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MASTER THESIS IN MATHEMATICS /APPLIED MATHEMATICS

VALIDATING THE WILLOW TREE MODEL USING JAVA AND COMPARING THE RESULTS WITH OTHER MODELS FOR SWEDBANK

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ABSTRACT

One type of risk that affects pricing of exotic options is the model risk.

Our main aim is to validate the willow tree model by developing a java program for the basic willow tree model. The results are used to calculate the term structure of interest rate which is then compared with the term structure on the market. This can help the Swedbank to be able to have the appropriate rates to price instruments correctly using the willow tree model and to quantify the amount of model risk in the willow tree.

EXECUTIVE SUMMARY

In July of 2009, the Basel Committee on Banking Supervision issued a directive[1] requiring that financial institutions quantify model risk. The Committee further stated that two types of risks should be taken into account: “The model risk associated with using a possibly incorrect valuation, and the risk associated with using unobservable calibration parameters”. The resulting adjustments must impact Tier I regulatory capital, and the directive must be implemented by the end of 2010.

On the surface, this seems to be a simple adjustment to the market risk framework, adding model risk to other sources of risk that have already been identified within Basel II. In fact, quantifying model risk is much more complex because the source of risk (using an inadequate model) is much harder to characterize.

Therefore, the understanding of how a particular financial models work is a must, for all banks and financial institution. To compare with built in models and with the prices on the market, It is valuable to create own models to verify and validate those.

Basel Committee on Banking Supervision. Revisions to the Basel II market risk framework - final version. July 2009.

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1.0 Introduction

Financial modelling is building an abstract representation of a financial decision making situation. These mathematical models are designed to represent the performance of a financial assets or a portfolio. The price process of the underlying is given by the geometric Brownian motion

$$S(t) = S(0)\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma z(t)\right),$$

where $\{z(t)\ ,t \geq 0\}$ is a standard Brownian motion.

With this formula many more options can be developed and priced using financial models.

Beginning with the financial crash of 1987 and the recent in 2008 deviations of option prices from the Black-Scholes model have been more pronounced. The weakness of the Black-Scholes model is central to understanding why models have to be validated. Black-Scholes assumptions of constant volatility and lognormal stock process are inconsistent with observations in the financial markets. This led to the use of 'implied volatility smile'. The implied volatility is the volatility used in Black-Scholes model in order to obtain correct quoted option prices. The word 'smile' refers to the curvature of the volatility function. Stock returns show that they have high kurtosis (i.e. higher central peak and fatter tails) compared to the lognormal distribution assumed in the Black-Scholes model. Since it has fatter tails, this translates into higher volatility on either side. Black-Scholes suggest that the market is complete. In order words, any contingent claim admits a replicating portfolio and hence can be hedge perfectly. Suppose there is a crash in the market, it is evident that there is no chance to carry out a continuously changing delta hedge and hence the impossibility of perfect hedging. This implies that the market is incomplete hence every option cannot be replicated by self-financing portfolio.

In risk –neutral world, option prices are arbitrage free and can be computed as discounted expected payoffs with respect to some measure Q. The risk neutral measure Q is a probability measure with the property that all assets have the same expected rate of return which is the risk-free rate that is:

$$E_Q[S_T] = S_0 e^{rT}$$

Under such a measure Q, any portfolio θ will always have expected rate of return r, that is

$$E_Q[V_T(\theta)] = V_0(\theta)e^{rT}$$

If θ is a replicating portfolio for any option X, when $V_T(\theta) = X_T$, then also $V_0(\theta) = X_0$ or otherwise there is arbitrage. This is true by the law of one price. Therefore we have:

$$X_0 = V_0(\theta) = \ e^{-rT} E_Q[V_T(\theta)] = \ e^{-rT} E_Q[X_T(\theta)].$$

Financial models are made possible because of option replication. A model is arbitrage free if there is a martingale measure. This measure is obtained by calibrating to the market data in particular, to liquid vanilla options.

Now the market consists of liquid traded options –vanilla options and exotic or illiquid options. Vanilla options are options with special features or restrictions. Vanilla options have standard strike prices, standard expiration dates and standard terms. Exotic options have nonstandard features and are more complex than options that trade on the exchange. The exotic options are traded over the counter (OTC). The market prices for vanilla options are available and these prices are determined by supply and demand in the market.

However, exotic options prices are determined by pricing models. These models act as arbitrage free 'extrapolation' rules, extending the pricing system from market quoted vanilla options to no quoted exotic options, one has to estimate the unknown parameters of the model which reproduces ' as closely' as possible the option prices quoted in the market. This is known as model calibration. This paper seeks to address this.

The use of financial models in pricing assets should be correctly done in order to reduce the risk of wrong pricing which in turn causes lose of huge amount of money. Model verification and validation (*v&v*) are essential parts of the model development process if models are to be accepted and use to price assets correctly. Verification of model is done to ensure that the model is programmed well and does not contain errors, oversights, or bugs. It ensures that the specification is complete and that mistakes have not been made in the implementing the model.

In model validation we assume the model is not deterministic (i.e. has random elements). Validation ensures that the model meets its intended requirements in terms of the methods employed and the results obtained. The ultimate goal of the financial model validation is to make the model useful in the sense that the model gives the right price, and to make the model actually work. There is risk in pricing financial instruments .Some of these risk are systematic risk, unsystematic risk and the model risk. In recent times the banks and other financial institutions has taken interest in the model risk so that it can be quantify and reduced as a result of creating a good model which is consistent with the prices in the market. Some traditional models like the Black-Scholes, the Hull and White are explain in details with examples .In this project enough time has been invested in the willow Tree model creation and its interest rate validation. Moreover we also show how to calibrate the prices with the market prices to see that we have similar prices with the same volatility.

2.0 The concept of Calibration of financial models

The calibration of a mathematical model in finance is the determination of the risk neutral parameters that govern the evolution of a certain price process $S(t)$

The martingale hypothesis assumes that there exists a probability measure Q, equivalent to $\mathcal{P}$, such that our discounted price process $\{S(t) \,/\, B(t)\}$ is a martingale (here $B(t)$ is the evolution of a riskless savings account usually given us $B(t) = B(0)\, e^{rt}$). $\mathcal{P}$ is the historical or physical probability measure. We use statistical procedures to fit $\mathcal{P}$ to the data, this reflects past evolution of prices of the underlying. Q is the risk neutral probability measure this is also calibrated through prices of derivatives on the underlying.

The calibration of a model is performed observing the prices of certain derivatives written on the underlying $\{S(t)\}$, and fitting the parameters of the model in such a way that it reproduces the observed derivative prices. The purpose of calibration is to compute prices of not so liquid derivatives instruments, or more complex instruments.

3.0 Binomial models

The binomial model was first proposed by Cox, Ross and Rubinstein (CRR-1979). It is therefore called the CRR model in some literature. The binomial model assumes that movements in prices of instruments follow the Binomial distribution. The model uses discrete-time model of the varying prices over time of the underlying instrument. For this reason the model is widely used since it is able to handle a variety of conditions for which other models cannot easily be applied. As a consequence it is used to value American Options that are exercise at any given time. Another advantage of the Binomial model is that it is relatively simple model and it is readily implementable in computer software.

The binomial model is more accurate particularly for longer-dated Options on securities with dividend payments. This model is computationally slower than the Black-Scholes formula. Options with several sources of uncertainty and Options with complicated features for example Asian Options, the binomial model is not commonly used to value due to several difficulties. Some other models we will talk about later in this paper are used to value instruments of such nature.

3.1 Building the BDT Binomial Model

The model we will use is based on a Black-Derman-Toy (BDT) interest rate Binomial tree approach which adjusts for the cost of the embedded option and the difference between model price and market price due to other risks, for example credit and liquidity risks.

The Black-Dorman-Toy (BDT) model is a single-factor short-rate model matching the observed term structure of forward rate volatilities, as well as the term structure of the interest rate. A binomial tree is constructed for the short rate in such a way that the tree automatically returns the observed yield function and the volatility of different yields. The model is described by a stochastic differential equation where the rates are log-normally distributed. Therefore, the interest rates cannot be negative. To adjust the theoretical price on the binomial tree to the actual price, a spread called option-adjusted spread (OAS) is added to all short rates on the binomial tree such that the new model price after adding this spread makes the model price equal the market price .The value of option-adjusted spread is that it enables investors to directly compare fixed income instruments, which have similar characteristics, but traded at significantly different yields because of embedded options.

The OAS model has three dependent variables:

- Option Adjusted Spread
- Underlying Price
- Volatility

The stochastic process for the short rate in the Black-Derman-Toy (BDT) model is given by stochastic differential Equation (SDE):

$$d\ln(\mathrm{r}) = \{\theta(t) + \rho(t)\ln(r)\}dt + \sigma(t)dz$$

Where the drift of the short-term rate is $\theta(t)$, $Z(t)$ is a Brownian motion and $\rho(t)$ is the mean reversing term to an equilibrium short-term rate that depends on the interest rate local volatility as follows:

$$\rho(t) = \frac{d}{dt}\ln[\sigma(t)] = \frac{\sigma(t)}{\sigma(t)}$$

$$d\ln(\mathrm{r}) = \{\theta(t) + \frac{\sigma(t)}{\sigma(t)}\ln(r)\}dt + \sigma(t)dz$$

Since the volatility is time dependent, there are two independent functions of time, $\theta(t)$ and $\sigma(t)$, chosen so that the model fits the term structure of spot interest rates and the structure of the spot rate volatilities. Jamshidian (1991) shows that the level of the short rate at time *t* in the BDT model is given by:

$$r(t) = U(t)\exp\{\sigma(t)z(t)\}$$

Where $U(t)$ is the median of the lognormal distribution of r at time t, $\sigma(t)$ the level of the short rate volatility and $z(t)$ the level of the Brownian motion, a normal distributed Wiener process that captures the randomness of future changes in the short-term rate. The Black-Derman-Toy model is a lognormal model that is able to capture a realistic term structure of the interest rate volatilities. According to the principle of risk-neutral valuation the expected return from a stock at time T,

$$E(S_T) = S_0 e^{rT}$$

S_0 is the current stock price and r is the continuous compounding risk-free rate.

We construct a binomial tree whose pricing is given by the figure below:

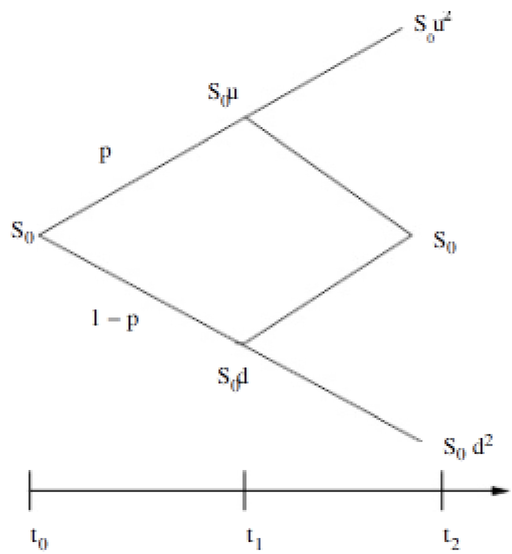


Figure 1.0: A simple binomial diagram

Restricting to only the single time step (i.e. t_0 and t_1). Initial price S_0has the option of moving to S_0u or S_0d at time t_1.
Let p be the probability of the price to rise to S_0u then the probability to move to S_0d is $1 - p$

Calculating the expected return from the stock at t_1 and making use of the risk-neutral valuation we have

$$E(S_{t_1}) = pS_0u + (1-p)S_0d = S_0e^{r(t_1-t_0)}$$

We get

$$p = \frac{e^{r(t_1-t_0)} - d}{u - d}$$

We choose appropriate values of u and d which can be obtained from equating the variance of the return to $\sigma^2\Delta t$.

The variance of the stock price return on the binomial tree is

$$pu^2 + (1-p)d^2 - [pu + (1-p)d]^2 = \sigma^2 \Delta t$$

Ignoring higher terms of Δt^2 and making use of $ud = 1$, we get:

$\Rightarrow u = e^{\sigma\sqrt{\Delta t}}$ And $d = e^{-\sigma\sqrt{\Delta t}}$

u and d are the ups and downs of the binomial tree.

Where $\sigma(t)$ is the volatility at time t. The risk-neutral probabilities of the binomial branches of this model are assumed equal to ½. (It by no means implies that the actual probability for an interest rate increase or decrease is equal to ½.) The tree uses the short-rate annual volatility of the benchmark rates which should be given in the Black-Scholes framework. The process can be illustrated using the following four short rates (all expressed with semi-annual compounding): $f_1, f_2\, f_3, f_4$

When the tree is built, the volatility spread factors, $Z_i's$ are kept constant and the tree is built with the following relationship between the nodes called the rates:

$$f_{i,j} = Z_i^{j-1}\, f_{i,1}$$

where $f_{1,1} = f_1$, so $f_{2,2} = Z_2 \,.\, f_{2,1}$ and $\frac{1}{2}f_{2,1} + \frac{1}{2}f_{2,2} = f_2$.

