Binomial Model

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Binomial model

The binomial model was first proposed by Cox, Ross and Rubinstein in 1979. In finance, binomial model is generally used for valuation options by numerical method. It is flexible, intuitive and popular approach to option pricing. Binomial model is base on random walks theory. The concept is that, over a single period of time (very short duration) the underlying asset can only move from its current price to two possible levels. It assumes that movement of the price of underlying asset follow a binomial distribution; for many trials (increase node of calculation), this binomial distribution approach the normal distribution assumed by Black-Scholes-Merton Model.

In this report, we will focus on how the accuracy of binomial model can be better. In order to compare the accuracy of each method which we apply to binomial model, we made the application which build binomial tree and calculate the option price to make a graph between the option price and number of steps of calculation. We can compare this value with Black Scholes model. So, it is able to show how many steps we use to get the same accuracy as Black Scholes model. Before showing the result of this application, we will introduce some important theory used in this application

Theory

We will introduce only theory used in our application. It is left to the reader to search more detail, if necessary.

Consider the financial market which consists of bond and stocks.

Deterministic bond with process: $\begin{cases} B(0) = 1 \\ B(1) = 1 + r \end{cases}$

Stock with stochastic process: $\begin{cases} S(0) = s \\ S(1) = \begin{cases} su \\ sd \end{cases} \quad prob \begin{matrix} p_u \\ p_d \end{matrix} \end{cases}$

Denote the true probabilities as $P = (P_u, P_d)$ and risk-free probabilities as $Q = (Q_u, Q_d)$

With probability measure Q $\begin{cases} Q(Z = u) = q_u \\ Q(Z = d) = q_d \end{cases}$

We can conclude that $S(0) = \frac{1}{1+r} E^Q[S(1)]$

If we use continuous compound interest rate, we get the following general pricing formula

$S(0) = e^{-rt}E^{Q}[S(1)]$

We will apply this formula to binomial model

1. The Cox-Ross-Rubenstein model

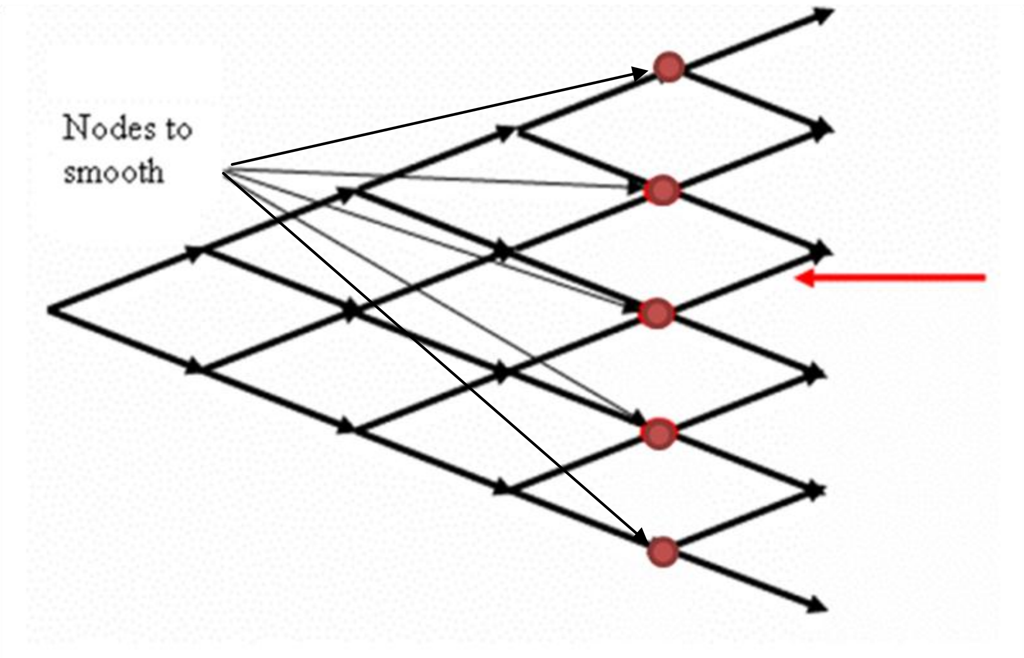
$$u = e^{\sigma \cdot \sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma \cdot \sqrt{\Delta t}}$$

$$p = \frac{e^{r \cdot \Delta t} - d}{u - d}$$

2. Black-Scholes smoothing

   We use CRR method to calculate price of options but we use black scholes value for the node closest to strike price. Actually, we can use Black-Scholes value only three node close to strike price but in order to get less oscillation, we apply Black-Scholes value to all node closest to strike price.



