Mälardalen University
Financial Engineering Program
Analytical Finance 1

Seminar Report

Hopscotch and Explicit difference method for solving Black-Scholes PDE

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1 Introduction

1.1 Objectives of the Seminar

The main objectives of the seminar are to study the mechanism of the Hopscotch and explicit difference methods for solving Black-Schoels PDE, and to learn how to build applications of these models in Excel Visual basic application (VBA). Our goal is to study how these different methods function in Excel/VBA for calculating European option prices, and to compare the results from these methods and Black-Schoels model by using graphs.

1.2 Brief Historical Review

In 1997, the Royal Swedish Academy of Sciences has decided to award the Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel to Professor Robert C. Merton and Myron S. Scholes.
Robert C. Merton, Myron S. Scholes and Fischer Black (1973) have developed a pioneering formula for the valuation of stock options. Their methodology has paved the way for economic valuations in many areas. It has also generated new types of financial instruments and facilitated more efficient risk management in society.[1]

Gordon (1965) and Gourlay (1970) have introduced a class of so called Hopscotch algorithms to solve parabolic and elliptic partial differential equations in two or more state variables. The purpose of this paper is then to present Hopscotch methods and to demonstrate how they can be used to solve financial models with two state variables.[2]

1.3 Methodology

In order to understand how the Hopscotch and explicit finite difference method works, we have formulated a numerical example of European options and solved it by help of Excel/VBA. We have also used the Black-Scholes model to calculate the price of the same options in VBA.

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[1] http://nobelprize.org/nobel_prizes/economics/laureates/1997/press.html

[2] "Hopscotch methods for two-state financial models"(Article by Adam Kurpiel, Thierry R oncalli: 1999)

2 Theoretical Framework

2.1 Hopscotch Method

Hopscotch method can solve parabolic and elliptic partial differential equations in two or more state variables but their utility in financial applications has not yet been realized. I will introduce how the hopscotch method can be used to solve financial models with two-state variables.

The basic idea is to divide the mesh points in the two-dimensional *x–y* mesh (*ih, jh*) as follows:

$$i + j \text{ odd}$$
$$i + j \text{ even}$$

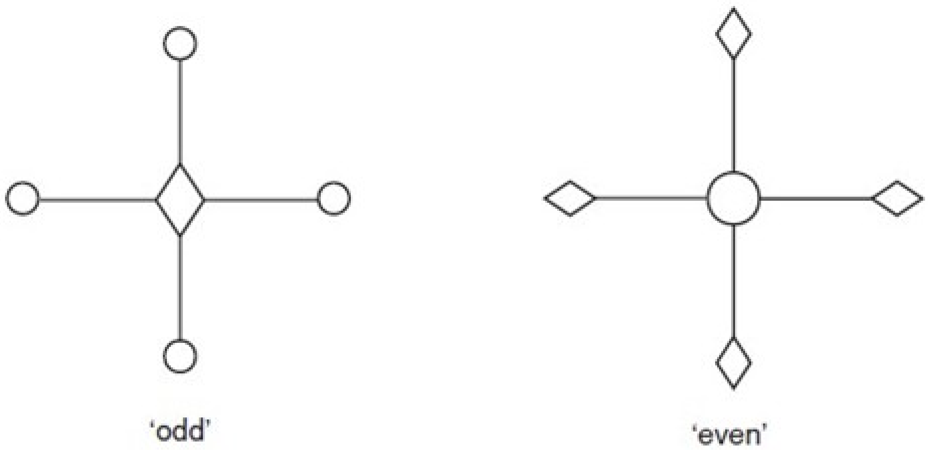
The Hopscotch consists of two 'sweeps'. In the first sweep (and subsequent odd-numbered
sweeps) the mesh points that are marked by a diamond, that is for which $i + j$ is odd, are calculated based on current values (time level n) at the neighboring points.