Therefore $f_{2,1} = \frac{2f_2}{1+Z_2}$

Also $f_{3,3} = Z_3^2 \,.\, f_{3,1}$ and $f_{3,2} = Z_3 \,.\, f_{3,1}$ so $\frac{1}{4}f_{3,1} + \frac{1}{2}f_{3,2} + \frac{1}{4}f_{3,3} = f_3$

Therefore $f_{3,1} = \frac{4.f_3}{1+2Z_3 + Z_3^2.}$ etc

Generally the rates a given by:

$$f_{n,1} \,.\, \sum_{i=0}^{n-1} \binom{n-1}{i} . Z_n^i = 2^{n-1} \,.\, f_n \Rightarrow f_{n,2}, \dots, f_{n,n}$$

3.2 Calibration of the BDT Binomial tree.

Before the tree we have built can be use it has to be calibrated with the market data. This process involves raising or lowering the estimates of the rates in the tree by a sufficient amount so that the value for the cash flows given by the tree exactly equals the values given by the discount function. The relationship between the different nodes $Z_i's$ must be maintained during the calibration process. The price at cash flow equal 1 is given by

$$\left(\frac{\frac{1}{2}}{1+f_{2,1}.(t_2-t_1)}+\frac{\frac{1}{2}}{1+Z_2.f_{2,1}.(t_2-t_1)}\right).\frac{1}{1+f_{2,1}.(t_1-t_0)}$$

And the price of the same cash flow equal 1is again given by $P(t_0,t_2)$, by the discount function $P(T, t)$ (with equal probabilities $\frac{1}{2}$) discounting from $t = t_2$ to $t = t_0$. Therefore the following must hold:

$$\left(\frac{\frac{1}{2}}{1+f_{2,1}.(t_2-t_1)}+\frac{\frac{1}{2}}{1+Z_2.f_{2,1}.(t_2-t_1)}\right).\frac{1}{1+f_{2,1}.(t_1-t_0)}=\mathrm{P}(t_0,t_2)$$

From the expression $f_{2,2} = Z_2 . f_{2,1}$, $f_{2,2}$ can be calculated if the value of $f_{2,1}$ is known. The nodes at time 2 are calibrated as follows:

$$\frac{1}{2}\left\{\left(\frac{\frac{1}{2}}{1+Z_3^2.f_{3,1}.(t_3-t_2)}+\frac{\frac{1}{2}}{1+Z_3.f_{3,1}.(t_3-t_2)}\right).\frac{1}{1+f_{2,2}.(t_2-t_1)}\right.$$
$$\left.+\left(\frac{\frac{1}{2}}{1+Z_3.f_{3,1}.(t_3-t_2)}\frac{\frac{1}{2}}{1+f_{2,1}.(t_3-t_2)}\right).\frac{1}{1+f_{1,2}.(t_2-t_1)}\right\}\frac{1}{1+f_{1,1}.(t_1-t_0)}$$
$$=\mathrm{P}(t_0,t_3)$$

These equations are solved numerically by a Van Winjgaarden-Decker-Brent method. When $f_{3,1}$ is known $f_{3,2}$ and $f_{3,3}$ can also be calculated. The rates in the calibrated tree are then compared with the rates from the un-calibrated and the necessary adjustment made.

3.3 Pricing Options in the BDT Binomial model

Options prices are evaluated from the end of the tree and working from backwards (i.e. backward induction). A put option is worth $\max[K-S(t),0]$ and a call option is worth $\max[S(t)-K,0]$ where $S(t)$ is the stock price at time T and K is the strike price. The risk- neutral probabilities and the discounting factors of the tree are given by $p=\frac{e^{\sigma\Delta t}-d}{u-d}$ and $e^{r\Delta t}$ from Hull (2003):

To illustrate this approach, let us consider an example:

> Consider a five-month American put option on a non-dividend-paying stock when the stock price S =\$50. The option is at the money, i.e., K = \$50, the risk-free interest rate is

10% per annum, and the volatility is 40% per annum. Suppose we divide the life of the option into five intervals of length one month:

(i.e. $\Delta \mathrm{t} = \frac{1}{12}$ year).

We can easily find that:

$u = e^{\sigma\sqrt{\Delta \mathrm{t}}}$ = 1.1224, $d = e^{-\sigma\sqrt{\Delta \mathrm{t}}}$ =0.8909

$e^{r\Delta \mathrm{t}}$ =1.0084 $p = \frac{e^{\sigma\Delta \mathrm{t}} - d}{u-d}$ =0.5076

1- P = 0.4924

Note that \$50 = 5000 cents, and the stock prices at each node are calculated by multiplying the previous stock price by *u* or *d*, for the upward movement and the downward movement of stocks, the prices of the option in the tree at each node are calculated as follows:

$$P = \max\left[\, (K - S(t),\ \frac{1}{e^{r\Delta \mathrm{t}}}\left(P(f_u) + 1 - P(f_d)\right)\right]$$

Where f_u, f_d are the up and down prices at each node respectively. For instance the price at time step *t* = 4 months with stock price *S*(*t*) = 5000cents is calculated as:

$$P = \max\left[\, (5000 - 5000),\ \frac{1}{1.0084}(0.5076(215) + 0.4924(695))\right]$$

Which gives *P* = 266 cents. We do the same for the rest of the nodes. The stock price is represented on the top of the node and the option price at the down of the node as shown in figure 1.1 below.

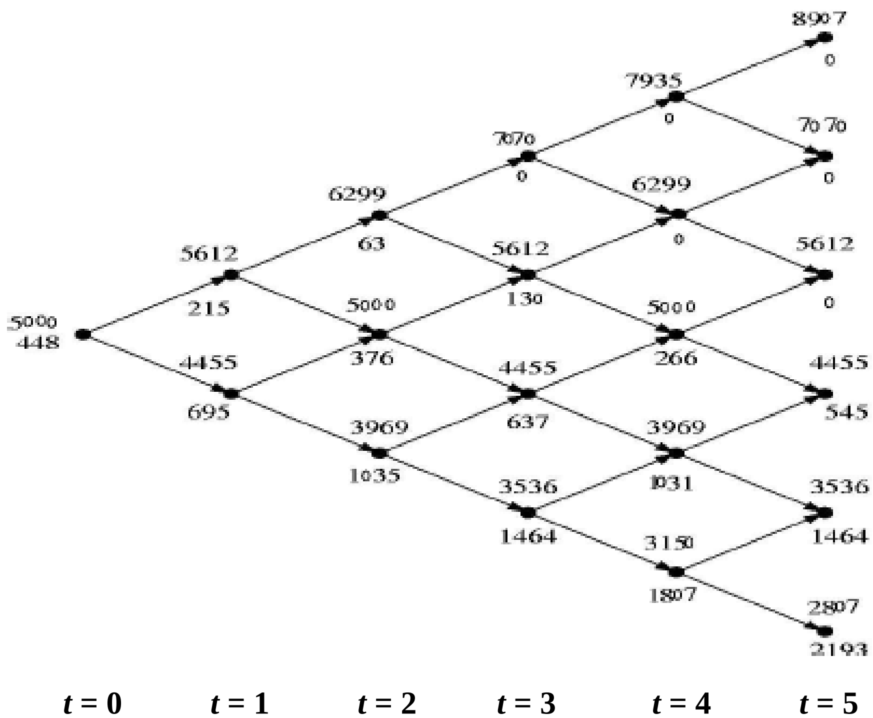


Figure 1.1: BDT Binomial tree for American put option in cents.

4.0 The Black –Scholes model.

The Black –Scholes model is a mathematical model of a financial market containing derivative of certain investment instruments. The model is widely used in the option market. Many empirical tests have shown that Black-Scholes price is close to the observe prices. This model was first proposed by Fischer Black and Myron Scholes in 1973.

The main idea was to perfectly hedge the option by buying and selling the underlying assets in the right way and also to reduce risk associated with it. This hedge is called delta hedging. From the Black-Scholes model (equation) we can deduce the Black scholes formula which gives the price of European –style options. There are number of assumptions in the Black- Scholes world:

- Options can be exercised only at expiration(European-style options)
- The stock price follows a geometric Brownian motion with constant drift and volatility.
- The underlying stocks do not pay dividends.
- There is no arbitrage opportunity(no profit without risk)
- It is possible to borrow and lend cash and also buy and sell any amount of stock.

- The transactions do not incur any cost or fees.
- Stock returns follow a lognormal distribution.
- Interest rates do not change in the life of the option (and are known).

4.1 Derivation of the Black-Scholes model (PDE)

We let:

S = the price of the stock

$V(S,t)$ = the price of the derivative as a function of time t and stock price *S*.

$C(S,t)$, $P(S,t)$are the prices of European call and put options respectively at time *t*.

r = the annualized risk- free interest rate.

μ = the drift rate of *S*

σ = the volatility of the stock returns

Π = the value of a portfolio.

From [1], and the assumptions given above it follows that price of the underlying assets follows a geometric Brownian motion. That is $\frac{dS}{S} = \mu dt + \sigma dz$

$$\Rightarrow dS = S\mu dt + S\sigma dz \quad \text{.......... (1)}$$

Where *W* is the Brownian motion and a simple random walk. Also

$$dS^2 = (S\mu dt + S\sigma dz)^2 = S^2\sigma^2 dt \quad \text{.......... (2)}$$

Since $(dt)^2$=0 and $d(z)^2 = dt$

The payoff of the option at maturity *T* is known to be V(*S*, *T*), to find it value at time *t*, we apply Itōs lemma to V(*S*, *t*) and we have:

$$dV = \frac{\partial V}{\partial S}.dS + \frac{\partial V}{\partial t}.dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}.dS \quad \text{..........(3)}$$

When we substitute equations (1) and (2) into (3) we have:

$$dV = \left(\frac{\partial V}{\partial S}S\mu + \frac{\partial V}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2}\right)dt + S\sigma\frac{\partial V}{\partial S}dz \quad \text{.......... (4)}$$

Now we consider a delta –hedging portfolio consisting of short one option and long $\frac{\partial V}{\partial S}$ shares at time t, the value of this portfolio is:

$\Pi = -V + \frac{\partial V}{\partial S} S$ over time period $[t, t + \Delta t]$ the total value of the portfolio becomes

$$\Delta\Pi = -\Delta V + \frac{\partial V}{\partial S}\Delta S \quad \text{(5)}$$

We now make equations (1) and (4) discrete by replacing differentials with deltas, we have:

$$\Delta S = S\mu\Delta t + S\sigma\Delta z$$

and

$$\Delta V = \left(\frac{\partial V}{\partial S}S\mu + \frac{\partial V}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2}\right)\Delta t + S\sigma\frac{\partial V}{\partial S}\Delta z$$

We then substitute these two discrete equations into (5) we have:

$$\Delta\Pi = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2}\right)\Delta t \quad \text{(6)}$$

We realize that Δz has been eliminated which means that there is no uncertainty in the portfolio which makes it effectively riskless in infinitesimal short period of time. Over a time period $[t, t+\Delta t]$ under the risk free rate r, the price process $\Delta\Pi$ becomes

$$\Delta\Pi = r\,\Pi\,\Delta t = r\left(-V + \frac{\partial V}{\partial S}S\right)\Delta t \quad \text{(7)}$$

Equating (6) and (7) we have:

$$-\left(\frac{\partial V}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2}\right)\Delta t = r\left(-V + \frac{\partial V}{\partial S}S\right)\Delta t$$

Simplifying and rearranging we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

This is the Black-Scholes partial differential equation.

4.2 The Black- Scholes formula

If $C(S,t)$ and $P(S,t)$ are the prices of European call and put options respectively at time t.

K is the strike price

T is the time to maturity

$N(x)$ is the cumulative distribution function of the standard normal distribution.

We use the risk-neutral valuation approach to calculate for the Black-Scholes formula. The value of expected value of call option in risk-neutral world is:

$$\hat{\mathrm{E}}[\max(S-K,0)]$$

Where Ê denotes the expected value in the risk-neutral world. The European call option price is calculated by discounting the expected value at risk-free rate of interest.

$$C(S,t) = e^{-r(\mathrm{T-t})}\hat{\mathrm{E}}[\max(S-K,0)]$$

From the assumption that Black-Scholes price process is lognormal

$$\Rightarrow \hat{\mathrm{E}}(S) = Se^{-r(\mathrm{T-t})}$$

And the standard deviation for

$$\ln S = \sigma\sqrt{T-t}$$

Therefore

$$C(S,t) = e^{-r(\mathrm{T-t})}\left[\,(Se^{-r(T-t)}N(d_1) - KN(d_2))\right]$$

or

$$C(S,t) = SN(d_1) - Ke^{-r(\mathrm{T-t})}N(d_2)$$

This is the Black-Scholes price for European call Option at time *t*.

