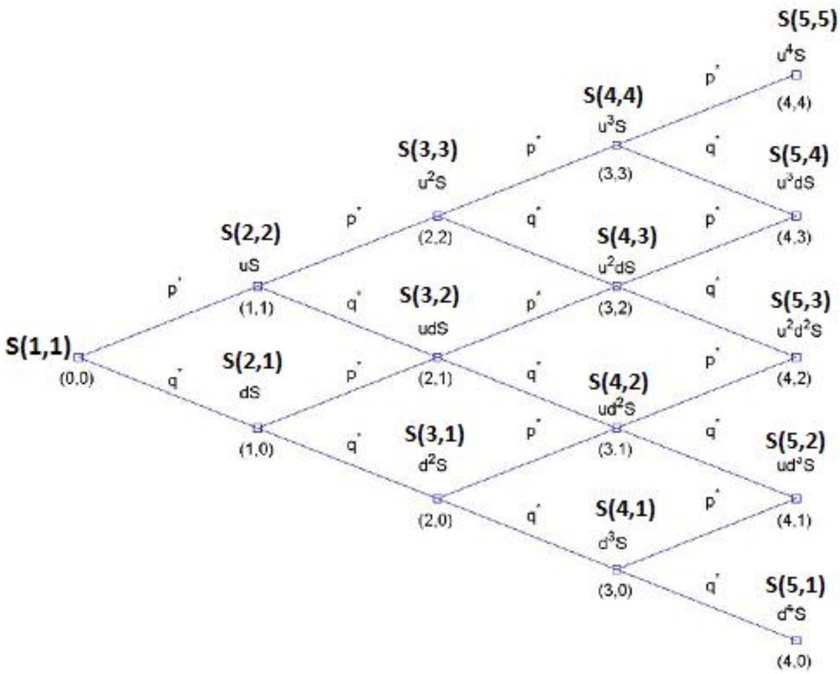
2.3 Barrier option valuation

As introduced above, barrier options are specific type of plain vanilla option, with extra conditions. In order to evaluate barrier option, we discuss it on an example of american option as european option is particular case of it.

Up and down factors and risk-neutral probability are calculated as in the formulas presented in previous part. We start our calculations by multiplying initial stock price S by u^n (n is the number of time steps the price changes). This way we get the highest stock price that can happen at the option maturity (Picture 1). Later we multiply $S*u^i$ by (d/u) and assign the result to the next lower level stock price at the time step i where i gets value from 1 to n. Thus we get all possible stock prices that can happen by maturity.

When we have all possible stock prices we compare them all with our barrier level h. In case of down-and-out and up-and-in options, we assign zeros to the stock prices lower than h and in case of down-and-in and up-and-out options we assign zeros to the stock prices higher than h barrier, as in that cases the option's value is zero. Thus these zero values will contribute to a lower option value in an earlier period while calculating the expectation. Because the zero value of underlying stock price affects put option value positively, for the case of put option we assign zeros to the option value at the node at which its stock price does not satisfy barrier condition. In case of double barrier condition, the stock price at every node must be compared with the barrier conditions needed.

The call option value in last nodes (at maturity) is calculated by $\{S(t)-k,0\}^+$ and for put option {k-S(t), 0}+. In all other nodes we calculate option value using $\{S(t)-k, e^{-r\Delta t}[S(t+1)]\}^+$ or by $\{k-S(t), e^{-r\Delta t}[S(t+1)]\}^+$ for call and for put options respectively. Using this method we calculate C(1,1) in the end, which is the price of our option.



Picture 1. Binomial model with stock price calculations

3. References

[1] Jan Roman, Analytical Finance I lecture Notes, MDH, 2010

[2] John C. Hull, Option, Futures and Other Derivatives, 7 th Edition, Prentice Hull New Jersey, 2009