DEPARTMENT OF MATHEMATICS AND PHYSICS
MÄLARDALEN UNIVERSITY
SE-72123 VÄSTERÅS, SWEDEN

ANALYTICAL FINANCE 1

Using the Hopscoth method to solve the Black-Scholes PDE

Authors:
Mahsima Ranjbar
Sophia Abdi Hassan
Michail Kalavrezos

Tutor: Jan R. M Röman

In this paper we will use the Hopscotch method to solve the Black and Scholes PDE in excel to obtain option prices.
The first spreadsheet will contain the input and output data.
Type the input parameters listed below in the spreadsheet.

- Initial stock Price
- Strike price
- Maturity
- Volatility
- Interest Rate

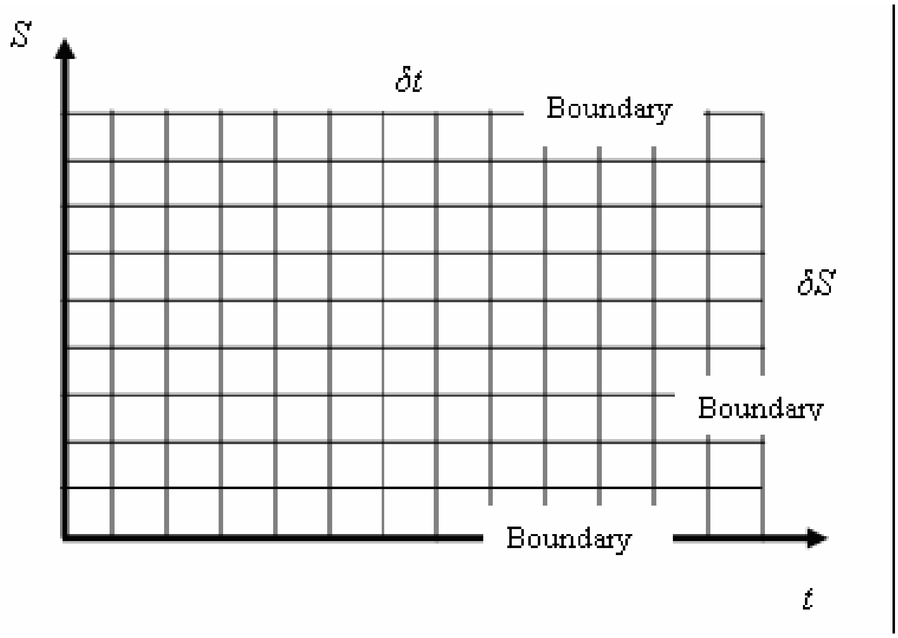
The following spreatsheets will contain calculations for the stock prices at different time intervals using the following formulas:
The dt is calculated by dividing the maturity by the time step chosen.
Movement up and Movement down

$$u = e^{\sigma\sqrt{dt}} \ldots\ldots\ldots\ldots\ldots 1$$
$$d = e^{-\sigma\sqrt{dt}} \ldots\ldots\ldots\ldots\ldots 2$$

Maximum and Minimum stock price are caculated by using the following formulas.

$$S_{\max} = S_0.u^n = S_0 e^{n.\sigma\sqrt{dt}} \ldots\ldots\ldots\ldots\ldots..3$$
$$S_{\min} = S_0.d^n = S_0 e^{-n.\sigma\sqrt{dt}} \ldots\ldots\ldots\ldots\ldots 4$$

Using the intial stock price and the up and down factors we create a binomial tree which gives the maximum and minimum stock prices. Thereby, creating a grid with boundaries that we use in the Hopscotch method to solve the Black-Scholes partial differential equation.



The hopscoth method combines both the explicit and implicit finite difference methods by calculating the explicit nodes for one time and then using the values obtained to calculate the implicit nodes. Each node after the known option price is calculated starting by explicit method and finishing by explicit, losing the highest and the lowest nodes each time.

Finite difference methods

We consider a parabolic boundary value problem of the Black and Scholes type

$$-\frac{\partial C}{\partial t} = \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 C}{\partial S^2} + (r-\delta) S \frac{\partial C}{\partial S} - rC$$

We let $x = \ln(s)$……………….5

$$\frac{\partial C}{\partial S} = \frac{\partial C}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{S}\frac{\partial C}{\partial x}$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}(\frac{1}{S}\frac{\partial C}{\partial x}) = -\frac{1}{S^2}\frac{\partial C}{\partial x} + \frac{1}{S}\frac{\partial}{\partial x}\frac{\partial C}{\partial S} = -\frac{1}{S^2}\frac{\partial C}{\partial x} + \frac{1}{S}\frac{\partial}{\partial x}(\frac{1}{S}\frac{\partial C}{\partial x})$$

$$= -\frac{1}{S^2}\frac{\partial C}{\partial x} + \frac{1}{S^2}\frac{\partial^2 C}{\partial x^2}$$

