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Monte Carlo Simulations for Pricing Asian Basket Multi-Digital Options

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Abstract

We use an application of Monte Carlo Methods to price an Asian Basket Multi-Digital Option. The first section describes the characteristics of this type of option, and the model we used to implement it in Matlab. The second section does an analysis of the Monte Carlo algorithm developed. In the last section we present our simulation results and compare them to real historical data from the Swedish and Russian market.

Contents

Chapter 1

Introduction

A Monte Carlo simulation can be regarded as any method which solves a problem by generating suitable random numbers and observing that fraction of the numbers obeying some property or properties. The method is useful for obtaining numerical solutions to problems which are too complicated to solve analytically. It was named by S. Ulam, who in 1946 became the first mathematician to dignify this approach with a name, in honor of a relative having a propensity to gamble (Hoffman 1998, p. 239). Nicolas Metropolis also made important contributions to the development of such methods [7].

In the following report we will apply a defined monte carlo simulation to an exotic option. An Asian option (or average value option) is a special type of option contract. For Asian options the payoff is determined by the average underlying price over some pre-set period of time. This is different to the case of the usual European option and American option, where the payoff of the option contract depends on the price of the underlying instrument at maturity. One advantage of Asian options is that these reduce the risk of market manipulation of the underlying instrument at maturity [8].

We will be dealing with a variation of Asian options, namely, Asian Basket Multidigital, that means, that instead of having a single underlying stock, there will be more (in our case we have decided to work with 10). The digital part in the name implies that the payoff of such a function is 1 money unit if some conditions are met. In the rest of this report we will go in further depth to cover this concepts more thoroughly.

1.1 Task

Our main task was to "Build an application in Excel/VBA to solve option prices for an Asian-Basket-Multi-Digital option. That is; an option with maturity in 6 months on a basket of N (say 10) shares (this is the Basket part). The strike on each option is given as the starting price $S_i(0)$. The final price is given by the average of the price during the last month (this is the Asian part). If n (say 5) of the shares average price is above or equal to the strike the entire option pays 1 cash unit (this is the Digital part). Calculate the fair price of this exotic option."

Chapter 2

Problem Description

The general procedure was to implement an algorithm which would give us an estimate of Asian Basket Multi-Digital options by using Monte Carlo simulation. Since the number of iterations required to satisfy the minimum conditions of convergence of our estimators is very high, we decided it would be a better approach to build the application in Matlab, due to its better response time.

2.1 Asian Basket Multi-Digital Options

There are two variants of Asian Digital options:

Floating Strike Options are those in which at maturity, the average price over the life of an option is used as strike while the current price of an underlying is used to calculate the payoff of an option at maturity.

Floating Price Options are those in which at maturity, the average price is used to calculate the pay off of an option, while the strike price is preset.

The case that concerns us is then the second one. The main idea of our algorithm is to generate randomly a price process for each of the 10 underlying stocks and compare the 30-day average of each with its strike price, thus obtaining the payoff (1 money unit or 0) of the option. Then this process is repeated a large number of times (say 1000 times) and the expected payoff is computed as the average of all the payoffs.

The monte carlo application is in the averaging process, it makes use of the strong law of large numbers to assume that the expected payoff can be approximated by averaging the payoffs of the iterations.

For the generation of random price paths of the underlying stocks, we assume that these follow geometrical brownian motion, and thus by generating random numbers with normal distribution with a certain mean and variance we can use the following value process:

$$S_t = S_{t-1} e^{\mu - \frac{\sigma^2}{2}\Delta t + \sigma\sqrt{\Delta t} N_t(0,1)} \tag{2.1}$$

As we will see in the following sections.

2.2 Monte Carlo Methods

Monte Carlo methods are based on the analogy between probability and volume. The mathematics of measure formalizes the intuitive notion of probability, associating an event with a set of outcomes and defining the probability of the event to be its volume or measure relative to that of a universe of possible outcomes. Monte Carlo uses this identity in reverse, calculating the volume of a set by interpreting the volume as a probability. In the simplest case, this means sampling randomly from a universe of possible outcomes and taking the fraction of random draws that fall in a given set as an estimate of the set's volume. The law of large numbers ensures that this estimate converges to the correct value as the number of draws increases. The central limit theorem provides information about the likely magnitude of the error in the estimate after a finite number of draws [3].

The scope of our work does not include the calculation of the error magnitude. However, for the part of price calculation we will take advantage of the properties the strong law of large numbers offers to conclude that the average of a large number of payoffs will converge to its expected value.

2.2.1 Laws of Large Numbers

The strong law of large numbers states that, if we have independent and identically distributed random variables, the average of "a large number" of these, converges to their (common) expected value almost surely, mathematically this is:

Theorem 1. *Strong Law of Large Numbers [1]*
Let X_1, X_2, ... be a family of IID random variables. Suppose that the mean $\mu = E[X_1]$ exists. Then,

$$\lim_{n\to\infty} \frac{X_1 + X_2 + \ldots + X_n}{n} = \mu \tag{2.2}$$

In our case, since we are randomly sampling numbers with normal distribution, with a given mean and variance and our pricing process is but a function of these numbers, we can assume independent paths. Therefore, if we average "a large number" of the prices obtained with each of these price paths we can infere that this average will converge to the correct expectation.

The proof of this theorem is beyond the scope of this work. It is however very important to understand how the previously described concept allows us to make use of monte carlo methods to estimate the correct price of our option.