3. Richardson Extrapolation

   We can get less oscillation from the model by applying Richardson Extrapolation with Black Scholes Smoothing method.

   Suppose we have a numerical method with a known error of order k

$$A = A(h) + a_0h^{k0} + O(h^{k1})$$

Using the step sizes h and h / t for some t, the two formulas for A are:

$$A = A(h) + a_0h^{k_0} + O(h^{k_1})$$

$$A = A\left(\frac{h}{t}\right) + a_0\left(\frac{h}{t}\right)^{k_0} + O(h^{k_1}).$$

Multiplying the second equation by t^{k0} and subtracting the first equation gives

$$(t^{k_0} - 1)A = t^{k_0}A\left(\frac{h}{t}\right) - A(h) + O(h^{k_1})$$

We can be solved for A to give

$$A = \frac{t^{k_0}A\left(\frac{h}{t}\right) - A(h)}{t^{k_0} - 1} + O(h^{k_1}).$$

In that way, we have increased the accuracy from order $O(h_k)$ to $O(h_{k+1})$. Typically we have $h_{k1} = h$ and $h_{k2} = h/2$:

$$A = \frac{2^p A(h/2) - A(h)}{2^p - 1}$$

4. The Tian model

The Tian model use the second order moments for normal distribution.

$$u = \frac{M \cdot V}{2}\left(V + 1 + \sqrt{V^2 + 2V - 3}\right)$$
$$d = \frac{M \cdot V}{2}\left(V + 1 - \sqrt{V^2 + 2V - 3}\right)$$
where
$$M = e^{r \cdot \Delta t}$$
$$V = e^{\sigma^2 \cdot \Delta t}$$
$$p = \frac{e^{r \cdot \Delta t} - d}{u - d}$$

The accuracy of Binomial model

To compare the accuracy of each method, we generate the data of option value and try to vary number of steps of calculation. After that, we use these data to make graph and compare with Black Scholes value. This procedure show us the number of steps we need to get the result close to Black Scholes value.

For example: European Call option

Current Stock price :	$S_0 = 50$
Volatility :	$\sigma = 40\%$
Risk free interest rate:	r = 5%
Strike price :	k = 55
Time to maturity	T = 1 year

We use our application to calculate option price given in picture below

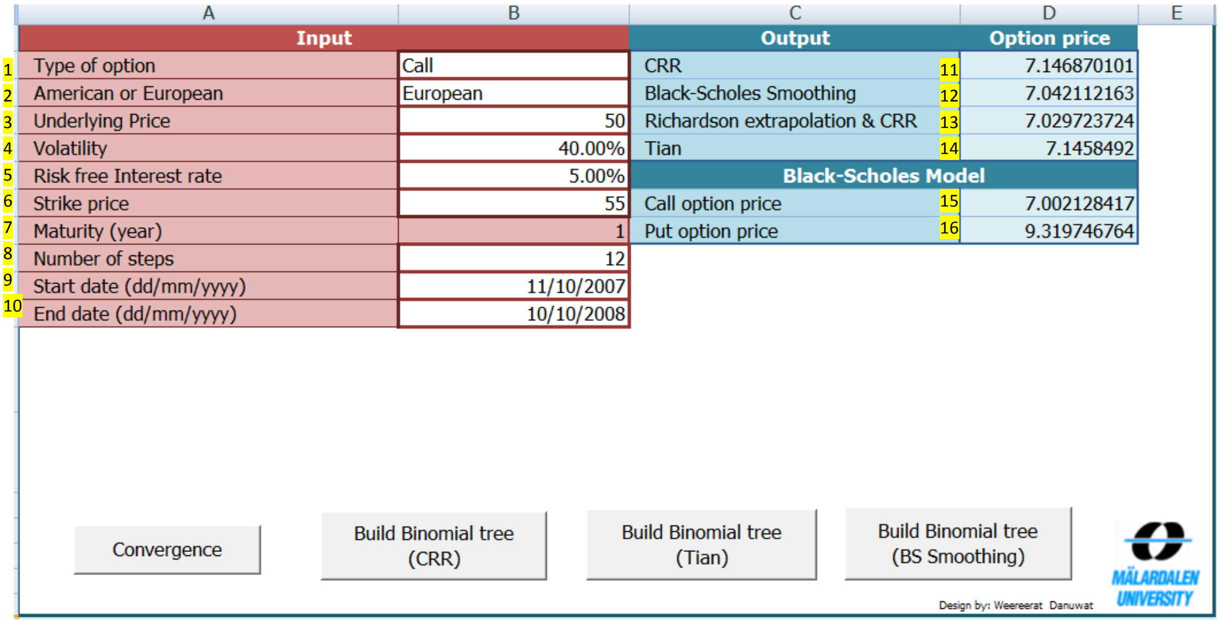


Figure 1: The Binomial tree application interface

1. Type of Option

   You can input call or put option. In the example, this is call option.