From the put call parity formula

$$P(\mathrm{S,t}) = Ke^{-r(\mathrm{T-t})} - S + C(S,t)$$

We have:

$$P(S,t) = Ke^{-r(\mathrm{T-t})} - S + SN(d_1) - Ke^{-r(\mathrm{T-t})}N(d_2)$$

$$\Rightarrow P(S,t) = Ke^{-r(T-t)}\big(1 - N(d_2)\big) - S\big(1 - N(d_1)\big)$$

Therefore we have

$$P(S,t) = Ke^{-r(\mathrm{T-t})}N(-d_2) - SN(-d_1)$$

Which is the Black-Scholes price of the European put option at time *t*.

Where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

These equations are called the Black-Scholes formula which is used to value financial instrument mostly Options of European type. There are some limitations to this model, some of them being the assumption of cost-less trading, yielding liquidity risk which is difficult to hedge and the assumption of constant volatility, which in real world varies over time. Very short-term options can be valued using Black-Scholes formula because volatility can change so much in only a few days, invalidation of these assumptions in longer term in the real world makes the Black-Scholes formula not work for mid-term and long-term options. Therefore proper application of Black-Scholes model requires the understanding of these limitations. Also interest rate is not always constant, it varies by tenor which gives an interest rate curve which can be interpolated to pick appropriate interest rate to be used for Black- Scholes pricing. The Black-Scholes model was later improved to deal with some limitations of the real world. For example the Generalized AutoRegressive Conditional Heterokedasticity (GARCH) model replaces the constant volatility with stochastic volatility.

4.2.1 Pricing Options in the Black-Scholes model

The Black-Scholes formula is used to calculate the value of option. We can demonstrate the working of the Black-Scholes formula with this example:

> Let us assume that the current price of shares of company *ABC* is \$100 and you would like to get an option to purchase one share of *ABC* company stock for \$95. The option expires in three months. We also assume that the stock pays no dividends. The standard deviation of the stock return is 50% per year, and the risk-free rate is 10% per year, we can calculate the value of the option as follows:

$T = \frac{3}{12} = 0.25$year, $t = 0$, S = \$100, K = \$95, r =10% = 0.10, σ= 50% = 0.50

We have:

$$d_1 = \frac{\ln\left(\frac{100}{95}\right) + \left(0.10 + \frac{0.50^2}{2}\right)(0.25)}{0.50 * \sqrt{0.25}} = 0.43$$

$d_2 = 0.43 - 0.50 * \sqrt{0.25}$

N (0.43) = 0.6664
N (0.18) = 0.5714

Therefore the value of the call option is

$C(S, T)$ = 100* 0.6664-95*$e^{-(0.10*0.25)}$ *0.5714 = 66.64 - 52.94 = 13.70

Using the put-call parity formula the price of the put option is:

$P(S, T)$ = 13.70 + 95*$e^{-(0.10*0.25)}$ -100 = 6.35

5.0 The Short rate models

The models discussed above are widely used to value instruments such as Caps, European bond options and European swap options. One limitation of these models among others is that they do not provide a description of how interest rate r evolves through time. Due to this, the models mentioned above are not used to value interest rate derivatives such as American-style swap options, callable bonds, and structure notes. *Term structure models* are used to value derivatives of this nature which take into account the movement of interest rates in the market. One difference between movement of stock price S and interest rate r is that interest rates appear to be pulled back over long-run average range of time. This phenomenon is known as **mean reversion.** When the short rate r is high, mean reversion tends to cause it to have negative drift: when r is low, mean reversion tends to cause it to have positive drift. This can be explain economically that when interest rate r is high there is low demand for funds from borrowers as a result interest rate decline inversely when rates are low the demand for funds increases and consequently rates tend to rise. The Interest rate models we talk about in this paper take into account the mean reversion phenomenon.

6.0 Term structure of interest rates

The term structure of interest rate is a yield curve displaying the relationship between spot rates of zero-coupon securities (e.g. zero coupon bonds) and their term of maturity as shown in the figure below in figure 1.2.

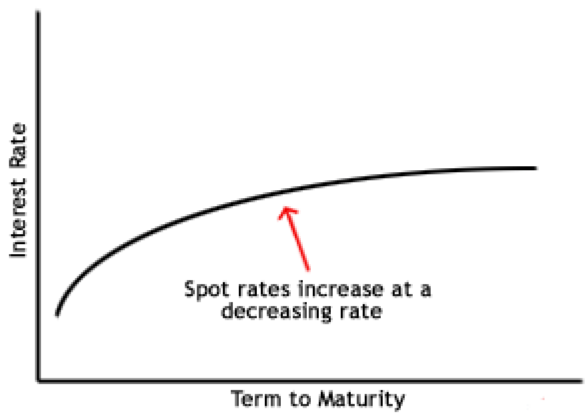


Figure 1.2 Term structure of Interest rates

This curve allows an interest rate pattern to be determined which can then be used to discount cash flows appropriately.

If the term structure $\{P(t,T): 0 \leq t \leq T, T > 0\}$ has the form

$$P(t,T) = F(r,t,T) = e^{A(t,T)-B(t,T)r}$$

Where *r* is dependent on time *t*. Then the model is said to possess an affine term structure (ATS). *A* and *B* are deterministic functions of two real variables *t* and *T*, *P* (*t*, *T*) is the price at time *t* of a zero-coupon bond that pays $1 at time *T*. The affine bond price can then be written as

$$P(t,T) = A(t,T)e^{-B(t,T)r} \quad \text{.......} (a)$$

If $R(t,T)$ is the continuously compounded interest rate at time t for a term of $T - t$ then

$$P(t,T) = e^{-R(t,T)(T-t)}$$

$$\Rightarrow R(t,T) = \frac{-1}{T-t} \ln P(t,T) \quad \text{.......(b)}$$

Substituting equation (a) into (b) we have

$$R(t,T) = \frac{-1}{T-t}\ln\left[A(t,T)e^{-B(t,T)r}\right]$$

Then,

$$R(t,T) = \frac{-1}{T-t}\ln A(t,T) + \frac{1}{T-t}B(t,T)r \text{ .... (c)}$$

Equation (c) is use to obtain the term structure of interest rates at any given time *t* from the values of *r* at that time.

This shows that the entire term structure can be written as a function of *r*

6.1 Equilibrium models of term structure

Equilibrium models derive process for the short rates r. These models also explain why the process for *r* affects bond prices and option prices. The process for *r* in one-factor equilibrium model involves only one sources of uncertainty. The risk neutral process for one factor equilibrium model is written in this form:

$$dr = m(r)dt + s(r)dz$$

Where *m* and *s* are instantaneous drift and volatility respectively, which are functions of r. A model is one-factor if all the rates move in the same direction over any short time interval. When the process for the short rates reverts to a long rate and follows a stochastic process, it defines two factor equilibrium model.

Equilibrium models do not automatically fit today's term structure of interest rates. Parameters have to be chosen cautiously so that they can fit into today's term structure of interest rates. Consequently traders have very little confidence in Equilibrium models since it does not really price the underlying bond correctly which may lead to a big error in the price of the bond option.

6.2 No-Arbitrage models of term structure

No-arbitrage models of term structure are popular among practitioners because they provide a description of the yield curve that is consistent with the prices of all actively traded bonds on a given date and can therefore be used for pricing less liquid bonds and fixed income derivatives on the same dates.

This model is designed in such a way that today's term structure of interest rates is an input and it is also consistent with the term structure. The drift part of these models (i.e. coefficient of dt)

usually is a function of time *t*. When time *t* is included in the drift of Equilibrium models they can be converted to no-arbitrage models.

7.0 The Vasicek Model

Vasicek model is a type of one-factor equilibrium short rate model, which has risk-neutral process for *r* as:

$$dr = a(b - r)dt + \sigma dz$$

Where a, b, and σ are constants. In this paper we will put $c = a * b$ for simplicity sake. Then the process can be written as:

$$dr = (c - ar)dt + \sigma dz \qquad (d)$$

The drift part $(c - ar)$ represents the expected instantaneous change in the interest rate at time *t*. The parameter a represent the "speed of reversion" and *c* represent the long run equilibrium value towards which the interest rate revert. In this model the short rate *r* is pulled to a level *c* at rate a.

7.1 Bond pricing using the Vasicek model

Assume that we have Q-dynamics

$$dr = \mu(t, r)dt + \sigma(t, r)dz$$

And assume that this process possess an affine term structure (ATS), then we observe that the drift μ and the variance σ^2 are both affine functions of *r* with time dependent coefficients then we can write

$$\mu(t, r) = a(t)r + b(t) \qquad (e)$$

$$\sigma^2 (\mathrm{t}, \mathrm{r}) = \mathrm{c}(\mathrm{t})\mathrm{r} + \mathrm{d}(\mathrm{t}) \qquad (\mathrm{f})$$

Comparing equation (*d*) with equations (*e*) and (*f*) we have:

a (t) = -*a*, *c* (t) = 0, *b* (*t*) = *b* and *d* (*t*) = σ^2

Considering $A(t, T)$ and $A(t, T)$ as a function of *t* we solve the ODE's:

$$A_t - \theta AB + \frac{1}{2}\sigma^2 AB^2 = 0 \text{ and } B_t - aB = -1$$

Subject to

$A(T,T) = 1$ and $B(T,T) = 0$

We get $\quad B(t,T) = \frac{1-e^{-a(T-t)}}{a}$

and $A(t,T) = exp\left[\left(\frac{\theta}{a} - \frac{\sigma^2}{2a^2}\right)(B(t,T) - T + t) - \frac{\sigma^2 B^2(t,T)}{4a}\right]$

8.0 The Hull-White model

In this paper we consider only the one-factor no-arbitrage Hull-White model which is an extension of the Vasicek model that provide an exact fit to the initial term structure. The stochastic differential equation describing the Hull-White interest rate model is

$$dr = [\theta(t) - ar]dt + \sigma dz$$

or

$$dr = a\left[\frac{\theta(t)}{a} - r\right]dt + \sigma dz$$

This means that at any given time, *r* reverts towards $\frac{\theta(t)}{a}$ at rate a. Its variance rate per unit time is σ^2. Here $\theta(t)$ is a function of time determining the average direction in which *r* moves and *a* is the mean reversion rate. This model assumes that the short-term rate in the future is normally distributed. We noticed that dr is negative if *r* is currently large and positive if *r* is currently small. This shows that in the Hull-White model the interest rate process can be negative. This model can also be considered as the Vasicek model with a time-dependent reversion level. The function $\theta(t)$ can be calculated from the initial term structure.

8.1 Bond pricing using the Hull-White model

The zero-coupon bond at a future time *t* in terms of the short rate *r* and prices of the bond today is evaluated as follows using the affine bond price

$$P(t,T) = A(t,T)e^{-B(t,T)r}$$

and

$$B(t,T) = \frac{1-e^{-a(T-t)}}{a}$$

This gives

$$A(t,T) = \frac{P(0,T)}{P(0,t)} e^{\left[\frac{B(t,T)F(0,t)-\sigma^2 B^2(t,T)(1-e^{-2at})}{4a}\right]}$$

Where $F(0,t)$ is the instantaneous forward rate that applies to time t as observed at time zero. This can be computed as $F(0,t) = \frac{\partial \log(P(0,t))}{\partial t}$

8.2 Calibration of the Hull-White model

The Hull-White model is calibrated by choosing the mean reversion rate a and the standard deviation σ in such a way that they are consistent with option prices observed in the marketplace. After this, $\theta(t)$ is calibrated against the theoretical bond prices. The calibration of the Hull-White model is largely an optimization in which the system finds values for the Hull-White volatility parameters σ and a, in which option prices calculated using Black-Scholes model match as far as possible.

9.0 The Black-Karasinski model

The Black-Karasinski model is a one factor no – arbitrage model that allows only positive interest rates *r*. Interest rates dynamics in this model is given by:

$$d\ln r = [\theta(t) - a\ln r]dt + \sigma dz$$

$\ln r$ Follow the same process as *r* in the Hull-White model. The short term interest rate in the Hull-White model can be negative but the short rates stays only positive in the Black-Karasinski model which gives it an advantage over the Hull-White model in real life. The future value of *r* in the Black-Karasinski model is lognormal. One disadvantage in this model is that it is not possible to produce formulas for valuing bonds in terms of *r* as such it does not have much analytical tractability as compared to the Hull-White model.

10.0 Trinomial Trees

The trinomial tree model improves upon the binomial model by allowing stock prices to move up, down or stay the same with certain probability. The trinomial tree can be applied to solve various European and American options, pricing barrier options and calculating the Greeks.