2.3 Model for the Underlying Assets Price Process

It is crucial to choose an appropriate pricing model in order to obtain adequate simulation results. Since our exotic option deals with 10 stocks, then we should look for an equation describing the behavior of stock prices in time. Obviously, price process is stochastic. As it can be seen from experience, many financial engineers assume that stock prices follow a Geometrical Brownian Motion (GMB) with drift and volatility. In fact, Black and Scholes had the same assumption, deriving European option pricing formula. LetŠs consider, that stock prices satisfy the following equation (GBM):

$$S_t = S_0 e^{\mu - \frac{\sigma^2}{2} t + \sigma Z(t)} \tag{2.3}$$

Where

S_t is a time t price of stock
S_0 is time 0 stock price
μ is a drift, which is an interest-rate
σ is diffusion, or volatility
Z is a Wiener process.

Then if we consider the ratio of two stock prices in two time moments: t and $(t-1)$, then we obtain:

$$\frac{S_t}{S_{t-1}} e^{\mu - \frac{\sigma^2}{2} \Delta t + \sigma (Z(t) - Z(t-1))} \tag{2.4}$$

But we know, that by definition and properties of the Wiener process, the increments $[Z(t)Z(t-1)]$ are independent and normally distributed. Thus the formula (2.4) can be rewritten as:

$$S_t = S_{t-1} e^{\mu - \frac{\sigma^2}{2} \Delta t + \sigma N_t(0, \Delta t)} \tag{2.5}$$

Now this recurrent equation can be used in Monte-Carlo simulations. That is because a random vector with size $N = [(timetomaturity)/(timeframe)]$, consisting of N random normal numbers, will result a price dynamics for one simulation when applied to the formula (2.5). But since MATLAB easily generates standard normally distributed numbers, then the formula (2.5) is slightly modified.

$$S_t = S_{t-1} e^{\mu - \frac{\sigma^2}{2} \Delta t + \sigma \sqrt{\Delta t} N_t(0,1)} \tag{2.6}$$

This is the formula used in our model. Now we have all necessary information to create an algorithm of a program pricing an exotic option.

Chapter 3

Algorithm Analysis

In the first section of this chapter we describe the process of generating a price path for each of the underlying stocks. The second section describes an iterative process to generate a large number of the previously obtained price paths to obtain a good estimate of the expected payoff.

3.1 Generating Price Processes

To generate the price paths we first generated T normally distributed random numbers with mean and volatility as described in the previous section for each of the 10 stocks. We then proceeded to use the pricing formula described in the model. Finally we averaged out the last 30 days of data for each of the 10 stocks, and compared each of these to their respective strike prices, obtaining 1 cash unit as payoff if 5 or more of the averages were above their respective strike prices as stated in the task.

3.2 Averaging Process

The averaging process is when we can actually apply monte carlo and the strong law of large numbers. We repeated the previously described process N times, obtaining this way N different payoffs. By using the monte carlo concept, we then averaged these payoffs and due to the strong law of large numbers we assumed that this average converged to the expected value of the payoff. Finally, we discounted this estimate to give us the correct arbitrage-free price estimate of the option.

Chapter 4

Results

The following chapter is devoted to presenting some charts and analysis comparing the results we obtained with our simulations to the real historical data. We will also describe the stocks used and how we obtained the volatility for them.

4.1 Description of Data

In this section we will explain how we obtained the volatility rates and the risk-free interest rate for use in our model. We decided to apply our model in two different markets with different dynamics to obtain a more clear result. The first one is the The Stockholm Stock Exchange (Swedish: Stockholmsbörsen). The second one is the more volatile Russian Trading System Stock Exchange RTS.

4.2 Stockholmsbörsen

We have picked ten stocks from Stockholmsbörsen namely ABB, Atlas Copco B, Electrolux B, Ericcsson B, Hennes & Mauritz, SEB, Scania, Svenska Hendelsbanken, SwedBank and Tele 2. We have used 6 months of historical data, from 1st April 2009 to 30th September 2009.

4.3 Moscow Interbank Currency Exchange

Let's study the hypothetical behavior of our option if it would be written on, say, 10 "blue chips" of a Russian stock exchange MICEX ("Moscow Interbank Currency Exchange"). So, we looked at price dynamics of 10 stocks traded on that exchange: Gazprom, Lukoil, Sberbank, VTB Bank, Rostelekom, Apotheke 36.6, Rosneft, Poluszoloto, Nornikel and Rusgidro. The companies deal in different industries (banking and finance, oil and gas production, telecommunication, health treatment goods, gold and nickel production, energy production). We can say that there is diversification. We considered two time interval with length 6 months: (October 14th 2008 - April 14th 2009) and (April 14th 2009 - October 14th 2009). There was a 1-hour time frame used. Data obtained from official sources [4] and [5]. Thus, we compared last month's average prices with initial ones for all 10 stocks on both time intervals. After that we obtained the number of stocks with average prices above strikes, which is necessary to state whether our option would have a payoff equal to 1 or to 0. ItŠs important to consider these two time intervals to apply our model to market in different conditions. The time interval (October 14th 2008 - April 14th 2009) is a crisis time, with extremely high volatility, while the second one (April 14th 2009 - October 14th 2009) corresponds to more stable market. As we will see further, in both cases option (if it ever existed) would have paid 1 cash unit.

