ANALYTICAL FINANCE I

Monte-Carlo simulation with Black-Scholes

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October 11, 2010

Abstract

This report shows hedging strategies performed on a portfolio conatining of stocks and different options. Options in this case are used to hedge the change in price of a certain amount of stocks. The Black-Scholes Model is used in the construction of the Matlab program that shows the Monte-Carlo simulation of the dynamic hedging process.

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1 Notations

r_f=Risk free interest rate
σ=Volatility
T=Time to maturity
S_0=Inital stock price
K_c=Strike price Call option
K_p=Strike price Put option
N_s=Amount of stocks
C_{tra}=Transaction costs
Δ_{tol}=Delta tolerance
N_{ts}=Amount of time steps
MC_{sim}=How many loops the Monte Carlo simulation runs
$Strategy$=Strategy is either Delta or DeltaGamma

2 Introduction

In this report we are going to analyse possibilities of hedging a portfolio consisting of options and stocks. We are going to use traditional methods that are distinguished and proven within the area of finance.
The report are divided into three parts. The first part gives a brief review of the applied theory. The second one describes the constructed Matlab program, written to initialize the dynamic hedging process. The final part consists of the investigation results and gives recommendations to improve the hedging strategy.

3 Problem formulation

Make a Monte-Carlo simulation with Black-Scholes with parameters given (initial underlying price, strike price, volatility, time to maturity and risk-free interest rate). Simulate the situation where you buy 10000 underlying stock and at start, hedge the position with an option. Each such option has 100 stocks as underlying. So, make the best hedge. During the price movements of the underlying, change the hedge each time you need to buy or sell an option (on 100 underlying stocks). Finally, calculate the outcome of this trading strategy. Make 100 000 simulation and present a histogram of the result and calculate the mean value and the variance. What happen if you also introduce a trading cost of 2%?

4 Theory

The theoritical approach to this report is presented in order to connect the theories with the empirical work and to clearly mark the used theories in this report. There are different kinds of hedging methods and in this report we will concentrate on the delta and gamma from the greeks.

4.1 The greeks

The mathematical characteristics of the Black-Scholes model are named after the greek letters used to represent them in equations. The five Option Greeks are used to measure the sensitivity of the options price with respect to four different factors: changes in the underlying stock price, interest rate, volatility and time-decay. Option Greeks allow option traders to calculate changes in the value of the option contracts in their portfolio with changes in the factors that affects the value of stock options. It gives option traders the ability to hedge their portfolio or to construct positions with certain replicating portfolios.
Therefore, hedging a portfolio in an option trading strategy requires to consider four forms of risk: a)direct risk (Δ), b) velocity of direct risk (Γ), c) volatility risk (ν) and, d) time decay risk (θ). Traders do not normally perform hedging for interest rate risk (ρ) as its impact is very small.
Hedging a portfolio requires understanding of what the biggest risk in that portfolio is. If time decay should be concerned, then theta neutral hedging should be used. If a drop in the value of the underlying stock is of greatest concern, then delta neutral hedging should be used. In our model we performed delta-hedging and delta-gamma hedging strategies.

4.1.1 Delta

Delta measures the sensitivity of an option's price to a change in the price of the underlying stock. It tells you how much money an option will increase or decrease in value with a \$1 rise or drop in the underlying asset. It also translates to the amount of profit you will make when the underlying stock rises. It means that the higher the delta value a stock option has, the more it will rise with every \$1 rise in the underlying stock. Stock options with options delta of 0.6 is expected to increase \$0.60 with a \$1 rise in the underlying. Stock options value is influenced most by changes in the price of the underlying stock, making delta value of stock options the most important parameter of hedging.
Options delta can either positive or negative. Call Options have positive delta values suggesting that it will gain in value proportionately with a gain in value in the underlying stock. Put Options have negative delta values suggesting that its value will fall down as the underlying stock rises. Delta value increases as options gets more and more In The Money (ITM) and reduces as the options gets more and more Out Of The Money (OTM). At The Money Options, no matter call or put options, have delta value of 0.5, suggesting a 50% chance of either ending up In The Money or Out Of The Money.
Moreover delta is define as $\Delta = \frac{\delta C}{\delta S}$ mathematically.

4.1.2 Gamma

Gamma is the rate of options delta change with respect to a small rise in the price of the underlying asset. It depicts how much the options delta changes as the price of the underlying stock changes. Of the five options greeks, Delta and Gamma are the only ones that are related to each other. In hedging, one would want an overall position Gamma to lean towards the direction of interest so that options delta expands as the trade develops. One would, as well, want as low an overall options gamma as possible so that the options trading position remains as neutral to changes in the underlying stock as possible.
Moreover gamma is define as $\Gamma = \frac{\delta^2 C}{(\delta S)^2}$ mathematically.

4.2 Hedging

Hedging is any technique designed to reduce or eliminate financial risk. Hedging is the installation of protection or insurance into a portfolio in order to offset any unfavorable moves in the price of securities.
Sometimes, the cost of hedging just doesn't justify the real assessed downside risk of a portfolio because hedging involves executing more trades, which costs more money. So, the model should take in the consideration this fact as well.
A mathematical formula designed to price an option as a function of certain variables-generally stock price, striking price, volatility, time to expiration, and the current risk-free interest rate. Before invention of the Black-Scholes Model there was no standard option pricing method. The Black-Scholes Model turned that guessing game into a mathematical science which helped develop the options market into the lucrative industry it is today. Options traders compare the prevailing option price in the exchange against the theoretical value derived by the Black-Scholes Model in order to determine if a particular option contract is over or under valued, hence assisting them in their options trading decision. The Black-Scholes Model was originally created for the pricing and hedging of European Call and Put options as the American Options market, the CBOE, started only 1 month before the creation of the Black-Scholes Model.

4.2.1 Delta neutral hedging

Delta neutral hedging is the construction of positions that do not react to small changes in the price of the underlying stock. No matter if the underlying stock goes up or down, the position maintains it's value and neither increases nor decreases in price.

4.2.2 Gamma Delta Hedging

When the delta value is in a absorbable state and the gamma value are close to zero independent of how the underlying stock moves it is suitable to use the gamma delta hedging.

4.3 Black-Scholes

The Black-Scholes model of calculating options pricing has several assumptions. The most significant is that volatility, a measure of how much a stock can be expected to move in the near-term, is a constant over time. The Black-Scholes model also assumes stocks move in a manner referred to as a random walk; at any given moment, they are as likely to move up as they are to move down. These assumptions are combined with the principle that options pricing should provide no immediate gain to either seller or buyer.
The Black-Scholes formula is defined as:

$$C_0 = S_0 N(d_1) - Ke^{-rT} N(d2)$$

where,

$$d_1 = \frac{ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

4.4 Monte Carlo Simulation

In order to simulate different kinds of scenarios in our report with random variables attached to the functions, we will have to use Monte Carlo simulation. The Monte Carlo simulation will loop the given functions which contains random variables so that each loop represents a unique event. In our model we can decide how many loops the constructed program will make, and the more loops we make the more accurate expectation of the function we get.

5 Pseudo Code

In order to construct an optimal program based on making a Monte Carlo simulation with Black-Scholes model we made 4 programs. The programs are connected to each other and in order to find a result we need input variables such as r_f, σ, T, S_0, K_c, K_p, N_s, C_{tra}, Δ_{tol}, N_{ts}, MC_{sim}, $Strategy$ Which is defined in chapter 1.
The main program starts by calling the second program, which will simulate stockprices as matrices where the rows are decided by the amount of Monte Carlo simulations. Each row shows a different scenarios of the movement on the stock price and the columns shows the value of the stockprice in each time step. The following formula describes how the stockprice is calculated

$$S_t = S_{t-i}r^{(r-\frac{1}{2}\sigma^2)dt+\sigma dtdZ}.$$

Each timestep will now be calculated and that is by dividing the Time to maturity in the initial time with the amount of time steps. That is in order to find each time step between each rehedging time.
In order to create a portfolio we need to create options and the following formulas will describe how the options are constructed.
The price of call option is described as:

$$P_{call} = S * N(d_1) - Ke^{-rT}N(d_2).$$

and where the price of put option is defined as:

$$P_{put} = Ke^{-rT}N(d_2) - S * N(d_1).$$

where d_1 and d_2 are defined as:

$$d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}.$$

the delta of the options are defined as:

$$\Delta_{call} = N(d_1), \qquad \Delta_{put} = N(d_1) - 1$$

while gamma is defined as:

$$\Gamma = \frac{N(d_1)}{S\sigma\sqrt{T-t}}$$

N_c and N_p are two new matrices which will define the amount of call options and put options in order to hedge the portfolio. The amount of the options will be calculated using another program, the constructed program is depended on the strategy which can either be Delta or DeltaGamma the following formula.
if the strategy is Delta hedging

$$N_c = -\frac{N_s}{\Delta_{call} - \Gamma_{call}\frac{\Delta_{put}}{\Gamma_{put}}}, \qquad N_p = 0$$

else i.e. when the strategy is DeltaGamma

$$N_c = -\frac{N_s}{\Delta_{call} - \Gamma_{call}\frac{\Delta_{put}}{\Gamma_{put}}}, \qquad N_p = -\frac{N_s}{\Delta_{put} - \Gamma_{put}\frac{\Delta_{call}}{\Gamma_{call}}}$$

Practically the program will have an 2 Matrices as output. The matrices consists of the amount of call options and the amount of put options in each time step.
In order to calculate the portfolio we need to add both options in each time period, which is calculated by using:

$$Portfolio = N_c(i, j-1)\Delta_{call} + N_p(i, j-1)\Delta_{put} + N_s$$

where i is equal to each scenario of the option while j is equal to the time. Then a small construction is made to detect when the portfolio have a delta less than Δ_{tol} and in that event no rehedge will take place. The level of tolerance can be good to use when dealing with transactions cost. It can also be connected to the transaction cost where we can find out if the rehedge is optimal according to the transaction cost.
Later on we decide the total prices for the call and put options. We will start by finding the initial value for the each option and then we take the increment of amount of the options in each time step. That will be multiplied by the price of the opions in each step. It will result in 2 matrices which is denoted as N_cP and N_pP.
In order to calculate the total value of portfolio we need to multiply the prices of each option bought in each time step by the interest rate and add the transaction cost. Total value of cost will be calculated by following formula

$$C_{total} = \sum_{i=1}^{k=T} N_cP(i) + N_pP(i) + abs(N_cP(i) + N_pP(i))C_{tra}$$

And finally the output of the program will be mean value of the portfolio, the standard deviation and market risk security.

6 Analysis

At first, we launch the simulation of the stock price that follows the Geometric Brownian motion in the Black-Scholes model. One-year period is considered with initial stock price equal to 100 points. The annual risk-free interest rate is taken on the level of 10%, historical volatility of the underlying is assumed to be 20% per year. Thus, for 100 000 simulations and 1000 points of time we obtain the graph of stock price trajectories. For readability, only 100 randomly chosen prices paths are shown below.

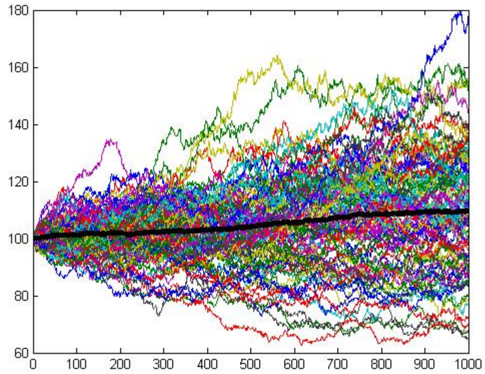

Figure 1: Simulation of the stock price with mean denoted by the black wide line.

Then, we start up the Monte-Carlo simulation of dynamic hedge process. On this step we investigate the behavior of our portfolio under delta-neutral hedging strategy with only call options $K = 100$, absence of transaction cost and delta-tolerance equal to 5%. For mentioned parameters we obtain 10,49% of profit in the end of period with standard deviation 599 points and variance coefficient 5,49%. The graph of portfolio value trajectories follows for 100 randomly taken simulations follows.

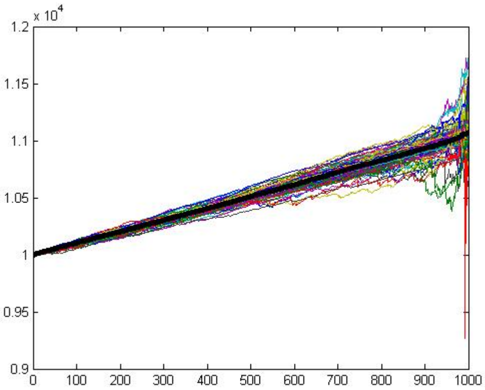

Figure 2: Delta-hedging portfolio value. Delta-tolerance=5%, cost=0.

On the graph it is clearly seen that the strategy doesn't provide good insurance and in some rare cases it even produces the losses. As a result, we decided to decrease the value of delta-tolerance parameter in order to make the re-hedging process be initialized more frequently.

After decreasing delta-tolerance to 1% we obtain 10,89% of profit, standard deviation of 355 points and variance coefficient 3,2%. Decreasing it to 0,5% causes the risk reduction to 1,9% level. But to treat these results correctly we should take in the consideration such a factor as transaction cost. After introducing it on the level of 2% we obtain the following graph of delta-hedging strategy.