Trinomial trees provide an effective method of numerical calculation of option prices. The trinomial model is considered to produce more accurate results than the binomial model when fewer time steps are modelled, it is often use when computation speed is of essence.

10.1 Building the Trinomial tree

To create the trinomial tree we first consider a single time step trinomial tree with the stock price at the beginning S_0 . During this time step the stock price can move up with probability p to the value S_u or move down with probability q to the value S_d or in the middle with probability $1 - p - q$ to the value S_m as illustrated in figure 1.3 below.

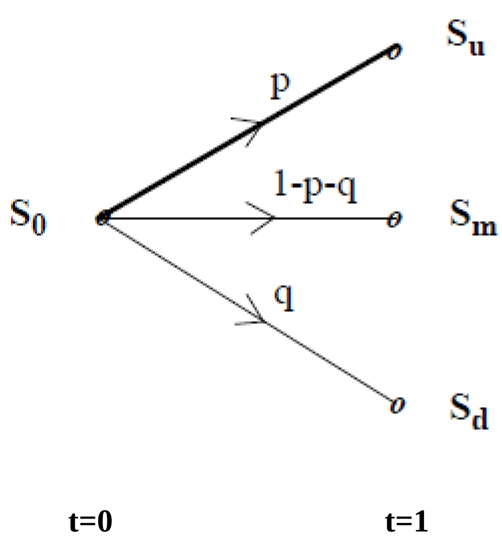


Figure1.3. Single step trinomial tree.

In the trinomial tree the jump sizes u, d, and m are match to the distribution of Geometric Brownian motion with transition probabilities p, q, P_m . The model can be summarize as follows

$$S(t) = \begin{cases} S_u & \text{with probability} \quad P_u \\ S_m & \text{with probability} \quad P_m \\ S_d & \text{with probability} \quad q \end{cases}$$

The jump sizes are:

$$u = e^{\sigma\sqrt{2\Delta t}}, \; d = e^{-\sigma\sqrt{2\Delta t}} \text{ and } m=1$$

The transition probabilities are given by:

$$P_u = \left(\frac{e^{\frac{r\Delta t}{2}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}}{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}}\right)$$

$$q = \left(\frac{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{\frac{r\Delta t}{2}}}{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}}\right)$$

$$P_m = 1 - P_u - q$$

r is the risk-free interest rate interest rate at an infinitesimal time. The standard trinomial tree looks like the figure below:

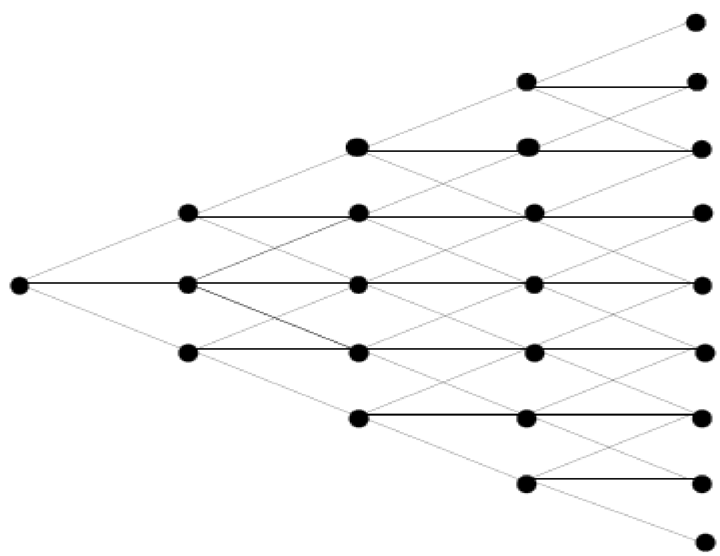


Figure1.4. Standard trinomial tree

When pricing financial instruments under the trinomial model *u*, *d* and *m* are evaluated with their corresponding transition probabilities P_u , P_m and q . If the instrument is option, the option type is also needed (i.e. call, put), we then apply the same methodology as used in pricing binomial option.

10.2 The Hull-White trinomial trees.

The Hull-White interest-rate tree is a process where we build a trinomial tree for the Hull-White model. This tree can be used to implement the Hull-White model and the Black-Karasinski model. This process is also used to develop new models such as the Willow tree model.

First of all we consider the interest rate process for the Hull-White model:

$$dr = [\theta(t) - ar]dt + \sigma dz$$

Where r is the instantaneous interest rate, a and σ are constants and $\theta(t)$ is a function of t chosen so that the model provides an exact fit to the initial term structure of interest rates.

For the consideration of the Hull-White tree we define a new variable r^* obtained from r by setting both $\theta(t)$ and the initial value of r equal to zero. The process for r^* is:

$$dr^* = -ar^*dt + \sigma dz$$

We then construct a tree for r^* that has a form as shown in figure 1.5 below:

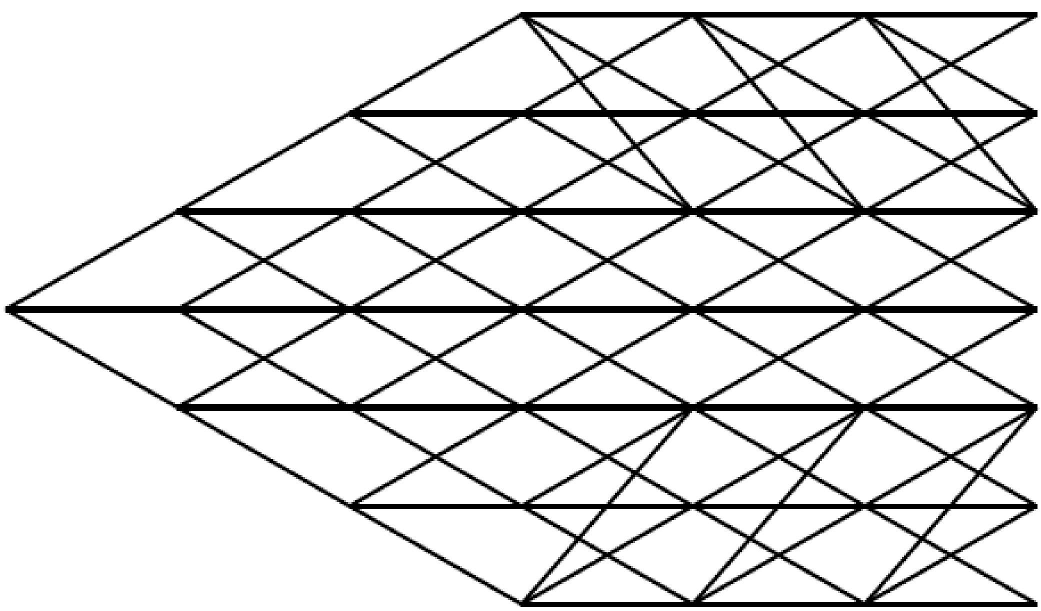


Figure1.5.The Hull-White interest rate tree in r^*

The central node at each time step has $r^* = 0$ and $r^*(t + \Delta t) - r^*(t)$ is normally distributed where the length of each time step is Δt. If we ignore higher terms of Δt we have

$$E[r^*(t + \Delta t) - r^*(t)] = ar^*(t)\Delta t$$

and

$$Var[r^*(t+\Delta t) - r^*(t)] = \sigma^2 \Delta t$$

We define Δr as the spacing between interest rates on the tree and we set $\Delta r = \sqrt{3V}$, where V is the variance of the change in r in time Δt .

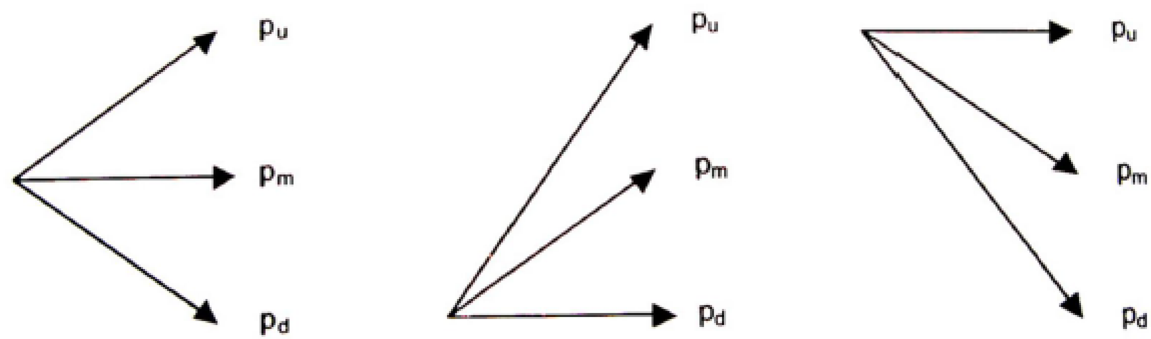
For each node (i, j) we define the expected change in r^* as Mr^* at node $j\Delta r^*$. From Hull and White [1994] the expected change in r^* and it's variance of the change in r^* in time Δt are given by:

$$E[dr^*] = Mr^* = (e^{-a\Delta t} - 1)r^*$$

$$Var[dr^*] = V = \frac{\sigma^2(1-e^{-2a\Delta t})}{2a}$$

For a node (i, j) a non-standard branching takes place at nodes $\pm j^*$ where $j^*$the smallest integer is greater than the value -0.184/M.

The probability at each node is chosen to match the mean and standard deviation of the change in r^* for the process for r^*. We have these 3 different branches in the tree.



Standard branch upward branch downward branch

We define the transition probabilities as P_u, P_m and P_d for the up, middle and down branching probabilities respectively.

$$\left.\begin{aligned} P_u &= \frac{1}{6} + \frac{j^2M^2 + jM}{2} \\ P_m &= \frac{2}{3} - j^2M^2 \\ P_d &= \frac{1}{6} + \frac{j^2M^2 - jM}{2} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11A)$$

At the top edge of the tree where the branching is non-standard the modified probabilities become:

$$\left.\begin{aligned} P_u &= \frac{7}{6} + \frac{j^2M^2+3jM}{2} \\ P_m &= -\frac{1}{3} - j^2M^2 - 2jM \\ P_d &= \frac{1}{6} + \frac{j^2M^2+jM}{2} \end{aligned}\right\} \qquad \text{(11B)}$$

At the bottom edge where the branching is no-standard the probabilities are:

$$\left.\begin{aligned} P_u &= \frac{1}{6} + \frac{j^2M^2-jM}{2} \\ P_m &= -\frac{1}{3} - j^2M^2 + 2jM \\ P_d &= \frac{7}{6} + \frac{j^2M^2-3jM}{2} \end{aligned}\right\} \qquad \text{(11C)}$$

The next stage for the construction of the Hull-White trinomial tree involves forward induction. We work from zero to the end of the tree adjusting the location of the nodes at each time step in such a way that the initial term structure is matched. The effect of this forward induction is to convert a tree for r^* into a tree for *r*. The conversion of the tree from r^* to *r* produces a tree as shown below in figure 1.6.

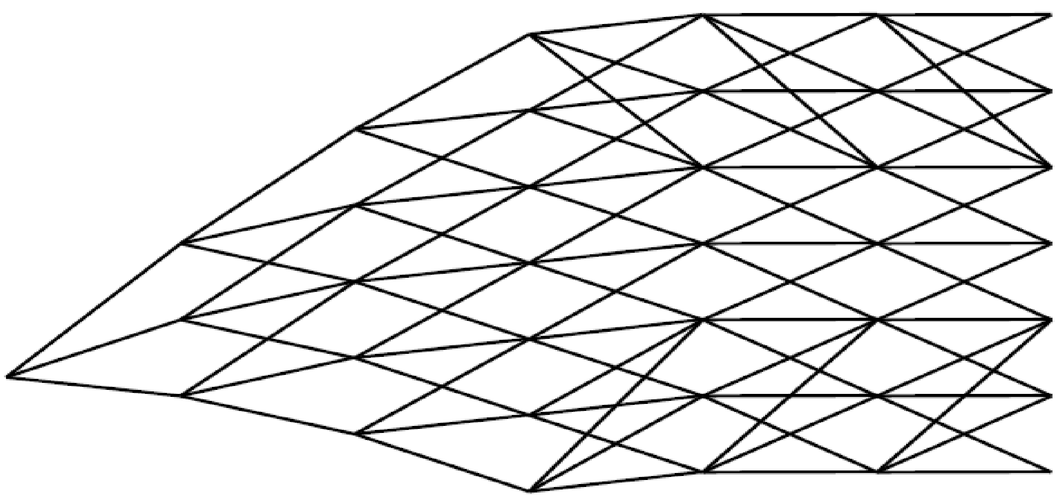


Figure1.6. Conversion of Hull-White tree from r^* to r

The Hull-White tree is analytically tractable. For instance Bond prices can be calculated analytically by using the affine bond pricing formula in equation (1) we have:

$$P(t,T) = A(t,T)e^{-B(t,T)r}$$

And the fact that when r is continuous we have:

$$P(t,T) = e^{-R(t,T)(T-t)}$$

Putting T=t+Δt and equating these two equations we have:

$$e^{-R\Delta t} = A(t, t + \Delta t)e^{-B(t,t+\Delta t)r}$$

Then

$$r = \frac{R\Delta t + \ln(t, t+\Delta t)}{B(t, t+\Delta t)} \quad \text{(11D)}$$

Given the Δt-period rate *R* at a node of the Hull-White tree we can calculate the instantaneous interest rate *r* by using equation (11D) and then use it to calculate for points on the term structure.