4.4 Choice of Volatility and Risk-Free Interest Rate

Our historical volatility is obtained by:

$$\sqrt{\frac{\sum_{i=1}^{n}(u_i)^2}{n-1} - \frac{(\sum_{i=1}^{n} u_i)^2}{n(n-1)}}\sqrt{k} \tag{4.1}$$

where

$$u_i = \ln(\frac{S_t}{S_{t-1}})$$

S_t is a time t stock price S_{t-1} is a time $(t-1)$ stock price N is a number of prices in the interval the volatility is calculated on k is an amount of time frames in 1 year (253 trading days)

The risk free interest rate has been chosen in correspondence with coupon interest rate on long-term government bonds. We have taken it equal to 7% ([6]).

4.5 Comparison of Simulated Results with Historical Data

In the following section we present some figures with our results of simulation and compare them to real historical data.

After simulations we get the graphs of price dynamics, amount of stocks above strikes with correspondence to iterations, etc. Let's compare real price dynamics on Swedish market (figure 4.4) and the one generated by our program (figure 4.3) using formula (2.6). They look pretty similar. Furthermore, if we increase time interval, simulations number and try to compare average of historical data with an average of simulated, then simulated and real price processes look very alike. That's why the GBM is good enough for adequate modeling. If we look at the graph (figure 4.1) then it can be noted that in most simulations more than 4 stocks have last months' average prices about strikes, with a very few extreme cases of 0 and 10. That is what would have happened if the option would have been written on ten MICEX blue chips. In both time intervals the payoff is 1 cash unit, because more than 4 stocks end 6-months path with last month's average higher then the starting price. So, the simulations give two prices of an exotic option: 0,020026 and 0,067265 for the 1st and 2nd time intervals respectively. It's not a lot in comparison with 0,496585 obtained from real data, but this is explained, because our historical application contained no uncertainty.

The chart depicted in (4.5) shows the values from the real historical data obtained from the Russian market (MICEX).

Let's now compare 0,020026 and 0,067265 for (October 14th 2008 - April 14th 2009) and (April 14th 2009 - October 14th 2009). Remember, that the main difference between two time intervals is volatility. And the option costs 3 times as more on a stable market. Thus, we can make a hypothesis that the price of an Asian Basket Multi-Digital Option is reversely proportional to volatility. In other words, the higher the volatility is, the fewer the price of the option is. This hypothesis is a subject for further research.