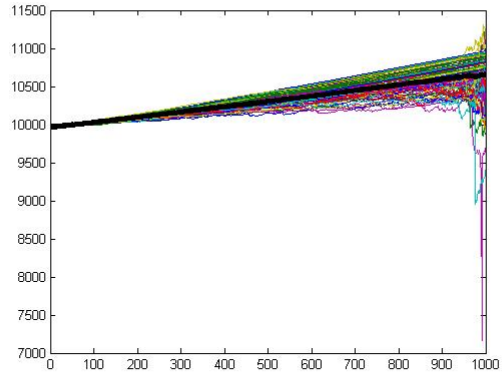

Figure 3: Delta-hedging portfolio value. Delta-tolerance=5%, cost=2%.

Cost	0%			2%		
Delta-tolerance	5%	1%	0,50%	5%	1%	0,50%
Mean	10,49%	10,89%	10,74%	7,83%	6,50%	6,61%
Std deviation	599,82	355,23	211,00	585,93	314,36	201,61
Var coefficient	5,43%	3,20%	1,90%	5,33%	2,95%	1,89%

Figure 4: Results of delta-hedging strategy.

The results of all simulations of delta hedging strategy could be compared in the table above. It's readily seen that making portfolio less delta-tolerant decreases the risk. More frequently we re-hedge the position - less risk we have, which is obvious. But introduction of realistic transaction cost eats up 3% of profit, if delta-tolerance is 5% and more than 4%, if we re-hedge more often and have acceptable risk. It draws to a conclusion that the strategy is not suitable under realistic conditions.

Therefore we introduce another strategy called delta-gamma hedging, that we'll hopefully improve the results. In this strategy the put option with strike price equal to 105 points is used in addition to the call option, to make the portfolio gamma-neutral at each point of time. All the other parameters remain the same.

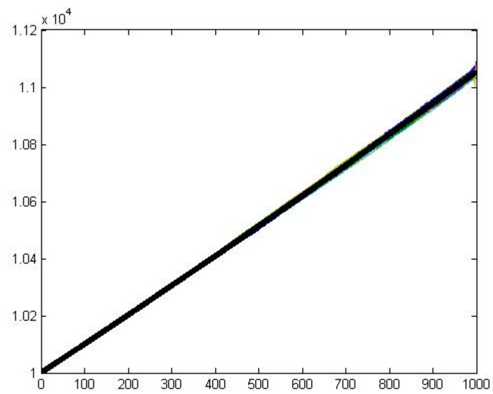

Figure 5: Delta-gamma-hedged portfolio value. Delta-tolerance=5%, cost=0.

The graph of the delta-gamma-neutral portfolio value shows us that even not frequent re-hedging provides very stable results, as the payoff of our replicating portfolio made of two options almost absolutely reproduces the payoff of underlying stock. The results are given in table 2 on the following page:

After comparing the results it can be seen, that transaction costs don't have a great influence on mean profit.

Cost	0%			2%		
Delta-tolerance	5%	1%	0,50%	5%	1%	0,50%
Mean	10,55%	10,53%	10,52%	9,98%	9,99%	9,95%
Std deviation	51,53	10,84	6,08	44,70	12,79	18,92
Var coefficient	0,46%	0,10%	0,06%	0,41%	0,12%	0,17%

Figure 6: Results of delta-gamma-hedging strategy.

Moreover, if the position is re-hedged frequently, the profit doesn't decrease as well. Thus, the efficiency of this strategy is much higher, than of delta-neutral. So, we can conclude that delta-gamma neutral strategy is the most convenient tool for hedging such an underlying asset as stock.

7 Conclusion

In this report we can study the impact of the Delta hedging and the Delta Gamma hedging and the reasons when to use them. The Black-Scholes model makes it easier for us interprate a real life situation and by the help of Monte Carlo simulations we will get different types of scenarios based on random variables.
The results are presented in section 6, where we also can see the different outputs when changing variables. It is also interesting to see the difference of the portfolio when chosing different strategy i.e. Delta hedging or Delta Gamma hedging. When the Δ_{tol} is changed we can also observe different outputs, that is because when the tolerance is set to be higher then the portfolio will not rehedge as often as if the tolerance was set to a lower value.
This report has opened our minds and enriched our knowledge within the field of analytical finance. Suggestions on further research within the subject is highly recommended. We have seen that there is opportunity to get deeper in the subject and to analyse more hedge parameters such as vega(ν), theta(θ), rho(ρ) etc and we also hope that this report can be a base of deeper studies.