2. American or European

   Choose between American or European option. In the example, we input European option.

3. Underlying Price

   Input current underlying price.

4. Volatility

   Input Volatility of underlying asset (%).

5. Risk free interest rate

   Input risk free interest rate (%) to use for discount in the calculation.

6. Strike price

   Strike price at time to maturity

7. Maturity

   It is time to maturity in unit years. This cell is calculated from ‘End date – Start date

8. Number of Steps

   Choose number of steps of binomial tree

9. Start date

   Input date start date of calculation.

10. End date

    Input maturity date

11. CRR

    This cell shows the result of calculation using the Cox-Ross-Rubenstein model.

12. Black schools smoothing

    This cell shows the result of calculation using Black scholes smoothing model.

13. Richardson extrapolation

This cell shows the result of calculation using Richardson extrapolation method with Black scholes smoothing model

14. Tian

This cell shows the result of calculation using the Tian model

15. Call option price

This cell shows calculation option price for call option using Black scholes model. It returns 0 when using with an American option.

16. Put option price

This cell shows calculation option price for put option using Black scholes model. It returns 0 when using with an American option.

We can also build binomial tree by this application to see the value of option and underlying price at each nodes.

To compare the accuracy of each method, we use convergence function in the application. When we click at convergence button, it will calculate the option price with different methods of calculation and number of steps. Then, the result is shown in graph as the picture below

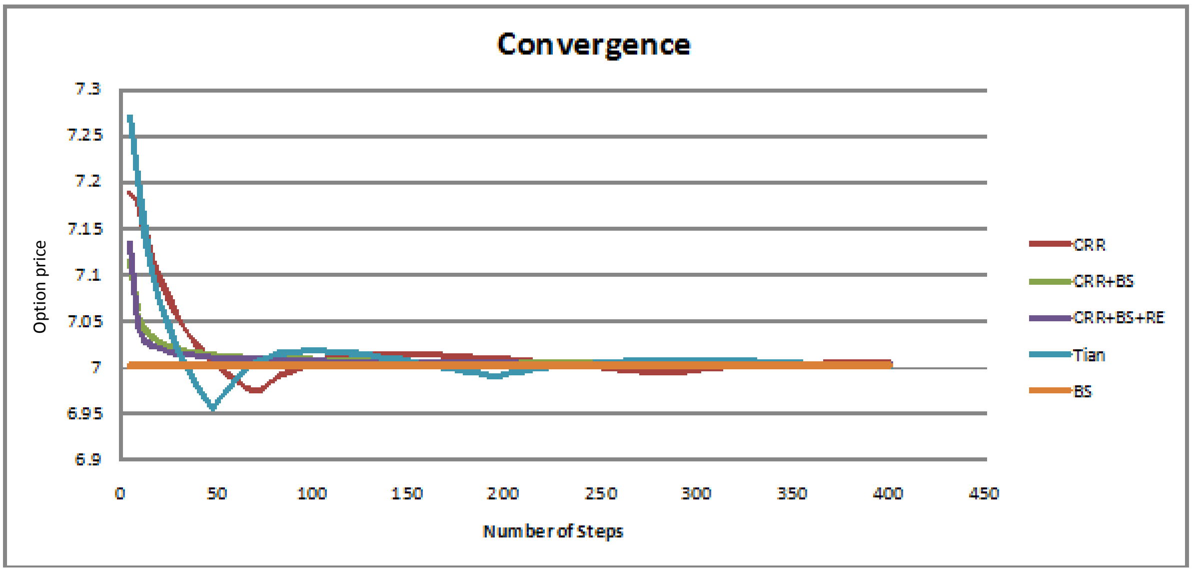


Figure 2: The convergence of each method using data from the example

From the picture, It implies that Richardson extrapolation method (purple line) will convert to Black scholes value faster than any other method. The convergence of this method is close to Black scholes smoothing because we apply Richardson extrapolation with Black scholes smoothing. Tian model has oscillation very huge but it converts faster than the Cox-Ross-Rubenstein model. The Cox-Ross-Rubenstein model converts slowly and has too much oscillation. In this example, we need at least 250 steps of calculation to get the result from the Cox-Ross-Rubenstein model close to Black scholes value.

Reference

Jan R. M. Röman. Lecture notes in Analytical Finance I. Mälardalen University, 2007.
Wikipedia, Richardson Extrapolation. Wikipedia, 2007.