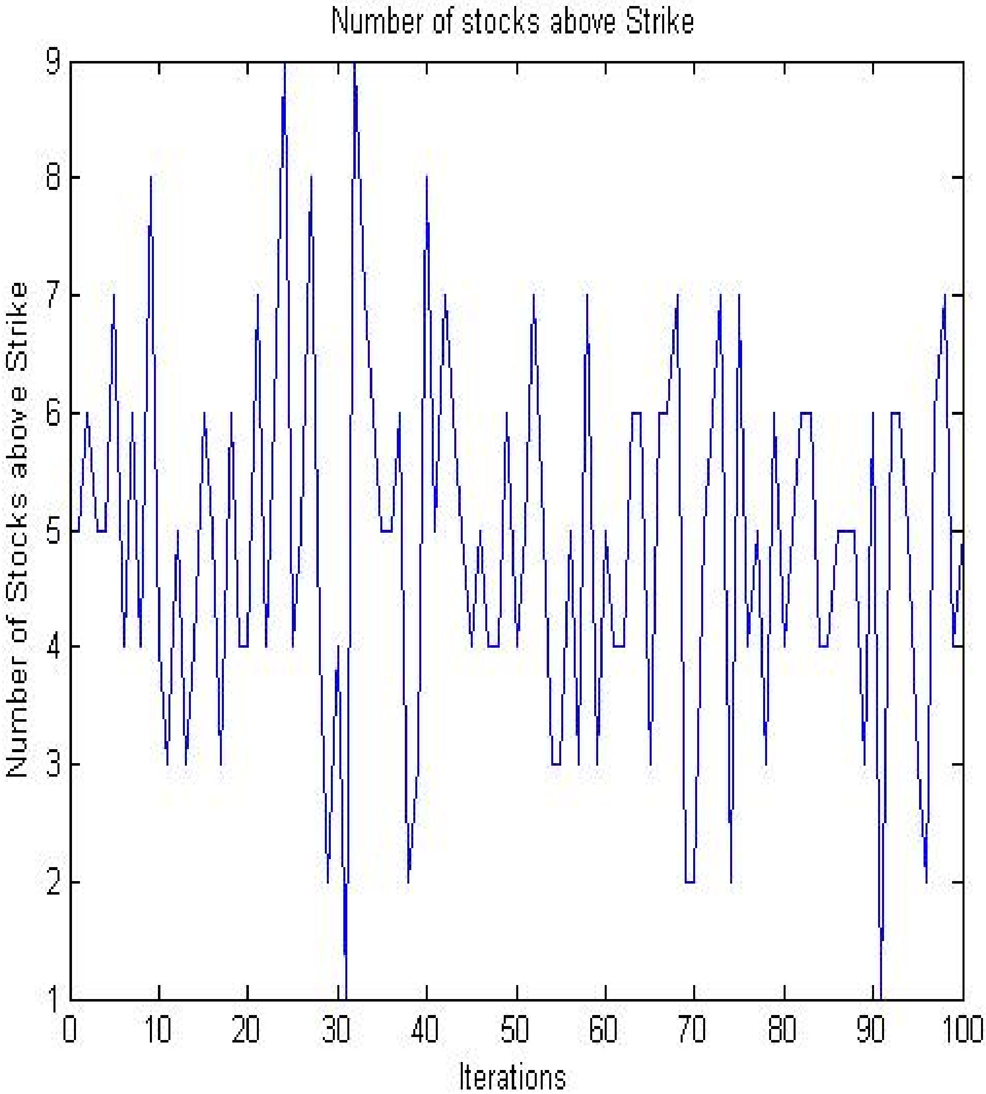

Figure 4.1: Number of stocks above strike in Russian market

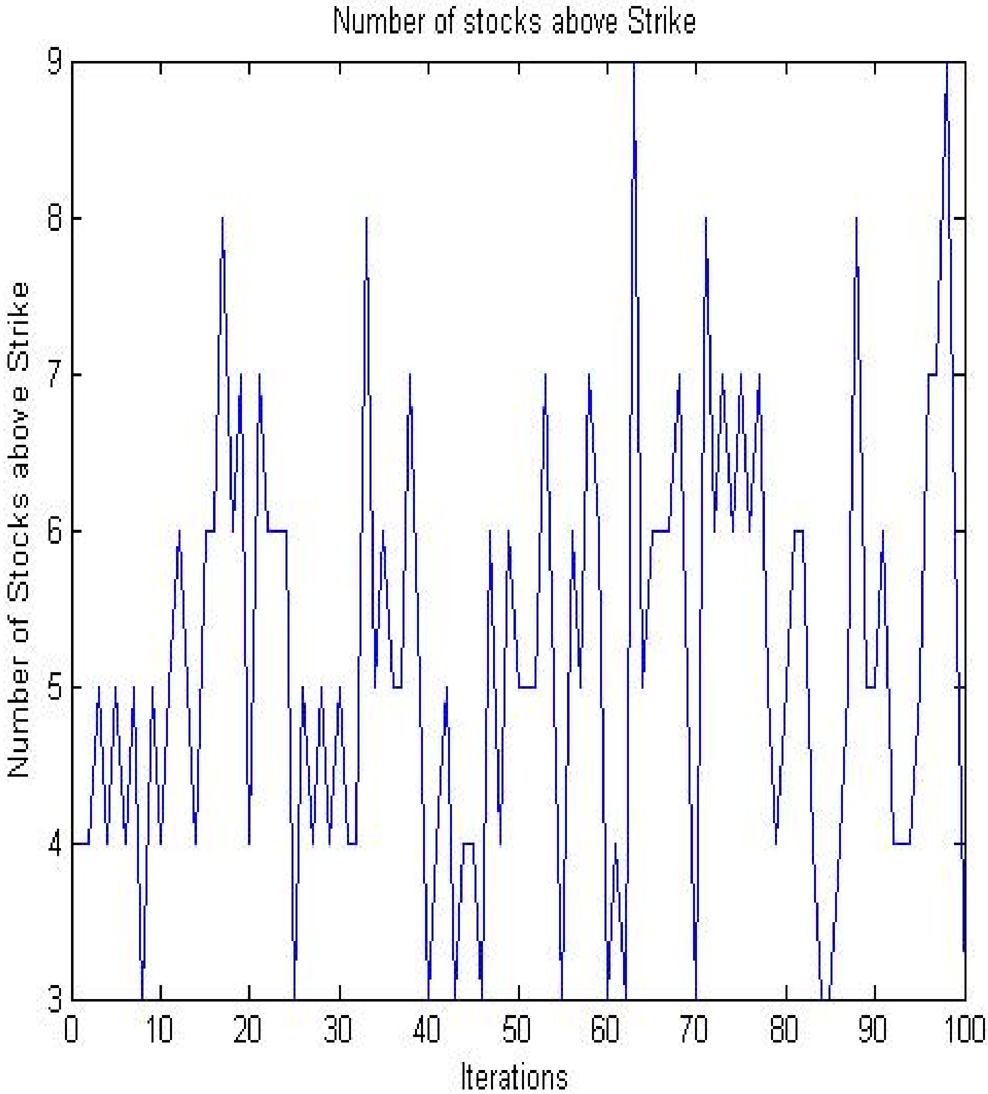

Figure 4.2: Number of stocks above strike in Swedish market

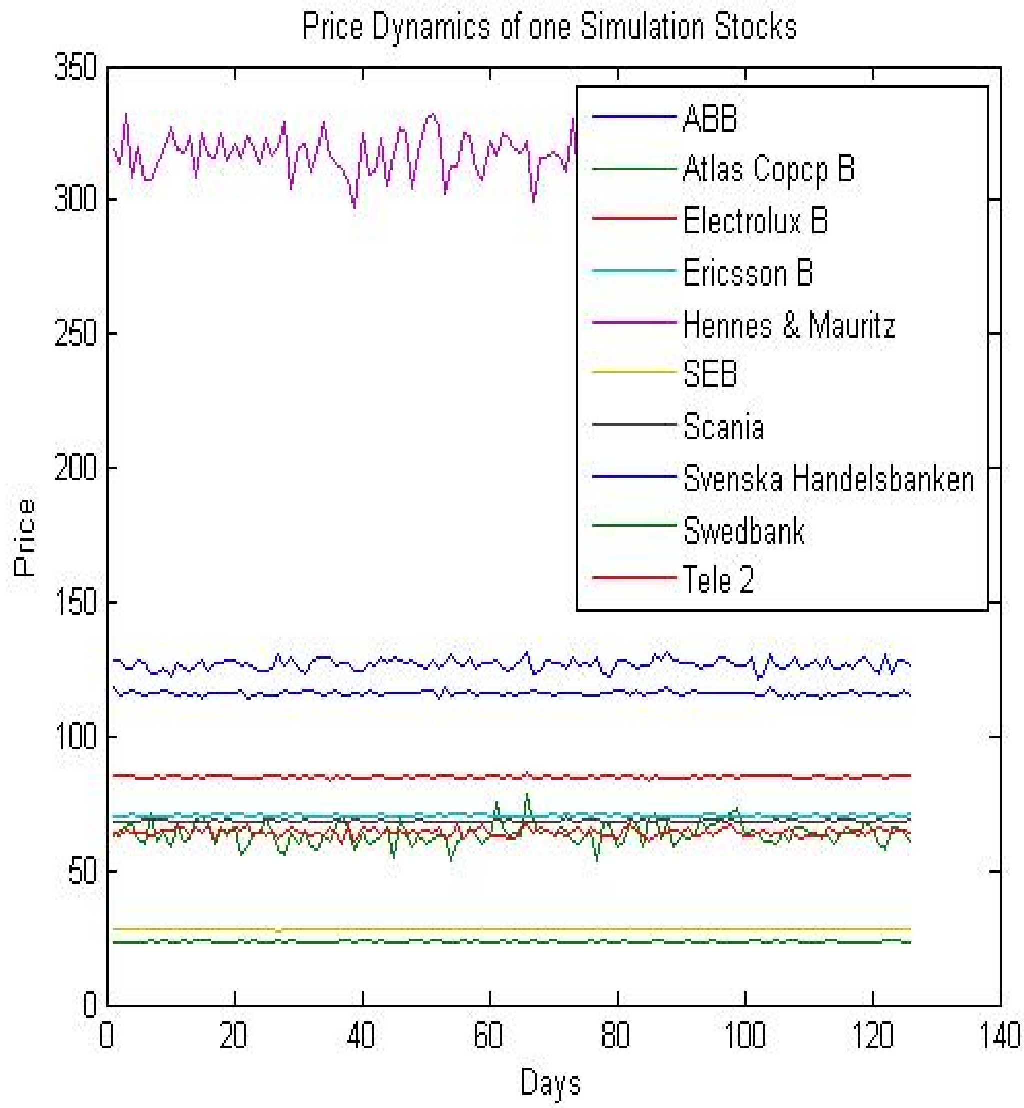

Figure 4.3: Simulation of Stock Prices in Swedish Market