Where $P(t, T)$ is the price at some time *t* of a zero coupon bond maturing at time *T*. The variable *r* is the instantaneous short rate while the variable *R* is the interest rate on the Hull-White tree at Δt-period. Variables *r* and *R* are not the same and therefore cannot be interchanged.

The bond price at each node (i, j) for each branch is calculated as follows:

$$V_{i,j} = \left(P_u V_{i+1,j+1} + P_m V_{i+1,j} + P_d V_{i+1,j-1}\right)e^{-R_{i,j}\Delta t}$$

$$V_{i,j} = \left(P_u V_{i+1,j+2} + P_m V_{i+1,j+1} + P_d V_{i+1}\right)e^{-R_{i,j}\Delta t}$$

$$V_{i,j} = \left(P_u V_{i+1,j+1} + P_m V_{i+1,j-1} + P_d V_{i+1,j-2}\right)e^{-R_{i,j}\Delta t}$$

We can see from the two functions of $dr(t)$ and $dr^*(t)$ that these functions differ only by some function of time. We define this difference as

$$\alpha(t) = \mathrm{r}\,(\mathrm{t}) - r^*(t) \quad \text{(11E)}$$

This is the difference between the central or mean values of *r* (*t*) and r^* (*t*) at time *t*. Differentiating equation (11E) we have:

$$\frac{\partial \alpha(t)}{\partial t} = \theta(t) - a\alpha(t)$$

This is because the expected value of r^* (t) is zero and $\alpha(t)$ is the expected value of $\mathrm{r}\,(\mathrm{t})$. We can also write

$$\alpha(t) = exp\left\{-at\left[\mathrm{r}(0) + \int_0^{\mathrm{t}} \theta(q)e^{aq}\,\mathrm{dq}\right]\right\}$$

Substituting the analytical expression for $\theta(t)$ given in Hull and White [1994a] this reduces to

$$\alpha(t) = F(0, T) + \frac{\sigma^2}{2a^2(1 - e^{-at})^2}$$

This expression is used to find central nodes so that we don't have to go through forward induction to find them. Since the tree is a discrete representation of the underlying continuous stochastic process it does not provide an exact fit to the initial term structure. The forward induction procedure matches the initial term structure exactly to the tree.

10.2.1 Construction of 3-step Hull-White tree

We construct a 3-step Hull-white tree if the zero coupon curve was used to price a 3-year option on a zero coupon bond. We choose the day count to be 365 then the size of the time step is

$$\Delta\mathrm{t} = \frac{\frac{\frac{3*365}{3}days}{365days}}{year} = 1.0years\ .$$

The parameters a and σ are chosen as $a = 0.1$ and $\sigma = 0.01$. These values were chosen based on rough representation of the values that are observed in the market.

Therefore the expected change in r^* is given as

$$E[dr^*] = Mr^* = (e^{-a\Delta\mathrm{t}} - 1)r^*$$

Therefore

$$E[dr^*] = M = e^{-a\Delta\mathrm{t}} - 1$$

$$\Rightarrow M = e^{-0.1*1} - 1 = -0.095162581$$

The variance of the change in r^* is given as

$$Var[dr^*] = V = \frac{\sigma^2(1-e^{-2a\Delta\mathrm{t}})}{2a}$$

Therefore

$$V = \frac{(0.01)^2(1-e^{-2*0.1*1})}{2*0.1} = 0.00009063462346$$

The step size $\Delta r = \sqrt{3V} = \sqrt{3 * 0.00009063462346}$

$$\Rightarrow \Delta r = 0.016489507$$

Non-standard branching takes place at nodes

$$\pm j^* = \frac{-0.184}{M} = \frac{-0.184}{-0.095162581} = 1.93353 \quad \Rightarrow \quad j^* \simeq \pm 2$$

We then calculate the transitional probabilities as follows:

For standard branching we use equation 11A, for instance, at $j = 0$

$$P_u = \frac{1}{6} + \frac{0^2 * -0.095162581^2 + 0 * -0.095162581}{2}$$

$\Rightarrow \qquad P_u = 0.1667$

The same procedure is repeated for *j*=1 and *j*=-1 at P_u and *j*=-1, 0, 1 for $P_m\ and\ P_d$.

For non-standard branching we use equations *11B* and *11C* for top and bottom branches respectively.

At *j*=2, using equation 11B we have

$$P_u = \frac{7}{6} + \frac{2^2 * -0.095162581^2 + 3 * 2 * -0.095162581}{2}$$

$\Rightarrow P_u = 0.899291$

We repeat the same for $P_m\ and\ P_d$. At *j* = -2, using equation (11C) we have

$$P_u = \frac{1}{6} + \frac{-2^2 * (-0.095162581)^2 - (-2) * -0.095162581}{2}$$

$\Rightarrow\ P_u = 0.0896$

The same is repeated for $P_m\ and\ P_d$. We then calculate the rates at each node as follows:

Rate = $j\Delta$r, For instance the rate at *j*=2 is

Rate = 2(0.016489507) = 0.0329790

Combining all these data for the 3-time step tree in r^* , we have the initial Hull-White tree as shown in a table 1.1 below.

J	Rate= $j\Delta$r	P_u	P_m	P_d	Equation
2	0.03298	0.89929	0.01109	0.08962	11B
1	0.01649	0.12361	0.06576	0.21878	11A
0	0.00000	0.16667	0.16667	0.16667	11A
-1	-0.01649	0.21878	0.65761	0.12361	11A
-2	-0.03297	0.08962	0.01109	0.89929	11C

Table1.1. 3steps Hull-White tree

The rates in the tree at each time step are then shifted by amount α , which is chosen so that the revised tree correctly prices discount bond.

11.0 The willow Tree model

The willow tree model is a high performance library for the development of lattice –based models. In the same way as the binomial or trinomial models the willow tree is intended to describe continuous –time stochastic process with discrete-time stochastic process. One distinct feature of the willow tree is that the number of nodes at each time step is constant. This is in contrast to the binomial tree where the number of nodes becomes (2^M) where M is the time steps at the end of the tree. The willow tree provides better coverage of high probability regions of the process space and gives less waste in the low probability regions as shown in the figure 1.7 below.

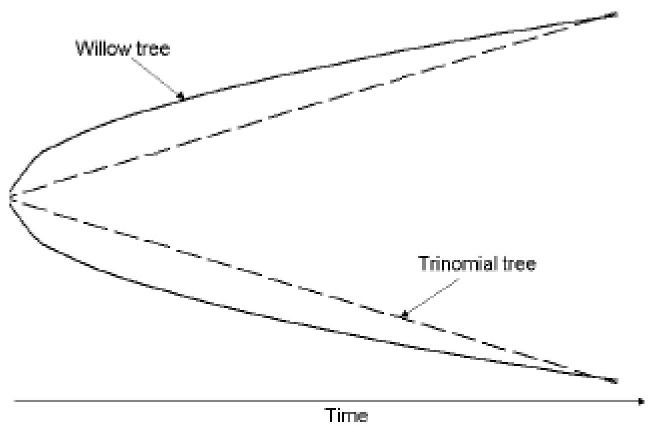


Figure 1.7.The space between trinomial and the willow tree

In this figure the space between the willow and trinomial envelopes is relatively a high probability region, which is neglected by the trinomial tree but taken into consideration by the willow model.

11.1 Features of the willow tree

1. Multiple interest rate processes: The Hull-White and Black-Karasinski processes are fully supported both in pricing and calibration.
2. Option-adjusted spread (OAS): The callable bond model supports calculation of OAS from market quotes, the use of an OAS in valuation and Option-adjusted sensitivities.
3. Market -to-future compliance: Swaptions can settle into underlying swap legs, early exercise is supported for Bermudan and American options. Callable bonds are called (or put) when optimal.
4. Performance/accuracy control: it trade accuracy and performance by specifying the number of days per time step in the lattice and the number of nodes at each time step (i.e. using possible values of lattice such as 7, 9, 11 etc)

.

11.2 Models implemented in the willow tree method

These models can be implemented using the willow tree method:

- One –factor European, American and Bermudan fixed maturity Swaptions: For Bermudan Swaptions, ability to explicitly specify an exercise schedule or choose to implicitly define one through cash flow dates (fixed, floating or fixed, or floating). It provides an ability to specify different lattice step size in option and swap period.
- Callable bonds/callable convertible bonds: For callable bonds schedules may be either continuous or discrete. Look-out periods are supported. Trade day rules are supported. It provides the value of the embedded call and put options. In callable convertible bonds it is a 2-factor model where factor 1 is the short rate and factor 2is the underlying asset.
- Flexible and limit cap/floor: This is a 1-factor short rate model. The underlying curve index determines the index rate to be used in computing cash flows. Three types of instruments are available: plain Vanilla caps/floors, regular flexible caps/floor, and limit type caps/floors.
- Equity options: it supports European, Bermudan and American type equity options.
- Interest rate index(IR) linked note: This is a 1-factor model directly using willow lattice to model complex interest rate products that depend on the short rate.
- Callable range accrual: This is an extension of IR index linked note. The interest only accrues if the index rate lies within the specified bounds.

11.3 Creating the willow tree model

The main method use to create the willow tree model is to approximate Brownian motion, which is widely used as a standard stochastic process for stocks, interest rates, etc. The graph below shows the optimized node positioning by the willow tree method.

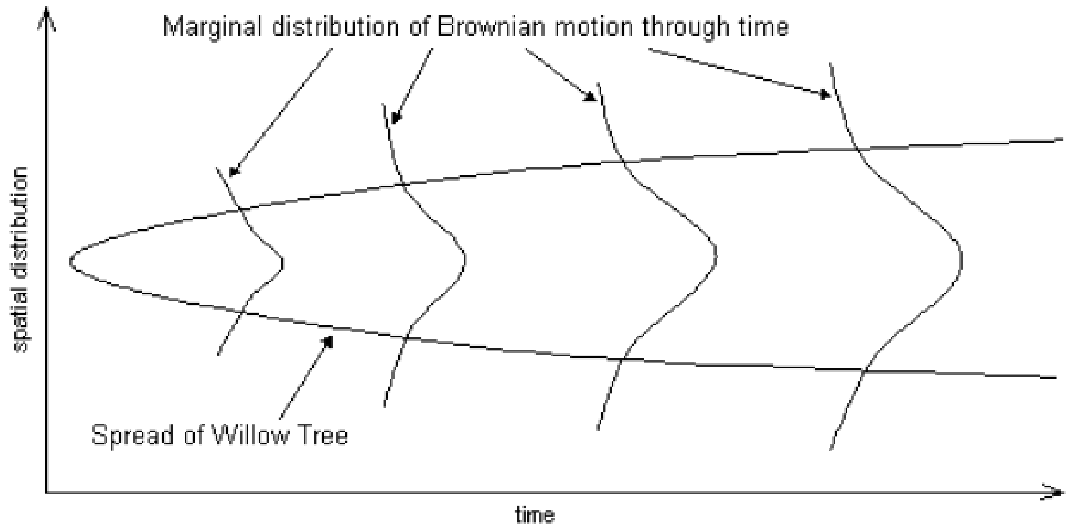


Figure1.8. Approximate Brownian motion method in willow tree

The final willow tree using 11 nodes with 3 equal time steps is the figure below:

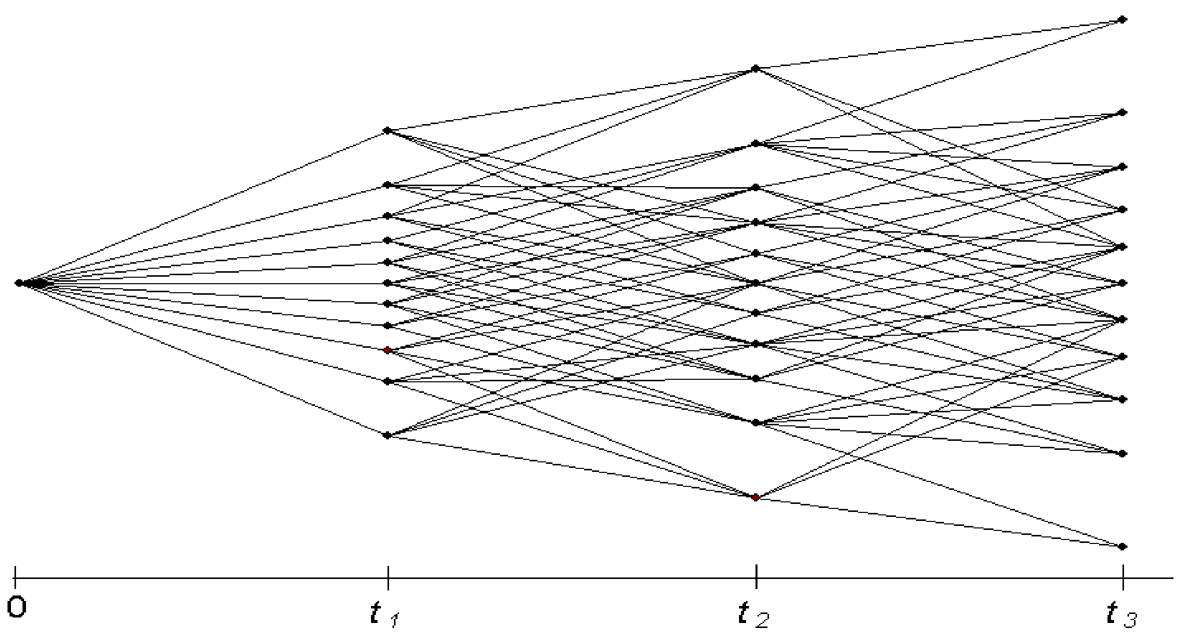


Figure 1.9.Willow tree with 11 nodes and with 3 equal time steps

11.4 Time-Inhomogeneous Markov Chain

Markov Chain is a class of stochastic processes which shares the Markov property, which means that given the present values of the process the future is independent of the past.