It can be defined as follows:

$$\frac{U_{ij}^{n+1} - U_{ij}^{n}}{k} = \Delta_x^2 U_{ij}^{n} + \Delta_y^2 U_{ij}^{n} \qquad \text{for} \quad (i+j) \text{ odd}$$

For the second sweep at the same time level $n + 1$ the same calculation is used at nodes marked with a circle. This second sweep is fully implicit. The scheme is:

$$\frac{U_{ij}^{n+1} - U_{ij}^{n}}{k} = \Delta_x^2 U_{ij}^{n+1} + \Delta_y^2 U_{ij}^{n+1} \text{ for } (i+j) \text{ even}$$



From this equation we can find the value at time level n+1 as follows:

$$U_{ij}^{n+1} = \frac{\left[U_{ij}^{n} + k\,\frac{U_{i+1.j}^{n+1} + U_{i-1.j}^{n+1}}{h_x^2} + k\,\frac{U_{i.j+1}^{n+1} + U_{i.j-1}^{n+1}}{h_y^2} \right]}{\left[1 + \frac{2k}{h_x^2} + \frac{2k}{h_y^2} \right]}$$

In the second and subsequent even-numbered time steps, the roles of the diamonds and circles are interchanged.

Now let K and T be the exercise price and the time to maturity of an European option on the underlying asset price $S(t)$. In the Black-Scholes framework, the call option price $C(T;S)$ satisfies the following equation

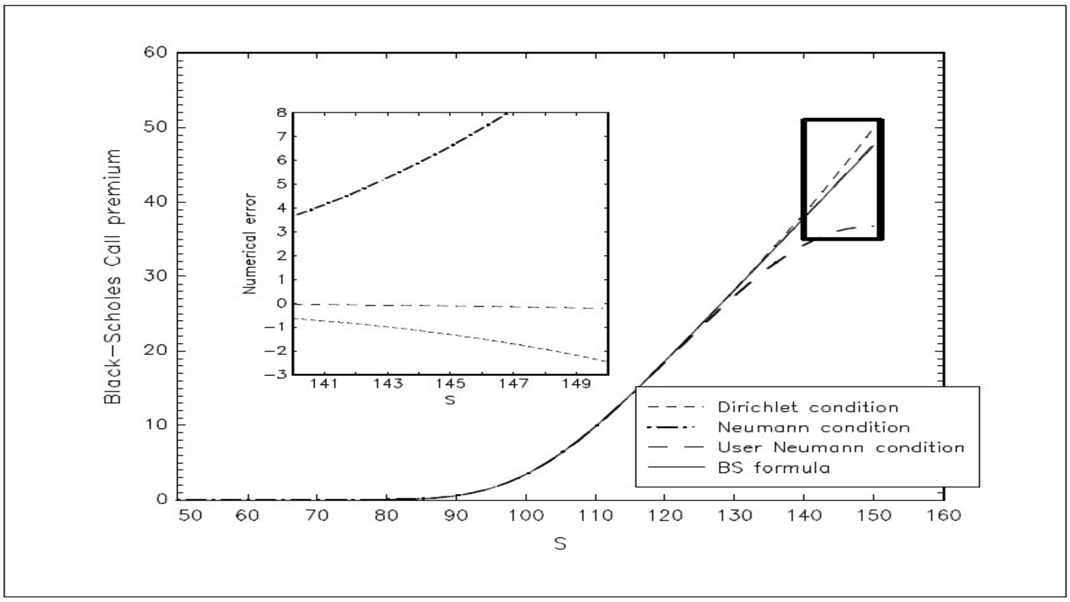
$$\frac{1}{2}\sigma^2 S^2 C_{ss} + bC_s = C_\tau + rC$$

$$C(0,S) = (S-K)_+$$

The parameter b is the cost-of-carry rate. To solve this problem numerically using Hopscotch methods, we have to add two boundary conditions for the extreme values S- and S+ taken by the S variable. For S equal to S-, we chose the following condition

$$u(t, S^-, y) = 0$$

because the option price tends to be zero when the underlying asset price decreases.