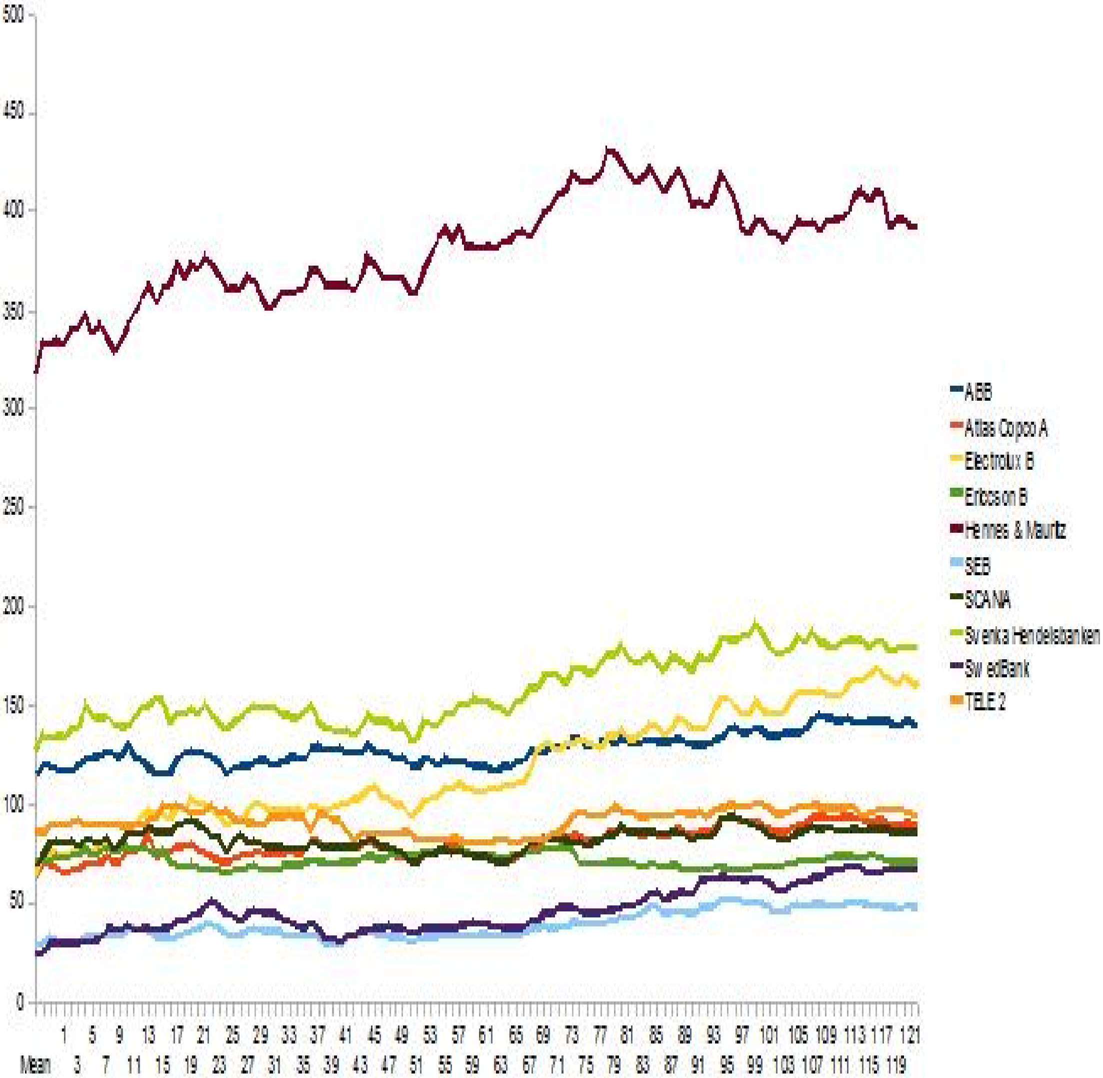

Figure 4.4: Historical Data of Stock Prices in Swedish Market

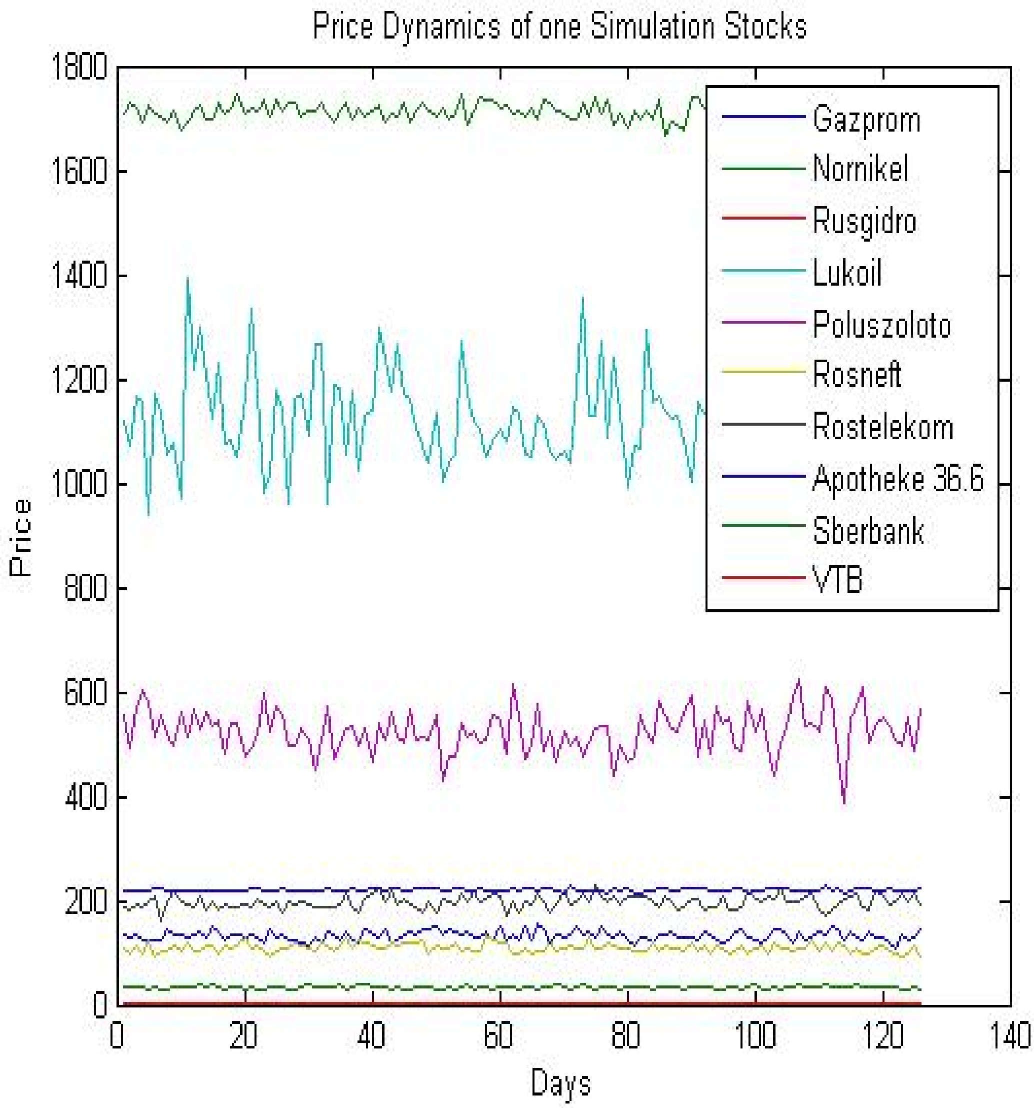

Figure 4.5: Simulation of Stock Prices in Russian Market

October 14 2008 – April 14 2009

Payoff=1	strike	Last month average	volatility	H
Gazprom	133,10	134,4002	79,63%	1
Nornikel	1712,91	2292,789	108,21%	1
Rusgidro	0,55	0,790288	95,98%	1
Lukoil	1133,90	1394,419	88,09%	1
Poluszoloto	526,10	1396,544	83,19%	1
Rosneft	109,50	163,7162	89,35%	1
Rostelekom	198,00	309,1153	70,85%	1
Apotheke 36.6	220,00	86,15606	128,08%	0
Sberbank	33,65	23,84727	98,37%	0
VTB	0,04	0,029897	84,81%	0

April 14 2009 – October 14 2009

Payoff=1	strike	Last month average	volatility	H
Gazprom	144,99	179,69	45,87%	1
Nornikel	2680,02	3749,91	54,79%	1
Rusgidro	0,98	1,07	54,00%	1
Lukoil	1538,90	1698,86	44,58%	1
Poluszoloto	1440,00	1391,73	40,51%	0
Rosneft	176,99	230,22	47,57%	1
Rostelekom	320,44	148,64	44,38%	0
Apotheke 36.6	135,60	304,80	47%	1
Sberbank	28,07	61,76	63,74%	1
VTB	0,03	0,06	63,43%	1

Figure 4.6: Data Obtained from MICEX

Chapter 5

Conclusion

The expression "Monte Carlo method" is actually very general. By now we have developed an understanding that the Monte Carlo (MC) methods are stochastic techniquesŰmeaning they are based on the use of random numbers and probability statistics to investigate and solve problems relating future predictions. We have found that MC methods are widely being used in everything from economics to nuclear physics to regulating the flow of traffic. Of course the way they are applied varies widely from field to field. But, strictly speaking, to call something a "Monte Carlo" experiment, all you need to do is use random numbers to examine some problem. In our case the problem was to price a complicated financial instrument Ű Asian Basket Multi-Digital Option. That is the very beginning og report Ű the explanation of the problem. The report contains good theoretical part of what is Monte-Carlo, so that all steps of our research are given reasonable explanation. First of all, being mathematically strict, we have proved that due to law of large numbers, we can use Monte-Carlo methods to price an option. Secondly, underlying assets pricing process is described, Geometrical Brownian Motion model is chosen, simulation formula derived. Next, we made an algorithm for the program that prices a contingent claim. Furthermore, the program is created and tested it on real data from Swedish and Russian stock market. The final parts of report show comparison of simulated results with calculations based on historical financial information. Since such complicated contingent claims not only exist, but are even traded (for example, in Swedbank), then results of research are of a great importance. Moreover, after carrying the research, authors make a hypothesis that the price of an Asian Basket Multi-Digital Option is reversely proportional to stocks' volatility. For these two reasons, the research will be continued, the model will be improved and tested no new data of different assets and markets

Chapter 6

Appendix

In the following sections we first present our developed algorithm implemented in Matlab, and we then present the flow-chart used to create this implementation.

6.1 Matlab program

```
% Script to generate Random Price Processes for N Underlying Stocks,l
% average the price of the last month of each Stock, to obtain the Payoff
% of Asian Digital Option

Y=0;                % Initial Value of Accumulated Payoff of one simulation

N = 10;              % Number of Underlying stocks for which the price processes
                    % will be generated.

T = 6;               % Maturity time of the Asian Digital Option in months.

n = 21;              % n is the number of prices per month.

t = T*n;             % Number of prices to generate until maturity

∆=(1/t)*ones(t,N);           % Times used in Stochastic Pricing Process (ones(t,N)

mu=0.05;                         % Fixed risk-free interest rate.

sigma=[0.079 0.77 0.276 0.035 0.259 0.07 0.053 0.176 0.12 0.06];     % Volatility r

CurrentTime=1;

Initial_Price=[116 64 64.2 70.5 317.5 28.10 68.5 127 23.41 85];     % Initial Price


for s=1:(t-1)

    timetomaturity(s)=21*6-s;
```

```
end


for k=1:10000

    epsilon(:,:,k) = randn(t,N);

for j=1:N

Price(:,j,k)=Initial_Price(j)*exp((((mu-(sigma(j)^2))/2)*Δ(:,j))+sigma(j)*epsilon(:,j,k).*sc


end


P_red(:,:,k) = Price(t-n+1:t,:,k);     % S_red is the Matrix that contains the prices for
                                      % the last month of each stock.

M(k,:) = mean(P_red(:,:,k));      % The matrix M denotes the mean price of the last month
                                 % of each stock.


H(k) = 0;            % Counter that indicates the number of stocks whose mean of
                     % the last month is higher than their respective strike prices.

i = 1;            % Counter for the for-loop


K=Initial_Price;        % K is the strike price.

for i=1:N
    if M(k,i)>K(i)
        H(k) = H(k)+1;
    end
end

P(k) = 0;           % This is the payoff of the Asian Digital Option. If there are
                % 5 or more stocks whose average price in the last month is
                % higher than their respective strike price then the payoff
                % is one cash unit, otherwise it is zero.




if H(k) > 4
        P(k) = 1;
else P(k) =0;

end

    Y=Y+P(k);
```

```
    expectedy=Y/k;


ExpectedPayoff=(exp(-mu*timetomaturity(CurrentTime)))*expectedy;


end

plot(Price(:,:,1));
xlabel('Days')
ylabel('Price')
legend('Stock 1','Stock 2','Stock 3','Stock 4','Stock 5','Stock 6','Stock 7','Stock
Title('Price Dynamics of one Simulation Stocks')

plot(H)
xlabel('Iterations')
ylabel('Number of Stocks above Strike')
```