Markov processes are important models of security prices because they are realistic representations of the true prices.

Given a filtration $F = \{F_t; t = (1,2,3, \dots, T)\}$ generated by $X = \{X_t; t = (1,2,3, \dots, T)\}$. This process takes values in some finite set *E*, called the state space. The process is in state *j* at time *t* if $X_t = j \epsilon E$

The filtration is the history of the present and past values of the process *X* in all states. The dynamics of interest rates or Equity prices for the willow tree method is described as discrete stochastic process in the form of a lattice.

In real world the nodes *j* and time step *k* define a state and the finite set of all possible states (*j*, *k*) which define state space *E*. The stochastic process *X* is said to be Markov chain if:

$$P\left\{X_{t+1} = j \middle| F_t\right\} = P\left\{X_{t+1} = j \middle| X_t\right\}$$

This simply means that having the whole information set of what happened before today has exactly the same predictive power as having the information today. The Markov chain *X* is said to be time –homogeneous if the conditional probabilities $P\left\{X_{t+1} = j \middle| F_t\right\}$ do not depend on time *t*; otherwise it is called time-inhomogeneous. We define the transition probabilities of a time-inhomogeneous process as:

$$P_t(i,j) = P_t\left\{X_{t+1} = j \middle| X_t\right\}, \ i, j \epsilon E$$

Or in a matrix form this can be written as

$$P_t = P_t(i,j)$$

11.5 The basic willow tree model

We consider dividing the normal distribution into *n* individuals and assigning a single value in each interval to represent the corresponding stratum of the distribution.

If $z_1, z_2, z_3, \ldots, z_n$ are the representative normal variants, with probabilities $q_1, q_2, q_3, \ldots, q_n$ i.e. $\{(z_i, q_i)\}$ is a discrete approximation of the standard normal function where $q_i = P(Z = z_i)$. Curran (May 2000) suggest that

$$z_i = N^{-1}\left(i - \frac{0.5}{n}\right) \qquad (11.5)$$

$$q_i = \frac{1}{n}$$

Where $N(x)$ is the normal distribution. In setting up the willow tree lattice for modelling option prices, we first need to set up a discrete Markov process that converges to Brownian motion in the limit as explain below.

11.6 Technique to approximate Brownian motion

Given $i, j \in E$, let $\{X_k; k = (1, 2, 3, \ldots, T)\}$ be a time-inhomogeneous Markov chain with state space E={1, 2, ..., n}. We will say that the process

$\{Y_{t_k}; k = (1, 2, 3, \ldots, T)\}$ has the value $\sqrt{t_k} \, . \, z_i$ when X_k is in state i

Where $t_k = \sum_{j=1}^{k} h_j$ for some $h_j > 0$ and j= 1, 2 … k

h_j represent the interval and Y_{t_k} is a discrete Markov process.

Subject to certain conditions on the transition probabilities the stochastic process, Y_{t_k} converges to Brownian motion as $k \to \infty$ and $h_k \to 0$ for all k.

Let P_{ij}^k denote the transition probability from node i to node j at time step k. The transition probabilities must be parameterized by t and h in order to achieve convergence to Brownian motion. This parameterization must satisfy the usual requirements for discrete-time models, consisting of the following three conditions:

1. The process must constitute a martingale

   $$E[Y_{t_k + h_{k+1}} \mid Y_{t_k}] = Y_{t_k} \quad \forall k$$

2. The variance of the process must be equal to the length of the time step

   $$Var[Y_{t_k + h_{k+1}} \mid Y_{t_k}] = h_{k+1} \quad \forall k$$

3. Transition probabilities from each node must sum to one

$$\sum_{j=1}^{n} P_{ij}^{k} = 1 \quad \forall i, k$$

4. Finally we impose the restriction:

$$\sum_{j=1}^{n} q_i P_{ij}^{k} = q_j \quad \forall j, k$$

This condition inductively guarantees that the unconditional probability of each state j at time step k+1 is given q_j, that this is the case at time step k.

Writing $\alpha_k = \frac{h_{k+1}}{t_k}$ and letting $h_k = h$ (For convenience, the subscript of α_k and the superscript k of P_{ij}^{k} will be dropped) conditions (1) and (2) can be rewritten as

$$\sqrt{1+\alpha}\left(\sum_j P_{ij} z_j = z_i \right) \qquad \forall i \ldots\ldots\ldots(5)$$

$$(1+\alpha)\left(\sum_j P_{ij}[z_j]^2 - z^2 = \alpha \right) \qquad \forall i \ldots\ldots\ldots(6)$$

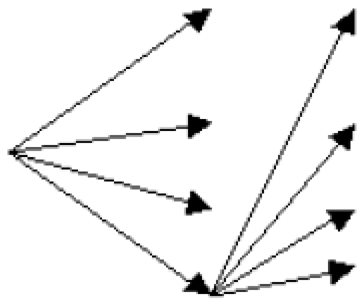
A unique solution of the transition probabilities P_{ij} can be solved using linear programming (LP). The objective function is chosen by the expectation of the absolute values of the third power of all increments

Minimize

$$\sum_i \sum_j P_{ij} \left|\sqrt{1+\alpha}\; z_j - z_i\right|^3 \ldots\ldots(11.6)$$

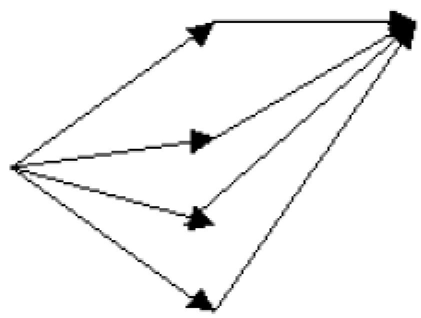
subject to conditions (3), (4), equations (5), (6) and $P_{ij} \geq 0 \;\; \forall i, j$. The solutions of this linear programming equation for different values of α are computed for P_{ij} . Since the conditions (1) and (2) and constraints (5) and (6) are assured, linear interpolation of P_{ij} based on $\frac{1}{\sqrt{1+\alpha}}$ is possible for a judicious choice of values of α for which the LP can be solved.

The graph below illustrates condition (3), which means that the sum of transitional probabilities going from any one of the nodes in the previous time step to all nodes in the next time step should be one.



Transition probabilities from each note sum to one

The graph below also illustrate condition (4) which means that the sum of transitional probabilities times the marginal probabilities going from all nodes in the previous time step to the node j in the next time step should be equal to the corresponding marginal probability q_i at the node j.



Sum of transitional probabilities times marginal probabilities equal marginal probability.

11.7 One factor willow model

Most valuations for the willow type of instruments use the one factor interest rate model, but the power of the willow tree method is capable to handle multifactor models.

11.7.1 Case1. Construction of the normal short rates in the willow tree.

Under this section we consider how the Hull-White model is used to build up the short rates in the willow tree.
From the Hull-White interest rate dynamics

$$dr = [\theta(t) - ar]dt + \sigma(t)dz$$

Where the mean reversion constant $a \in [0,1]$. If r^* is the short rate of the willow tree, the forward induction process is as follows:

Let $r = \alpha(t) + r^*$ and r^* satisfies the equation

$$dr^* = -ar^* dt + \sigma(t)dz$$

with initial condition $r^*(0) = 0$

Solving this we have

$$r^* = e^{-at}\int_0^t \sigma(s)e^{as}dz(s)$$

We let

$$X = \int_0^t \sigma(s)e^{as}dz(s)$$

where X is normally distributed with mean 0 and variance $\int_0^t \sigma^2(s)e^{2as}ds$.

Therefore

$$X = \left(\int_0^t \sigma^2(s)e^{2as}ds\right)^{\frac{1}{2}}.z$$

where z is the normally distribution with mean 0 and variance 1 . X can be written in the discrete form at node j and time step k as follows:

$$X(j,k) = \left(\int_0^{t_k} \sigma^2(s)e^{2as}ds\right)^{\frac{1}{2}} z_j \;\simeq\; \left(\sum_{i=0}^{k-1} \sigma^2(t_i)\int_{t_i}^{t_{i+1}} e^{2as}ds\right)^{\frac{1}{2}} z_j$$

$$= \left(\sum_{i=0}^{k-1} \sigma^2(t_i)\frac{e^{2at_{i+1}} - e^{2at_i}}{2a}\right)^{\frac{1}{2}} z_j \quad \text{.......} (12.3)$$

here the z_j's are the z values we generated from the basic willow tree model.

We simplify this further by letting

$$T_k = \sum_{i=0}^{k-1} \sigma^2(t_i)\frac{e^{2at_{i+1}} - e^{2at_i}}{2a} \quad \text{.......} (12.4)$$

where j=1, 2, 3 ...K. k= 1, 2, 3... T

We then have

$$X(j,k) = \sqrt{T_k}\,.\,z_j$$

Finally we have

$$r^*(j,k) = e^{-at_k}.X(j,k) \quad (12.5)$$

11.7.11 Calibration of the normal short rates in the willow tree model.

We calibrate the short rate of the willow model by fitting the initial term structure. The drift term α is added to the short rate r^* by forward induction method so that the interest rate at each node matches the initial term structure of the interest rate.

We first let $Q_{j,k}$ be the Arrow –Debreu model (which suggest that under certain conditions there must be a set of prices such that supplies must be equal to demands for every security in the economy). We let $Q_{j,k}$ pays $1 only at node (*j*, *k*) or otherwise 0. We can then calculate α_k and $Q_{j,k}$ iteratively in such a way that the initial term structure is match exactly.

We let

$$Q_{j,1} = q_j.d(t_0, t_0, t_1) \ .$$

Where the $q_j's$ are the q values of the probabilities and

$$d(t_0, t_0, t_1) = \frac{1}{(1+\alpha_0.\Delta t_0)} \quad (12.6)$$

We can then find α_0 by finding solution for equation (12.3).

The drift α_k is calculated by finding the solution for the following equation

$$d(t_0, t_0, t_{k+1}) = \sum_{j=1}^{n} \frac{Q_{j,k}}{(1 + (r^*(j,k) + \alpha_k).\Delta t_k)}$$

k = 1, 2, 3... T

The discount factor $d(t_0, t_0, t_{k+1})$ is determined by the yield curve of the instrument which represents the initial term structure.

Since the value of α_k is known $Q_{j,k+1}$ can be determined by the equation below

$$Q_{j,k+1} = \sum_{i=1}^{n} \frac{P_{i,j}.Q_{i,k}}{(1 + (r^*(j,k) + \alpha_k).\Delta t_k)}$$

We then recover the short rates by using equation (12.7) below:

$$r(j,k) = r^*(j,k) + \alpha_k \ldots\ldots (12.7)$$

11.7.2 Case2. Construction of the log normal short rates in the willow tree model.

In this section we consider how the Black and Karasinski model is used to build the interest rates in the willow tree. Using the Black and Karasinski interest rate dynamics

$$d\ln r = [\theta(t) - a\ln r]dt + \sigma(t)dz$$

Where a is the mean reversion constant and $a \in [0,1]$. We let the short rates be R^*.