$$-\frac{\partial C}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2} - \frac{1}{2}\sigma^2 \frac{\partial C}{\partial x} + (r-\delta)\frac{\partial C}{\partial x} - rC$$

$$-\frac{\partial C}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2} - v\frac{\partial C}{\partial x} - rC$$

where $v = r - \delta - \frac{1}{2}\sigma^2$ …………….6

In the explicit finite difference method, approximations of the derivatives are inserted into the Black-Scholes PDE and rearranged as follows

$$C_{i,j} = \frac{1}{1 + r.\Delta t}(p_u.C_{i+1,j+1} + p_m.C_{i+1,j} + p_d.C_{i+1,j-1})$$

where

$$p_u = \frac{1}{2}.\Delta t.\left(\frac{\sigma^2}{\Delta x^2} + \frac{v}{\Delta x}\right) \text{……………….7}$$

$$p_m = 1 - \Delta t.\left(\frac{\sigma^2}{\Delta x^2}\right) \text{…………………….8}$$

$$p_d = \frac{1}{2}.\Delta t.\left(\frac{\sigma^2}{\Delta x^2} - \frac{v}{\Delta x}\right) \text{……………….9}$$

with the boundary conditions:

$$C_{i,N_j} - C_{i,N_{j-1}} = \lambda_U$$

$$C_{i,-N_{j+1}} - C_{i,-N_j} = \lambda_L$$

where the boundary conditions for a call option are as follows:

$$\lambda_U = S_{i,N_j} - S_{i,N_{j-1}}$$

$$\lambda_L = 0$$

The Δx is calculated using

$$\Delta x \geq \sigma\sqrt{3.\Delta t} \quad \text{......................10}$$

The ΔS is calculated using

$$\Delta S = \frac{S_2\alpha - S_2}{e^{\Delta x}} \quad \text{.....................11}$$

Since we know the value at the boundary where the option expires, we perform the calculation backwards in time until the valuation date.

In the implicit finite difference method, approximations of the derivatives are inserted into the Black-Scholes PDE and rearranged as follows

$$C_{i+1,j} = p_u.C_{i,j+1} + p_m.C_{i,j} + p_d.C_{i,j-1}$$

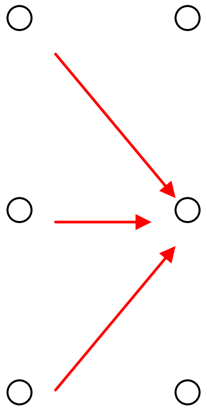
where

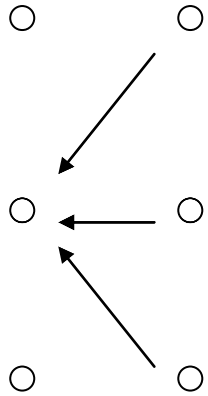
$$p_u = \frac{1}{2}.\Delta t.\left(\frac{\sigma^2}{\Delta x^2} + \frac{v}{\Delta x}\right)$$

$$p_m = 1 - \Delta t.\frac{\sigma^2}{\Delta x^2} + r.\Delta t \quad \text{.....................12}$$

$$p_d = \frac{1}{2}.\Delta t.\left(\frac{\sigma^2}{\Delta x^2} - \frac{v}{\Delta x}\right)$$

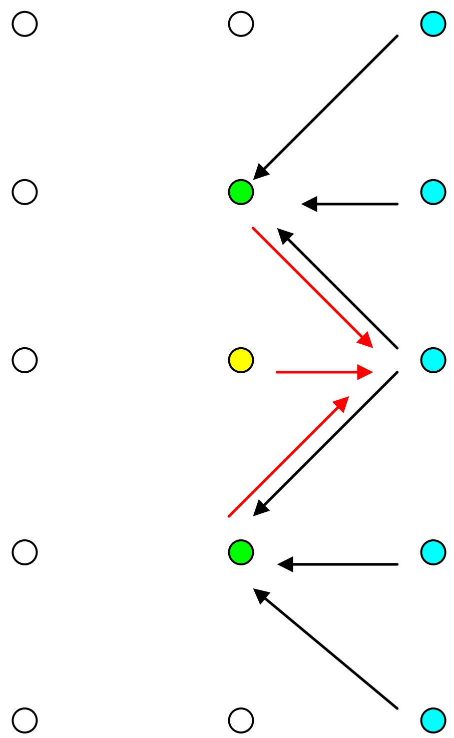
In the implicit method, we know the value of the option in the future and don't know the history of the option, the future depends on the past..





In our model: we know the option price at maturity we use the explicit method to calculate the price for the option price one step before maturity then we use the implicit method to solve for the yellow node below

$$C_{i,j} = \frac{C_{i+1,j} - p_d.C_{i,j-1} - p_u.C_{i,j+1}}{p_m} \quad \text{....................13}$$



Implicit finite method

Explicit finite method

Option price at maturity

Option price at one time step before maturity using explicit method

Option price at one time step before maturity using implicit method