6.2 Algorithm Flow-chart

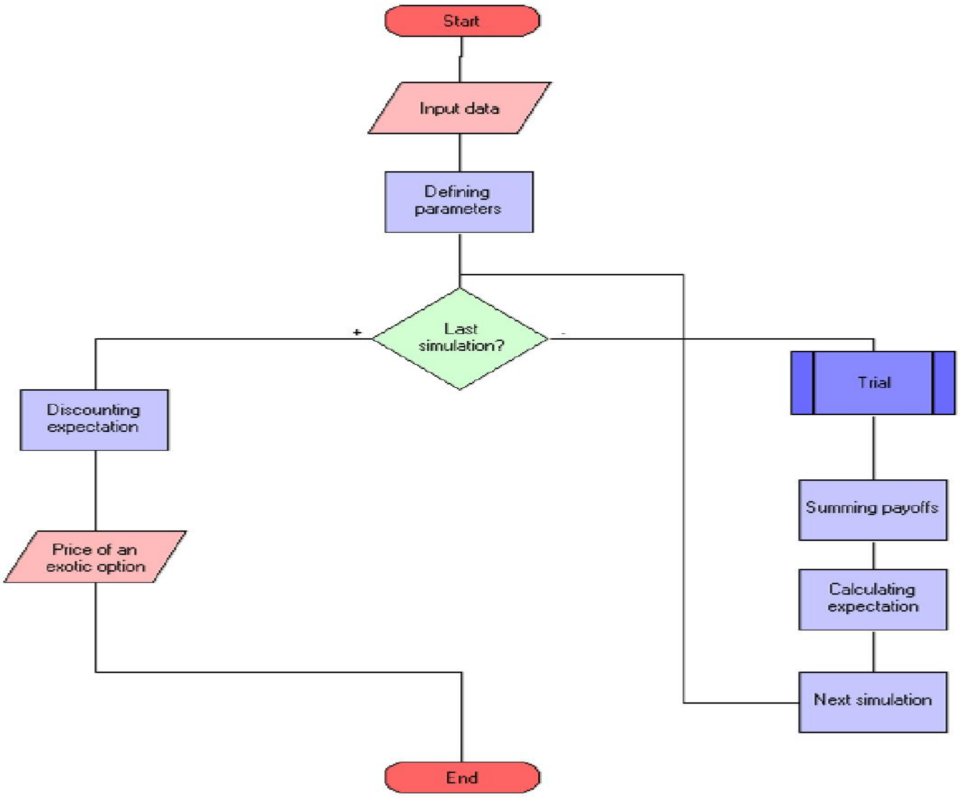

Figure 6.1: Flow-chart Averaging Process

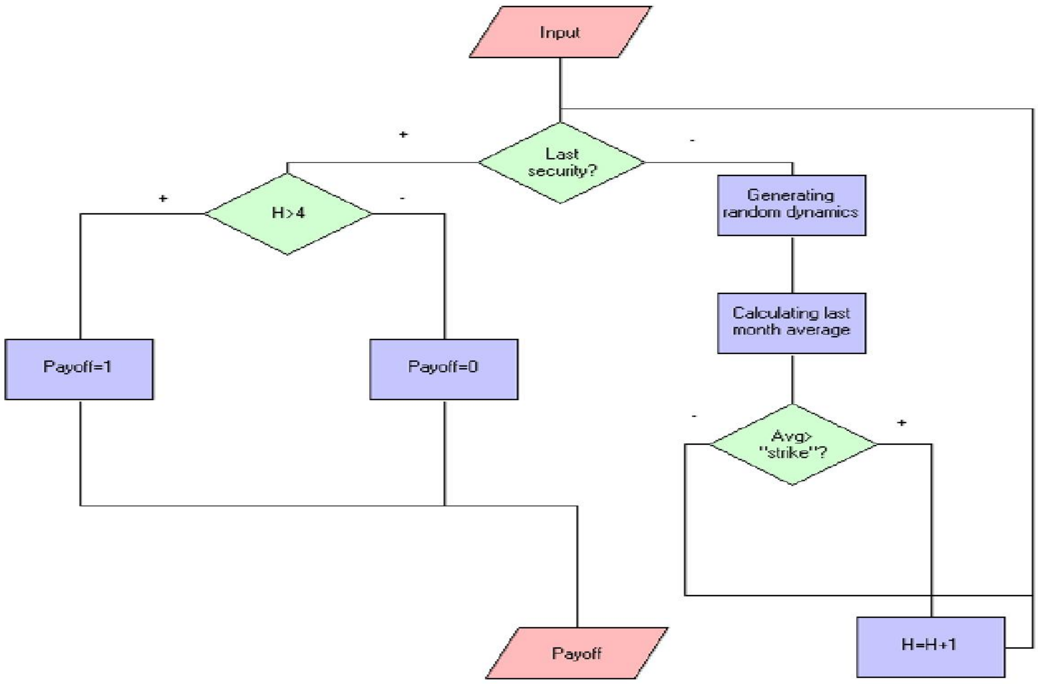

Figure 6.2: Flow-chart Price-path generation

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