Ho-Lee binominal trees. Analytical implementation of Ho–Lee model for the short interest rate.

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Ho & Lee model and binominal trees

The Ho & Lee model for short interest rates was founded in 1986. It was an improvement of earlier pricing models since it assumes the whole term structure follows a random progress. Their creation was the first model free of arbitrage and although not very realistic its analytical uses are of interest. The disadvantage is the existence of negative interest rates with a positive probability. The model assumes that the interest rates are normally distributed.

The short rate, in continuous time, has the following dynamics:

$$dr_t = \theta_t \, dt + \sigma \, dW_t$$

Where θ is the drift and σ the volatility of the short rate. The short-term interest rates are assumed to be normally distributed and the reason for no-arbitrage comes from the time dependence of θ.

In our project we have built an application in Excel that valuates bond contracts using a Ho & Lee binomial tree. The inputs in discrete time are pure discount bond prices and the volatility of future, one-period, short rates. In our project we have valued contracts by assuming the case of constant volatility. The pure bond prices have been calculated from bond data given, using linear interpolation of initial data to find yields to price additional maturities. To further calculate necessary input data we have been using bootstrapping to reduce coupon bonds into pure discount bonds. The only input left thus a user-optional value for the volatility.

Similar binomial trees, or lattices, can through adjustments also be created for time-varying volatilities rather than constant volatility.

The corresponding short rate dynamics in discrete time are:

$\Delta r(t) = \theta(t)\Delta(t) + \sigma(t)\Delta z(t)$ for all $t \geq 0$, where z is norm distributed variable

If we expand the equation we find:

$$r(t) = r(0) + \sum_{j=1}^{t-1} \theta(j-1) + \sum_{j=1}^{t-1} \sigma(j-1)\Delta z(j-1)$$

The bond prices P in general form, with no-arbitrage conditions, will thus be stated as:

$$P(n) = E_0^{Q}[e^{-\sum_{j=0}^{n-1} r(j)}]$$

Calculating the bond price at t=2, we get:

$$P(2) = E^{Q}[e^{-(r(0)+r(1))}] = e^{-r(0)} E_0^{Q}[e^{-r(1)}] = e^{-r(0)} e^{-E^{Q}[r(1)]+1/2\sigma^2(r(1))}$$

$$\ln P(2) = -r(0) - E_0^{Q}[r(1] + 1/2\sigma^2(r(1))$$

$$E_0^{Q}[r(1)] = -\ln P(2) - r(0) + 1/2\sigma^2(r(1))$$

Since

$$\ln P(2) = -f(0) - f(1) = -r(0) - f(1)$$

we can then write

$$E_0^{Q}[r(1)] = -f(1) + 1/2\sigma^2(r(1))$$ recall: $\Delta r(t) = \theta(t)\Delta(t) + \sigma(t)\Delta z(t)$

thus,

$$E_0^{Q}[r(1)] = r(0) + \theta(0)$$

$$\theta(0) = f(1) - r(0) + 1/2\sigma^2(r(1))$$

As a conclusion we can see that the drift is dependent on two things. First of all, on the difference between the forward and the short rate. Secondly, there exist a positive drift that will create no-arbitrage.

Further calculations are necessary to reach a simplified calculative structure for the binomial tree. See excel sheet. The formula for the short rates used in the binomial tree will finally have a very simple structure. The probabilities of going up or going down are both ½.

$$r(t) = \begin{matrix} r(t-\Delta t) + \theta(t-\Delta\theta)\Delta t + \sigma(t-\Delta t)\sqrt{\Delta t} \\ r(t-\Delta t) + \theta(t-\Delta\theta)\Delta t - \sigma(t-\Delta t)\sqrt{\Delta t} \end{matrix}$$

Short Interest Rate Lattice for Constant Volatility: