Hull-White

Trinomial Trees

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STAGE 1. Building the Tree

The Hull-white model is

$$dr = \left[\theta(t) - ar\right]dt + \sigma dz \qquad (1)$$

r ---instantaneous short rate
$\theta(t)$ ------some function of t
a and σ are constants

When consider the Δt rate R, the model can be written as:

$$dR = \left[\theta(t) - aR\right]dt + \sigma dz \qquad (2)$$

Then construct a tree for a variable R^* that is initially zero and followed the process:

$$dR^* = -aR^* dt + \sigma dz \qquad (3)$$

When $R^* = 0$, the variable $R^*(t+\Delta t) - R^*(t)$ is normally distributed, and

$$E\left[R^*(t+\Delta t)-R^*(t)\right]=-aR^*(t)\Delta t \quad (4)$$

$$Var\left[R^*(t+\Delta t)-R^*(t)\right]=\sigma^2\Delta t \quad (5)$$

We define ΔR as the spacing between interest rates on the tree and set

$$\Delta R=\sigma\sqrt{3\Delta t} \quad (6)$$

And we are going to build a tree as shown in figure 1

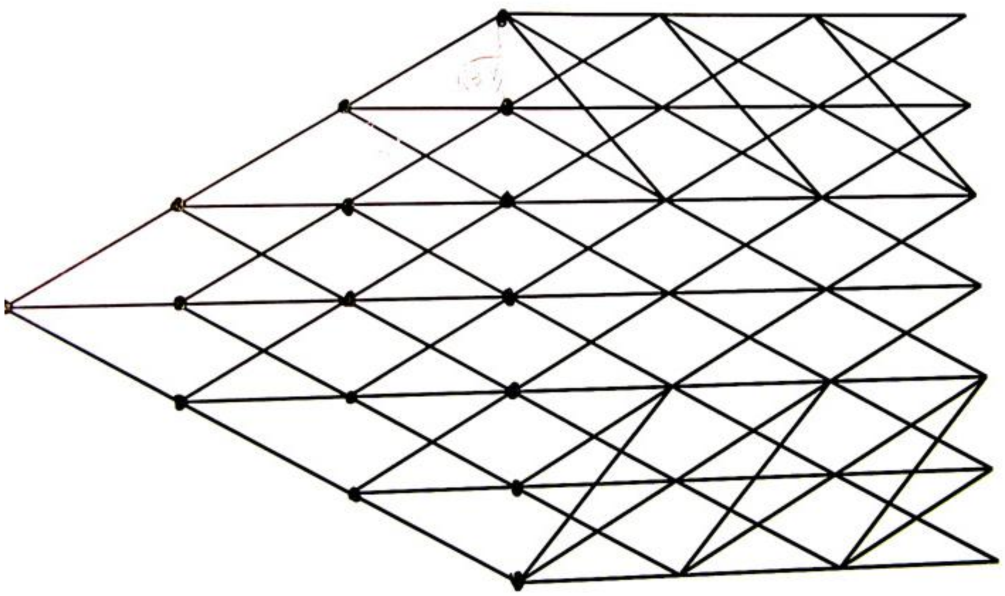


Figure 1

Then we will face three branching situations as shown in figure 2:

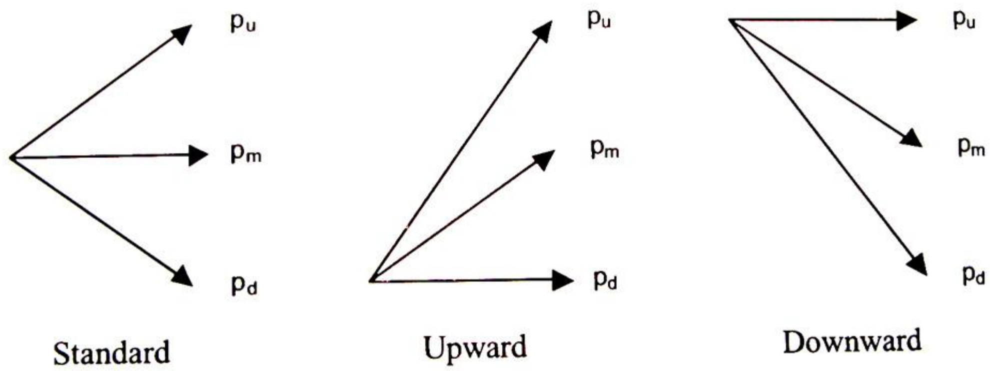


Figure 2

In order to calculate these three kinds of nodes, we need some formula.

First, define (i, j) as the node where $t = i\Delta t$ and $R^* = j\Delta R$. Hull and White show that the probabilities are always positive if we set $j_{\max}$ equal to the smallest integer greater than $0.184/(a\Delta t)$.

Second, define the transition probabilities as p_u, p_m, p_d which are used to match the expected value and variance of $R^*(t+\Delta t) - R^*(t)$ over time interval

Δt .through formula (4),(5),(6), p_{u}, p_{m}, p_{d} at node (i, j) for the standard branching must satisfy:

$$
\begin{aligned}
&p_u \Delta R - p_d \Delta R = -aj\Delta R\Delta t \\
&p_u \Delta R^2 + p_d \Delta R^2 = \sigma^2 \Delta t + a^2 j^2 \Delta R^2 \Delta t^2 \\
&p_u + p_m + p_d = 1
\end{aligned}
$$

Using $\Delta R = \sigma\sqrt{3\Delta t}$ to solve this equation system, we can get:

$$
\begin{aligned}
&P_u = 1/6 + \frac{a^2 j^2 \Delta t^2 - aj\Delta t}{2} \\
&P_m = 2/3 - a^2 j^2 \Delta t^2 \\
&P_d = 1/6 + \frac{a^2 j^2 \Delta t^2 + aj\Delta t}{2}
\end{aligned}
\tag{7}
$$

Similarly, we can get the probabilities for the up-ward branching:

$$
\begin{aligned}
&P_u = 1/6 + \frac{a^2 j^2 \Delta t^2 + aj\Delta t}{2} \\
&P_m = -1/3 - a^2 j^2 \Delta t^2 - 2aj\Delta t \\
&P_d = 7/6 + \frac{a^2 j^2 \Delta t^2 + 3aj\Delta t}{2}
\end{aligned}
$$

(8)

And the down-ward branching:

$$P_u = 7/6 + \frac{a^2 j^2 \Delta t^2 - 3aj\Delta t}{2}$$

$$P_m = -1/3 - a^2 j^2 \Delta t^2 + 2aj\Delta t$$

$$P_d = 1/6 + \frac{a^2 j^2 \Delta t^2 - aj\Delta t}{2} \quad (9)$$

STAGE 2 . Fitting the Tree

The objective of this stage is to convert the tree for R^* into a tree for R as shown in figure 3.

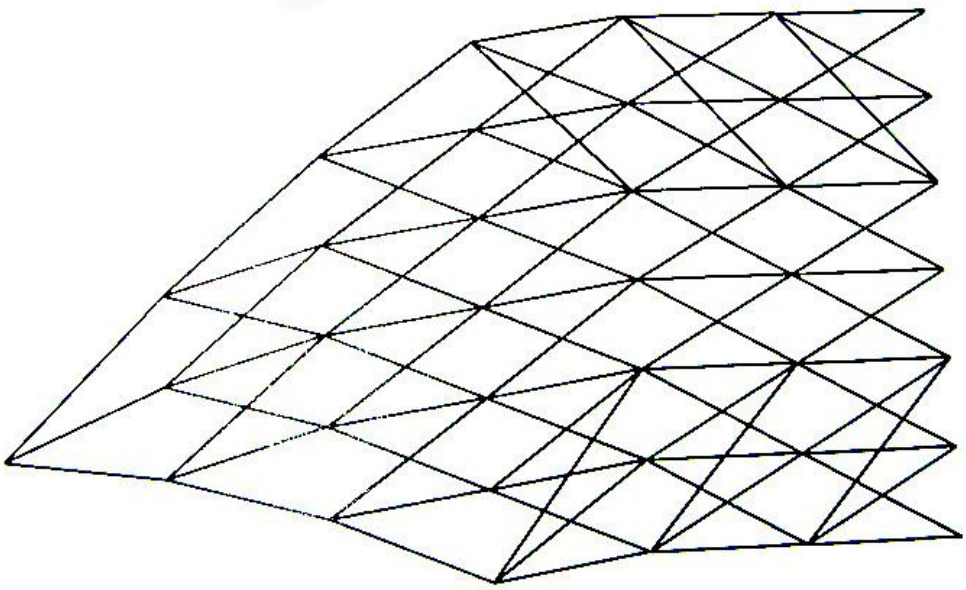


Figure 3

Define

$$\alpha(t) = R(t) - R^*(t) \tag{10}$$

$$d\alpha = \left[\theta(t) - a\alpha(t)\right]dt \text{ (From (2) and (3))} \tag{11}$$

$$\alpha(t) = F(0,t) + \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2 \tag{12}$$

Equation (12) can be used to create a tree for R from the corresponding tree for R^*. The approach is to set the interest rate on the R-tree at time $i\Delta t$ to be equal to the corresponding interest rates on the R^*-tree plus the value of α at time $i\Delta t$ to keep the probabilities the same. But it's not exactly consistent with the initial term structure. So we can calculate α iteratively to match the initial term structure exactly.

To illustrate this approach, we define

α_i =R at time $i\Delta t$ on the R-tree- R^* at time $i\Delta t$ on the R^*-tree.

α_0 =the right price for a zero-coupon bond maturing at time Δt which is equal to the initial Δt period interest rate.

$Q_{i,j}$ =the present value of a security that pays off $1 if node (i, j) is reached and 0 otherwise.

Suppose $Q_{i,j}$ have been determined for $i \leq m(m \geq 0)$ .The next step is to determine α_m so that the tree correctly prices a zero-coupon bond maturing at $(m+1)\Delta t$ .The interest rate at node (m, j) is $\alpha_m + j\Delta R$,so that the price of a zero-coupon bond maturing at time (m+1) Δt is given by

$$p_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j} \exp\left[-\left(\alpha_m + j\Delta R\right)\Delta t\right] \qquad (13)$$

Where n_m is the number of nodes on each side of the central node at time $m\Delta t$. The solution to this equation is

$$\alpha_m = \frac{\ln \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta R \Delta t} - \ln P_{m+1}}{\Delta t} \qquad (14)$$

Once α_m has been determined, the $Q_{i,j}$ for $i = m+1$ can be calculated using:

$$Q_{m+1,j} = \sum_k Q_{m,k} q(k,j) \exp\left[-(\alpha_m + k\Delta R)\Delta t\right] \qquad (15)$$

Where $q(k,j)$ is the probability of moving from node (m,k) to node (m+1,j) and the summation is taken over all values of k for which this is nonzero.

Since the above formulas are little abstract, we can illustrate them with some direct examples.

Consider the following tree:

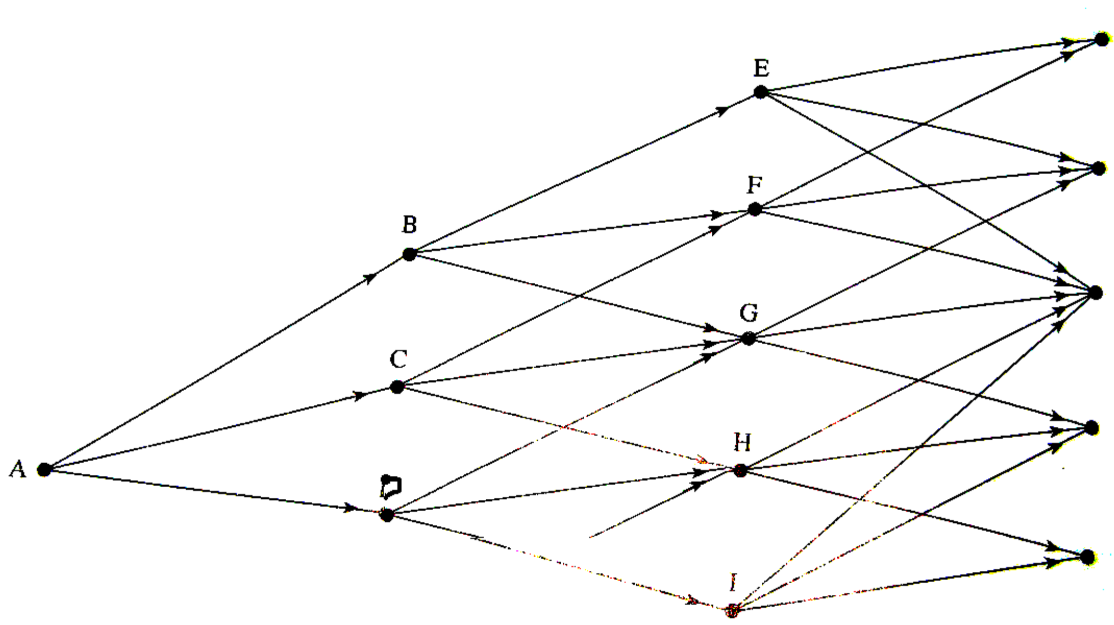


Table1 zero rate

maturity	Rate %
0.5	3.43
1	3.824
1.5	4.183
2	4.512
2.5	4.812
3	5.086

1. Specify Δt =1year, the initial rate is 3.824%. Then α_0 =3.824%. $Q_{0,0}=1$ and $\Delta R=\sigma\sqrt{3\Delta t}=0.01\cdot\sqrt{3}=0.01732$
2. Calculate the pond price of node B,C,D

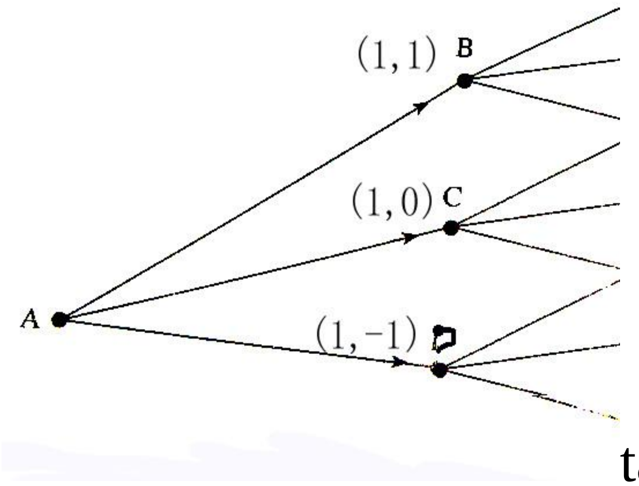


table 1

Node	A	B	C	D	E	F	G	H	I
R	3.824%								
Pu	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
Pm	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
Pd	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

$$Q_A = Q_{0,0} = 1$$

$$Q_B = Q_{1,1} = p_{Au}e^{-R_A\Delta t} = 0.1667e^{-0.03824\cdot 1} = 0.1604$$

$$Q_C = Q_{1,0} = p_{Am}e^{-R_A\Delta t} = 0.6666e^{-0.03824\cdot 1} = 0.6417$$

$$Q_D = Q_{1,-1} = p_{Ad}e^{-R_A\Delta t} = 0.1667e^{-0.03824\cdot 1} = 0.1604$$

3. Calculate α_1 which is chosen to give the right price for a zero-coupon bond maturing at time $2\Delta t$

Fitting the bond price to P

$$P_B = P_{1,1} = Q_B e^{-(\alpha_1+\Delta R)} = 0.1604e^{-(\alpha_1+0.01732)}$$

$$P_C = P_{1,0} = Q_B e^{-\alpha_1} = 0.6417e^{-\alpha_1}$$

$$P_D = P_{1,-1} = Q_B e^{-(\alpha_1-\Delta R)} = 0.1604e^{-(\alpha_1-0.01732)}$$

For the initial term structure, the bond price should be $e^{-0.0452\cdot 2} = 0.9137$

$$P_B + P_C + P_D = 0.9137$$

$$0.1604e^{-(\alpha_1+0.01732)} + 0.6417e^{-\alpha_1} + 0.1604e^{-(\alpha_1-0.01732)} = 0.9137$$

and α_1 can be solved:

$$\alpha_1 = \ln\left(\frac{0.1604e^{-0.01732} + 0.6417 + 0.1604e^{0.01732}}{0.9137}\right) = 0.05205$$

4. calculate the node rate for B,C,D

$$R_c = \alpha_1 = 0.05205$$

$$R_B = \alpha_1 + 1\cdot\Delta R = 0.05205 + 0.017$$

$$R_D = \alpha_1 + 1\cdot\Delta R = 0.05205 - 0.017$$

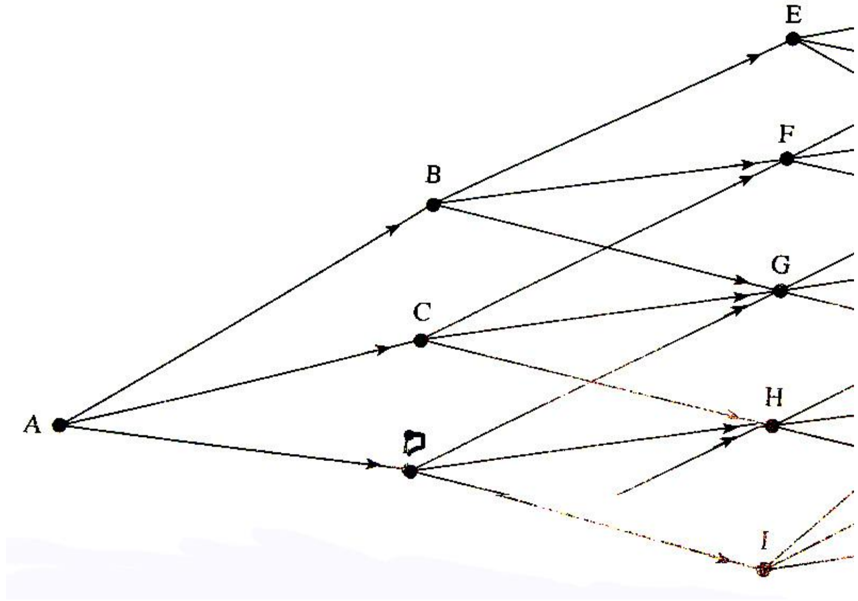
Fill in the table with the node rate above. The new table can be obtained:

Table 2

Node	A	B	C	D	E	F	G	H	I
R	3.824%	6.937%	5.025%	3.473%					
Pu	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
Pm	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
Pd	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

5. calculate the node E,F,G,H,I

for F, since it can be reached only from nodes B and C, then



$$Q_F = Q_{2,1} = p_{Bm}e^{-R_B}Q_B + p_{Cu}e^{-R_C}Q_C$$
$$= 0.6566e^{-0.06937} \cdot 0.1604 + 0.1667e^{-0.005205} \cdot 0.6407$$
$$= 0.1998$$

For G, since it can be reached from B,C,D,

$$Q_G = Q_{2,0} = p_{Bd}e^{-R_B}Q_B + p_{Cm}e^{-R_C}Q_C + p_{Du}e^{-R_D}Q_D$$
$$= 0.2217e^{-0.06937} \cdot 0.1604 + 0.6666e^{-0.05205} \cdot 0.6417 + 0.2217e^{-0.03473} \cdot 0.1604$$
$$= 0.4736$$

other nodes can be calculated in the same way.

We get:

$$Q_E = 0.0182$$
$$Q_H = 0.2033$$
$$Q_I = 0.0189$$

6. Fitting the bond price to P and calculate α_2

$$P_E = Q_E e^{-(\alpha_2+2\Delta R)}$$
$$P_F = Q_F e^{-(\alpha_2+\Delta R)}$$
$$P_G = Q_G e^{-\alpha_2}$$
$$R_H = Q_F e^{-(\alpha_2-\Delta R)}$$
$$P_I = Q_I e^{-(\alpha_2-2\Delta R)}$$
$$P_E + P_F + P_G + R_H + P_I = e^{-5.086 \cdot 3}$$

Solve these, and get α_2

And the node rate for E-F and be obtained.

Where

$$R_E = \alpha_2 + 2\Delta R$$
$$R_F = \alpha_2 + \Delta R$$
$$R_G = \alpha_2$$
$$R_I = \alpha_2 - \Delta R$$
$$R_H = \alpha_2 - 2\Delta R$$

7.continue calculation, find price and rate of each node use the formula(13),(14)and (15).