In many cases, we do not know four boundary conditions. Sometimes, a simple guess is used as a prior for a boundary condition. We may however use incorrect boundary conditions and still consider numerical solutions in the central region. We must be careful and we have to verify the behavior of the numerical solution when we change the boundary function.

2.2 Explicit Finite Difference Method

The main goal of finite difference techniques is solve numerically the Black-Scholes equation or one if its variations. The aim of build a numerical scheme for that equation is not to find the solution itself (we know that Black-Scholes for European options has an analytical solution) but to exploit such scheme to solve more general equations and inequalities. An easy way to start is to impose a coordinate transformation that permits to simplify the BS equation to one of its variances with constant coefficients.

Finite Differences

As with any class of option, the price of the derivative is governed by solving the underlying partial differential equation. The use of finite difference methods allows us to solve these PDEs by means of an iterative procedure.

We can start by looking at the Black-Scholes partial differential equation:

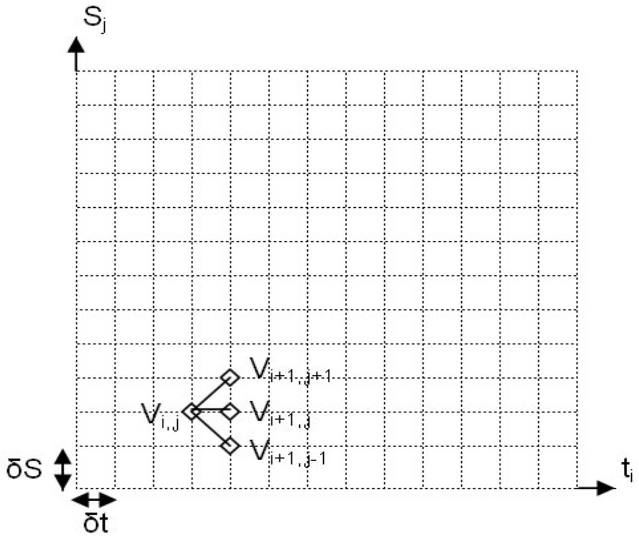
$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\mu S\frac{\partial V}{\partial S}-rV=0$$

Where dV is the change in the value of an option, dt is a small change in time. σ is the volatility of the underlying, S is the underlying price and μ is the carry (r-D).

By specifying initial and boundary conditions, one can attain numerical solutions to all the derivatives of the Black-Scholes PDE using a finite difference grid. The grid is typically set up so that partitions in two dimensions - space and time (in our case, we would be looking at the asset price and the change in time):

Once the grid is set up, there are three methods to evaluate the PDE at each time step. The difference between each of the three methods is contingent on the choice of *difference* used for time (i.e. forward, backward or central differences - more details in our financial mathematics glossary here). Central differences are used for the space grid (S).

Explicit Finite Differences



Explicit FD uses forward differences at each time node t. By splitting the differential equation into the time element and space elements, we can apply forward differences to time as follows:

First of all, the PDE as a reminder:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\mu S\frac{\partial V}{\partial S}-rV=0$$

if we substitute x = ln(S), the equation becomes:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}+\mu\frac{\partial V}{\partial x}-rV=0$$

Applying the finite differences method, the above equation can be broken down and approximated:

$\frac{\partial V}{\partial t}$ becomes $\frac{V_{i+1,j}-V_{i,j}}{\delta t}$

For the space grid, we can apply central differences for all order of derivatives:

$\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}$ becomes $\frac{1}{2}\sigma^2\frac{V_{i+1,j+1}-2V_{i+1,j}+V_{i+1,j-1}}{\delta x^2}$

and

$\mu\frac{\partial V}{\partial x}$ becomes $\mu\frac{V_{i+1,j+1}-V_{i+1,j-1}}{2\delta x}$

and

rV becomes $rV_{i+1,j}$

Combining the terms gives:

$$\frac{V_{i+1,j}-V_{i,j}}{\delta t}+\frac{1}{2}\sigma^2\frac{V_{i+1,j+1}-2V_{i+1,j}+V_{i+1,j-1}}{\delta x^2}+\mu\frac{V_{i+1,j+1}-V_{i+1,j-1}}{2\delta x}-rV_{i,j}=0$$

Which is the same as:

$$V_{i,j}=\frac{1}{1+r\delta t}\left(p_u V_{i+1,j+1}+p_m V_{i+1,j}+p_d V_{i+1,j-1}\right)$$

where the probabilities of each of the nodes is:

$$p_u=\frac{1}{2}\delta t\left(\frac{\sigma^2}{\delta x^2}+\frac{\mu}{\delta x}\right)$$

$$p_m=1-\delta t\frac{\sigma^2}{\delta x^2}$$

$$p_d=\frac{1}{2}\delta t\left(\frac{\sigma^2}{\delta x^2}-\frac{\mu}{\delta x}\right)$$

This case is actually equivalent to the trinomial tree where probabilities can be assigned to the likelihood of an up move, a down move as well as no move. It can also be shown that the following approximation holds:

$$\frac{1}{1+r\delta t}\approx\frac{1}{e^{r\delta t}}$$

2.3 Black-Scholes Formula

The Black-Scholes formula calculates the price of a call option to be:

$$\pi_c = S_t * N(d_1) - K * e^{-r(T-t)} * N(d_2)$$

Then the price of a put option is:

$$\pi_p = K * e^{-r(T-t)} * N(-d_2) - S_t * N(-d_1)$$

Where

$$d_1 = \frac{\ln\{S/K\} + \left(r + \frac{\sigma^2}{2}\right) * (T - t)}{\sigma * \sqrt{T - t}}$$

$$d_2 = d_1 - \sigma * \sqrt{T - t}$$

St= price of underlying stock
K=Option exercise price
r= Risk free interests rate

T=Expiring time
t= Current time
N ( ) = Area under the normal distribution curve

There are some assumptions of the model which are as followings[3]:

- The Black-Scholes model assumes that the option can only be exercised on the expiration date;
- It requires to use constant and known interest rates, the risk-free rate such as the discount rate on U.S Government Treasury Bills are usually used;
- It also assumes that the underlying stock does not pay dividends;
- It assumes the returns on the underlying stock are normally distributed.
- It assumes that the market is efficient.

3 Numerical Examples

The underlying stock price is 50 SEK today, and the strike price of the option is set as 50 SEK. The volatility of the market has been given as 20% and the risk-free interest rate is 5%. This is a European option with maturity date of 1 year from today. Please calculate the price of the call option as well as the price of the put options.

By using the Black-Scholes model we can calculate the both call and put option prices as followings:

Black-Scholes European call/put option prices

Stock Price (S):	50
Strike Price (K):	50
Time to maturity (T-t):	1
Volatility (σ):	0.2
Interest rate (r):	0.05

Calculate

Call price:	5.225329
Put price:	2.7868

The results shown on the figure above have been gained by using the Excel/VBA and the details of the application that build inside the VBA for this calculation can be found in the separate excel file called as "Seminar_BS_VBA ".

[3] http://hilltop.bradley.edu/~arr/bsm/pg04.html

By using Hopscotch and explicit difference methods to build applications in Excel, we will get similar prices of the options. These calculations and applications in VBA will be illustrated in a separated Excel file and this will be shown in the seminar presentation. (This part is not ready yet).

4 Conclusions

Black-Scholes theories are some of the most significant contributions in the development of finance theory. Black-Scholes model is well-known as sufficient instrument to price securities on the financial market together with Hopscotch and explicit finite difference method. Today, it has become much more powerful with help of the Excel/VBA which makes the complicating meth calculations to be done in a much simple way.

5 References

"Hopscotch methods for two-state financial models"(Article by Adam Kurpiel, Thierry R oncalli: 1999)

http://nobelprize.org/nobel_prizes/economics/laureates/1997/press.html
http://hilltop.bradley.edu/~arr/bsm/pg04.html