Let

$r = \alpha(t)R^*$ and R^* satisfies the following equation

$$d\ln R^* = -a\ln R^* dt + \sigma(t)dz$$

with initial the condition $R^*(0) = 1$. Solving this we have

$$R^* = e^{r^*}$$

where r^* is the interest rate in case 1. Therefore the formula of R^* at node *j* and time step *k* is

$$R^*(j,k) = e^{(r^*(j,k))} \ldots\ldots (12.8)$$

11.7.21Calibration of the lognormal short rate of the willow tree

We follow the procedure as in case 1 but here we let $R^*(j,k).\alpha_k$ replace $r^*(j,k) + \alpha_k$. We let $Q_{j,k}$ be the Arrow-Debreu function that represents the present value of a security that pays \$1 only at node (*j*, *k*), otherwise 0. We proceed by calculating for α_k and $Q_{j,k}$ iteratively.

First we calculate the drift part α_0 by solving this equation

$$d(t_0, t_0, t_1) = \frac{1}{(1+\alpha_0.\Delta t_0)}$$

where we let

$$Q_{j,1} = q_j . d(t_0, t_0, t_1)$$

Here the $q_j's$ are the q values of the probabilities. Then the drift α_k is calculated by solving this equation

$$d(t_0,\ t_0,\ t_{k+1}) = \sum_{j=i}^{n} \frac{Q_{j,k}}{(1 + (R^*(j,k).\alpha_k).\Delta\, t_k)}$$

k, k=1, 2, 3,...,T

The yield curve of the instrument which represents the initial term structure is used to determine the discount factor $d(t_0,\ t_0,\ t_{k+1})$.

Since we know α_k we can compute $Q_{j,k+1}$ by using the equation

$$Q_{j,k+1} = \sum_{i=1}^{n} \frac{P_{i,j}.Q_{i,k}}{(1+(R^*(j,k).\alpha_k).\Delta\, t_k)}$$

We then recover the short rates by using the equation (12.9)

$$r(j,k) = R^*(j,k)\alpha_k \quad (12.9)$$

12.0 The Java applet for the willow tree interest rates

We proceed by using Java program to create an applet which is use to calculate the interest rates in the willow tree by using equation (11.5) the basic willow tree approach and equations (12.3),(12.5) for the rates as in case1 and equation(12.8) in case2.

These values were chosen: a= 0.03, $\sigma = 0.1$ for this test. It should be noted that volatility changes with time; for the sake of the simplicity of our test we used constant volatility. An applet showing interest rates of the willow tree with total number of nodes n=5, with 5 equal time step, at node j=1and time step k=1is shown below.

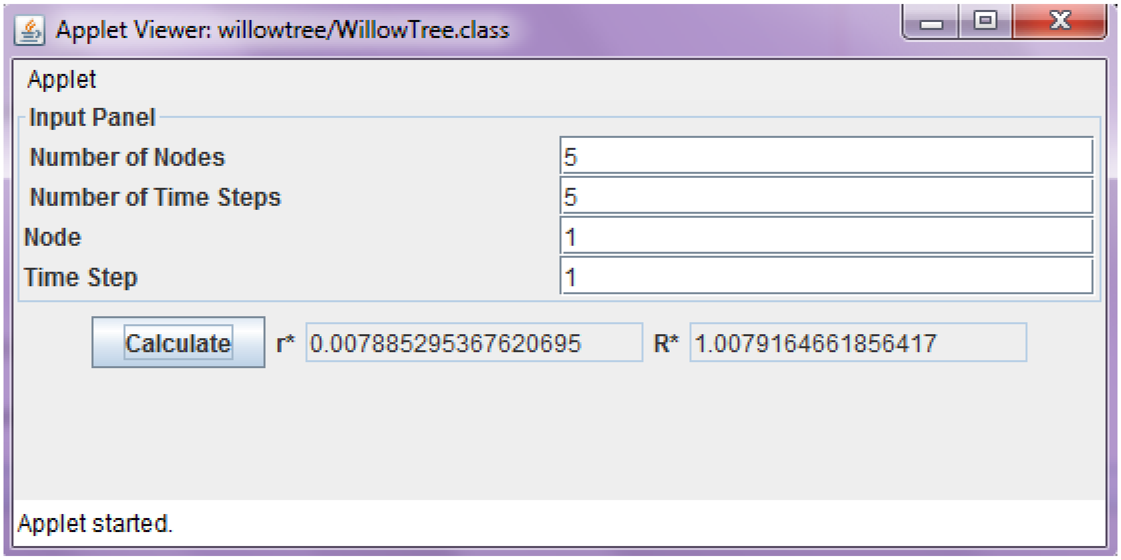


Figure1.10. Java applet showing willow tree interest rates

The applet above in figure1.10 shows $r^*(1,1)$ = 0.0078 and $R^*(1,1)$ = 1.0079 for the rates in the willow tree built by using the Hull and White and the Black Karasinski models respectively.

12.1 The components of the Java applet

The applet in figure 1.10 has two main components; the input panel and the calculate button.

The input panel contains four data sets:

1. Number of nodes- which is the total number of nodes in the willow tree.
2. Number of time steps- which is the total number of time step in the willow tree.
3. Node- this is a particular node in the tree (e.g. node 2).
4. Time step- this is a particular time in the tree (e.g. 2years).

The calculate button has two components:

1. r^* -which calculates the Hull-White rates in the willow tree.
2. R^* -which calculates the Black-Karasinski rates in the willow tree.

12.2 Using the applet

At the click on the calculate button, the button is highlighted as shown below and it performs the following actions:

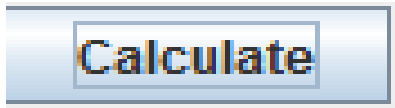


Figure 1.11 highlighted calculate button

1. Calculates the Hull-White interest rates for the willow tree as shown below:

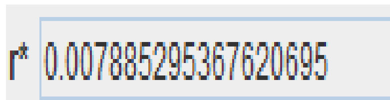


Figure 1.12 rate calculated by Hull-White

2. Calculates the Black-Karasinski interest rates for the willow tree as shown below:

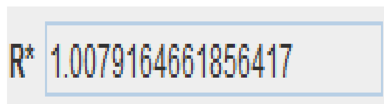
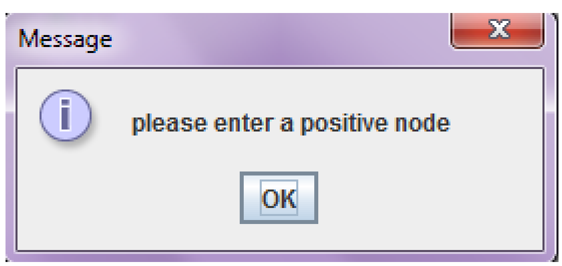


Figure 1.13 rate calculated by Black-Karasinski

12.3 Exceptions

Although this specialised Applet is use mainly by practitioners there is still a need to add some exception class in Java to make it user free.

All inputs in the Applet take only positive values. When a non-positive number is entered the menu below pops up to allow the user to enter the right number, for this example we entered a negative node.

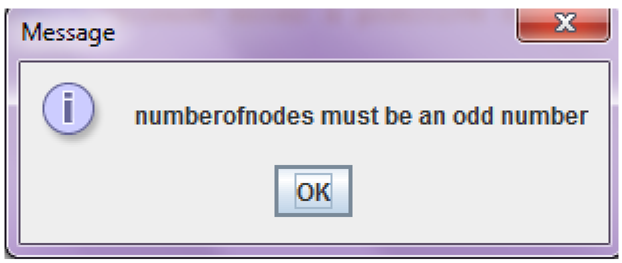


1.14 Exception for non positive number

All inputs must be numbers (integers). When alphabets or any other characters are entered, there is a pop up which tells the user to enter an integer. This is shown below by the menu

1.15 Exception for numerical values

Again, since number of nodes in the willow tree must be an odd number for equal distribution in the tree, when an even number is entered in the number of nodes panel the menu below pops up which helps the user to enter an odd number.



1.16 Exception for odd number of nodes

Finally, node must be less or equal to number of nodes and time step must be less or equal to number of time step. So when a user enters a node/time step greater than the number of time step or numbers of nodes this menu is displayed.

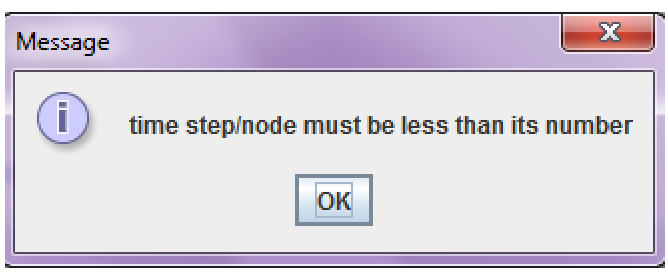


Figure 1.17 Exception for node/time step

13.0 Interest rate term structures for the willow tree

We calculate rates for 5 nodes, 5 time steps at the same node and same time step for the willow tree using the Hull-White model as in cas1. The table below shows the results:

Time to maturity = T	Short rates = $r^*(j,k)$
1	$r^*(1,1)$ =0.0079
2	$r^*(2,2) = 0.0815$
3	$r^*(3,3)$ =0.2809
4	$r^*(4,4)$ =0.6582
5	$r^*(5,5)$ =0.0000

Table 1.2. Hull-White willow tree interest rates

We then plot the interest rates against the time steps using table and we have the graph below:

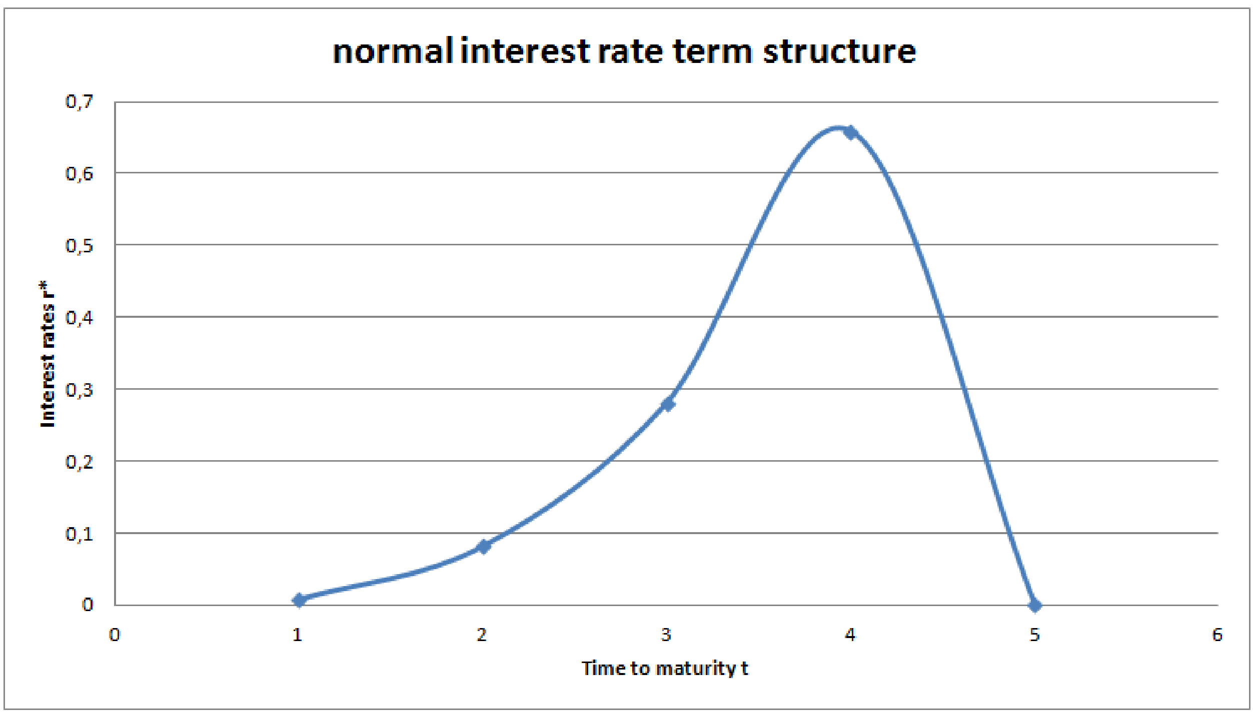


Figure1.18.Normal interest rates term structure for the willow tree.

We see from figure1.18 above that the movement of interest rates follow the normal distribution and the drift pulls it down when the rates grow to the maximum point which supports the theory of mean reversion.

For case2: We tabulate the results from the Black and Karasinski rates at the same node and time steps as we did for the Hull-White model. The table below gives the values:

Time to maturity = T	**Short rates = $R^*(j,k)$**
1	$\boldsymbol{R^*(1,1)}$ **=1.0079**
2	$\boldsymbol{R^*(2,2) = 1.0849}$
3	$\boldsymbol{R^*(3,3)}$ **=1.3243**
4	$\boldsymbol{R^*(4,4)}$ **=1.9313**
5	$\boldsymbol{R^*(5,5)}$ **=1.0000**

Table1.3. Black-Karasinski willow tree interest rates.

We again plot interest of the Black-Karasinski willow tree rates using table 1.3 against time step and we have the graph below:

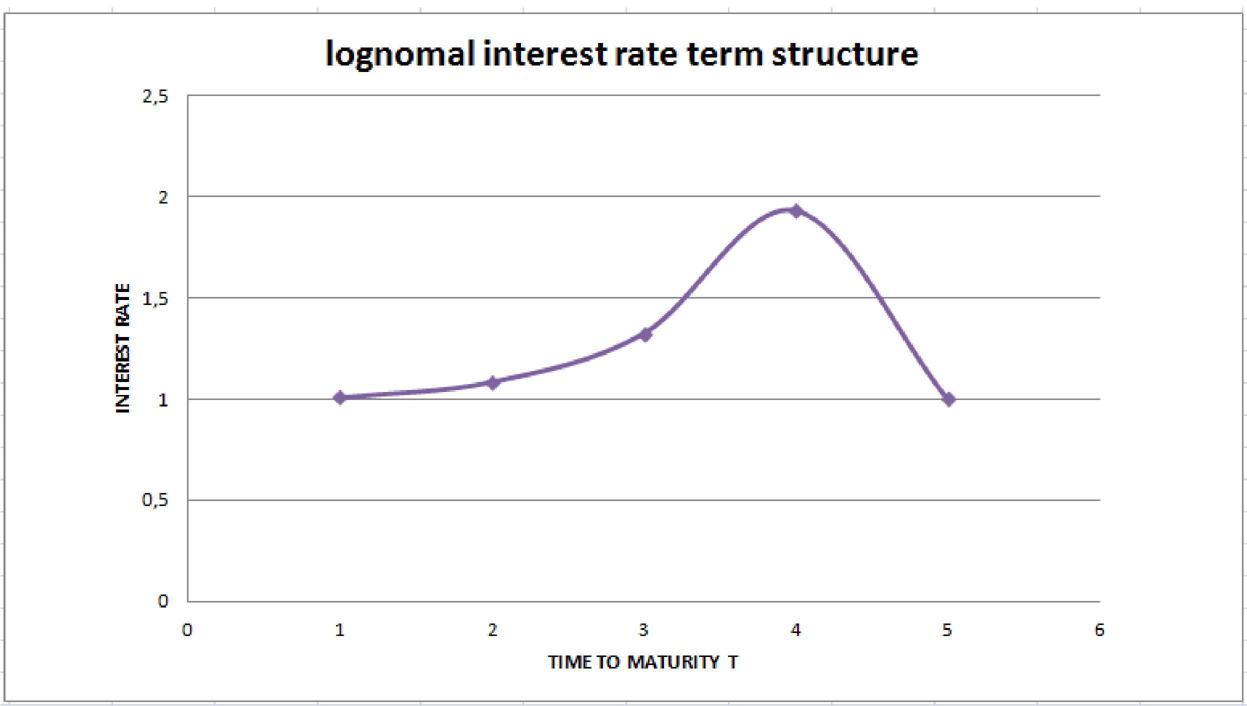


Figure 1.19 lognormal interest rate term structure for the willow tree.

These yield curves in figure 1.18 and figure1.19 move in the same way as the yield curve on the market, when the interest rate increases to the maximum, the drift part in the model pull it in such a way that it comes down and when the interest rate moves down the drift pulls it up as explain by the mean reversion phenomenon. This shows that the willow tree interest rate term structure has been validated and it behaves like the interest rate on the market. The willow tree model is used by the Swedbank to price cancellable bonds and other instruments. The software can be found from the RISKWATCH Company.

14.0 Conclusion

This paper has explained a new model called the willow tree model which is an alternative to the traditional binomial and trinomial trees.

The willow tree is an improvement over the binomial and the trinomial trees since it takes higher nodes than the binomial and trinomial it therefore gives more accurate prices. It can be use to implement other models like the Hull-White model and the Black and Karasinski models. The willow tree can be use to develop a multi-factor convertible bond model and normal and lognormal short-rate models. Some advantages of the willow tree is that it provides better coverage of high probability regions of the process space and the length of the time steps can be chosen arbitrarily which simplifies the implementation of the pricing models.

We then propose valuing securities (especially with long-term rates) using the Hull-White and the Black-Karasinski interest rate term structure for the willow Tree Model.

15.0 Recommendations

Further work can be done by any researcher (student) by creating a program to price options using the willow tree. This case the linear program in equation (11.6) should be solved subject to the constraints given, so that the values of the transitional probabilities P_{ij} can be found. The interest rates we have calculated with other parameters such as volatilities smiles (σ_i) can then be used to price the option.

The prices can then be calibrated against the market data. To calibrate the prices against the market we recommend the use of European plain vanilla Swap options. The prices are then fitted to the market price to get an accurate option prices.

The willow tree is use to price instruments such as European Swap options, callable bonds, callable convertible bonds, cancellable bonds, etc. The model can be used to price these instruments and many more by choosing a particular time step and the volatility at that time.

When we compare the performance of the willow tree to the trinomial tree (the industry standard), models based on willow tree are much faster, more accurate, and more stable.

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17.0 Appendix 1

Notation	Variable in Java Code
Time step	*n*
Node	*m*
Number of nodes	Node
Number of time step	timeStep
$\boldsymbol{T_k}$	*TK*
$\boldsymbol{R^*(j,k)}$	*R*
$\boldsymbol{r^*(j,k)}$	*r*
$\boldsymbol{\sigma}$	Vol
$\boldsymbol{X(j,k)}$	*X*
$\sqrt{\boldsymbol{T_k}}$	$TK[i]$
$\boldsymbol{z_j}$	$Z[i]$
$\boldsymbol{N(x)}$	$N[i]$

18.0 Appendix 2

```
package willowtree;

  import java.awt.*;
  import java.awt.event.ActionEvent;
  import java.awt.event.ActionListener;
  import java.text.*;
  import javax.swing.*;
  import static java.lang.Math.*;

 public class WillowTree extends JApplet {

    int m = 10, n = 10;

int node = 101;

double N [] = new double [node];

double k [] = new double [node] ;

double X = 0;

double r;

int timeStep = 102;

double [] TK = new double [timeStep];

 double[] Z = new double [node];

 double a1=0.319381530;

double a2=-0.356563782;

double a3=1.781477937;

double a4=-1.821255978;

double a5=1.330274429;

double y = 0.2316419;

double vol = 0.1;

double a = 0.03;

double temp = 0;

double R;

// panels
        private JPanel mainPanel = new JPanel(new java.awt.BorderLayout());
        private JPanel dataPanel = new JPanel(new java.awt.BorderLayout());
        private JPanel inputPanel = new JPanel(new GridLayout(4,6));
```

```java
    private JPanel controlPanel = new JPanel();

      // labels
  private JLabel nodeLabel = new JLabel(" Number of Nodes");
  private JLabel timeStepLabel = new JLabel(" Number of Time Steps");
  private JLabel particularNodeLabel = new JLabel("Node ");
  private JLabel particularTimeStepLabel = new JLabel("Time Step");
  private JLabel hullWhiteRateLabel = new JLabel("r*");
  private JLabel bKrateLabel = new JLabel("R*");

  // text fields
  private JTextField nodeField = new  JTextField();
  private JTextField timeStepField = new  JTextField();
  private JTextField particularNodeField = new JTextField();
  private JTextField particularTimeStepField = new JTextField();
  private JTextField hullWhiteRateField = new JTextField(15);
  private JTextField bKrateField = new JTextField(15);

    // button
  private JButton button = new JButton("Calculate");

  private DecimalFormat myFormatter = null;
   @Override
  public void init () {
           // Initialise formatter
          DecimalFormatSymbols symbols = new DecimalFormatSymbols();
          symbols.setDecimalSeparator('.');
          myFormatter = new DecimalFormat("###.#####",symbols);
          // get content pane
          Container contentPane = getContentPane();
           // add main panel to content pane
           contentPane.add(mainPanel);
           // add data panel
           mainPanel.add(dataPanel,BorderLayout.NORTH);
           // add input panel
           dataPanel.add(inputPanel,BorderLayout.CENTER);
           // add control panel
           dataPanel.add(controlPanel,BorderLayout.SOUTH);

          // create and add label
         inputPanel.setBorder(BorderFactory.createTitledBorder("Input
Panel"));
         inputPanel.add(nodeLabel);
         nodeLabel.setLabelFor(nodeField);
        // add text field
        inputPanel.add(nodeField);
        nodeField.setText(myFormatter.format(node));

      // create and add label

         inputPanel.add(timeStepLabel);

      // add text field
    inputPanel.add(timeStepField);
    timeStepLabel.setLabelFor(timeStepField);
    timeStepField.setText(myFormatter.format(timeStep));
```

```
        // create and add label
       inputPanel.add(particularNodeLabel);

       // add text field
        inputPanel.add(particularNodeField);
        particularNodeField.setText(myFormatter.format(m));
        particularNodeLabel.setLabelFor(particularNodeField);

        // create and add label
          inputPanel.add(particularTimeStepLabel);

       // add text field
       inputPanel.add(particularTimeStepField);
     particularTimeStepField .setText(myFormatter.format(n));
      particularTimeStepLabel.setLabelFor(particularTimeStepField);

      // add button
      controlPanel.add(button);

      // create and add label
      controlPanel.add(hullWhiteRateLabel);

      // add text field
      controlPanel.add(hullWhiteRateField);
     hullWhiteRateField.setEditable(false);
     hullWhiteRateLabel.setLabelFor(hullWhiteRateField);

      // create and add label
      controlPanel.add(bKrateLabel);

       // add text field
        controlPanel.add(bKrateField);
        bKrateField.setEditable(false);
        bKrateLabel.setLabelFor(bKrateField);

       button.addActionListener(new ActionListener() {

            @Override
             public void actionPerformed(ActionEvent evt) {

                 buttonActionPerformed(evt);
            }
         });

      }

    private void buttonActionPerformed(ActionEvent evt) {


       try{ node = (int)Float.parseFloat(nodeField.getText());

       } catch (NumberFormatException e)
{JOptionPane.showMessageDialog(null,"enter an integer");
          return;
       }

        try{ timeStep = (int)Float.parseFloat(timeStepField.getText());
```

```
        } catch (NumberFormatException e){
            JOptionPane.showMessageDialog(null,"enter an integer");
           return;
        }

        try{ if(node<=0){
            throw new ArithmeticException();
        }

        } catch (Exception e){
          JOptionPane.showMessageDialog(null,"enter a positive number of
nodes");
          return;
      }
        try{ if(timeStep<=0){
            throw new ArithmeticException();
        }

        } catch (Exception e){
          JOptionPane.showMessageDialog(null,"enter a positive number of time
step");
          return;
      }

        try{ if(node % 2 == 0){
            throw new ArithmeticException();
                   }
               }
        catch (Exception e){
            JOptionPane.showMessageDialog(null,"node must be an odd number");
            return;
        }

        try{  m = (int) Float.parseFloat(particularNodeField.getText());

        }

  catch(NumberFormatException ev){JOptionPane.showMessageDialog(null,"enter
an integer");
    return;

  }
        try{  n = (int) Float.parseFloat(particularTimeStepField.getText());

        }
  catch(NumberFormatException ev){JOptionPane.showMessageDialog(null,"enter
an integer");

    return;

  }
        try{  if (m <=0){

          throw new ArithmeticException();

              }
```

```
      }

      catch(Exception e){

        JOptionPane.showMessageDialog(null,"enter a positive node");

        return;

      }

        try{  if (n <=0){

          throw new ArithmeticException();

              }
      }

      catch(Exception e){

        JOptionPane.showMessageDialog(null," enter a positive time step");

        return;

      }
          try{ if(n > timeStep || m > node){
            throw new ArithmeticException();
             }
        }
        catch (Exception e){
         JOptionPane.showMessageDialog(null,"time step/node must be less than
its number");
                return;
        }

      for (int j=0;j<= timeStep;j++){

        for (int i=1;i<=node-1;i++){

    temp = (pow(vol,2)*i)*pow(((exp(2*a*i+1)-exp(a*i*2))/2*a),1/2);

    TK [i] += temp;

   k[i] = (1/(1+(y*i)));

   N [i] = (1- ((1/(sqrt(2*PI)))*exp((-1/2)*(pow(i,2)))*

       (a1*k[i]+a2*pow(k[i],2)+a3*pow(k[i],3)+a4*pow(k[i],4)+a5*pow(k[i],5))));

       Z [i] = (1/N[i])*((i-0.5)/node);
   }
     }

  X = TK[m]*Z[n];

  r = exp(-a*m)*X;

  R = exp(r);
```

```
 hullWhiteRateField.setText(String.valueOf(r));
 bKrateField.setText(String.valueOf(R));

        }
}
```