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Yield Curve Construction

An research of Interpolations for Bootstrapping

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Abstract

Interest curve construction is always the issue in the financial market due to its effect on the evaluating the price of the instruments, especially, after the financial crisis in 2008 - since the government can default.

In order to construct the best-fit curve, based on the bootstrap model, several chosen interpolations is introduced and applied in Excel VBA in using some liquid interest rate instruments data from the Swedish market. The results' comparison and analysis has been conducted to define the relatively best curve because there do not exist standard answer for the interest curve construction. To sum up, although the linear interpolation is simple, the clamped cubic spline displays the smoothest curve with poor locality and stability. The two monotonic cubic methods both present relatively continuous and smooth curve with good stability and locality.

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1. Introduction

In the financial world, the interest curves play an important role, especially, the discount curves and the forward rate curves; they provide the information for generating cash-flows for instrument valuation purpose. Prior to the credit crisis in 2008, interest rate modeling was generally well understood, however, after the crisis, the better way for constructing the interest curves is needed desperately, due to the change of the market situation –the government can default, some currency are more risky than others.

Under the concept of the Bootstrapping method, some typical interpolations are introduced and applied in Excel VBA, in order to find the best method for getting the best-fit interest curve – the curve reflects the exact market price. The linear interpolation is a quite popular method in the real world because its simplicity can save time during the trading process, but it gives the forward curve lacking the important continuity and smoothness. By using the cubic splines, smooth curves with continuity can be obtained, however, concerning to locality and forward stability, clamped cubic become non-valuable. Relatively, the two monotonic cubic methods display nice forward curve at the same time grantee forward stability and locality.

The plan of the paper is as follows: section 2, reviewing basic definitions and the bootstrapping theories of interest rate curve construction; section 3, linear and cubic spline interpolations for bootstrapping are introduced clearly; section 4, programming the interpolations in Excel VBA based on the same data of STIBOR¹; In section 5, analysis the results, and discuss the prospect for the curve fitting; the conclusion are collected in section 6.

¹ STIBOR: an abbreviation for Stockholm Interbank Offered Rate.

2. Some Definitions and Basic Theory

The curve construction is a complex subject, different kinds of interest rates and their computing ways lead to have various curve. Moreover, the instruments chosen, day count convention, and models for construction are all the important aspects. In the following subsections, the definitions and theories that related to this paper is presented.

2.1 Interest Rates

Simply, an interest rate is the rate of interest that a borrower paid to the lender for the use of the money. However, there are many kinds of interest rate in the market; some common and important interest rates are simple explained in below.

Simple rate, Compounding rate, Continuous compounding rate,

The Simple rate can be seen as the yield², expressed as percentages of the invested amount, the formula³: $\text{Investment} * (1 + r_{\text{simple rate}} * t)$. Compound interest: the interest is added to the principal and has been added also earns interest, the formula is $\text{Investment} * (1 + r_{\text{annual}})^t$. For compounding rate, the question of “How often the payments to be paid” is important. Let “f” be the period, and r_f be the interest rate of the period f, then $(1 + r_{\text{annual}})^t = (1 + \frac{r_f}{f})^{f*t}$. The relationship between the simple rate and the compounding rate is $(1 + r_{\text{annual rate}})^t = (1 + r_{\text{simple rate}} * t)$. The continuous compounding rate is: as “f” trends to “∞”, the formula of compounding rate becomes to $e^{r_{\text{continuous}}*t}$, so the continuous compounding is expressed as: $\text{Investment} * e^{r_{\text{continuous}}*t}$.

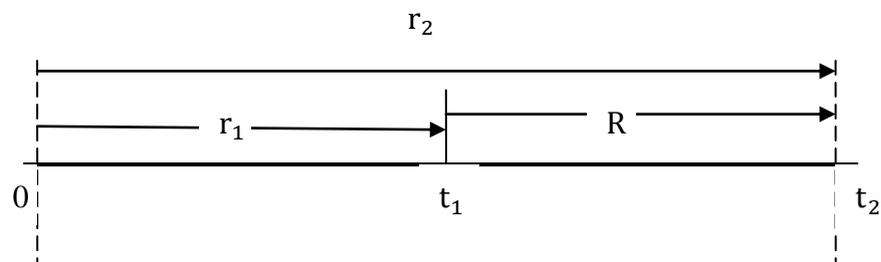
² Yield: the amount of in cash that returns to the owner of a investment.

³ “t” means time to maturity. It means the same in the following formulas in section 2.1

Zero rate, Spot rate, and Forward rate:

Zero rate does not mean the interest rate is “zero”, it means that the interest will be paid in once at the maturity date, such as the bond with zero-coupon rate. Spot rate is the rate start from today to at some future time, and defined as the theoretical profit given by a zero coupon bond. Forward rate is the rate starts from some day in the future t_1 , and ends in the future t_2 , and $t_1 < t_2$. It can be deducted from present interest rate by the method of the bootstrapping, see the example in Appendix 1. The forward rate formula:

Figure 1 The forward rate



Zero rate:

$$(1 + r_1 * t_1) * [1 + R * (t_2 - t_1)] = (1 + r_2 * t_2)$$

$$R = \frac{1}{t_2 - t_1} * \left(\frac{1 + r_2 * t_2}{1 + r_1 * t_1} - 1 \right)$$

Compounding rate:

$$(1 + r_1)^{t_1} * (1 + R)^{(t_2 - t_1)} = (1 + r_2)^{t_2}$$

$$R = \left(\frac{(1 + r_2)^{t_2}}{(1 + r_1)^{t_1}} \right)^{\frac{1}{t_2 - t_1}} - 1$$

Continuous rate⁴:

$$e^{r_1 * t_1} * e^{R * (t_2 - t_1)} = e^{r_2 * t_2}$$

$$R = \frac{r_2 * t_2 - r_1 * t_1}{t_2 - t_1}$$

Where t_1 is time from 0 to t_1 , and r_1 : interest rate for t_1
 t_2 : time from 0 to t_2 , and r_2 : interest rate for t_2
 R : forward rate from t_1 to t_2

⁴ In the paper, the formula of the continuous rate is applied, according to the market data.

To notice that, the forward rate is an estimation of the interest rate from time t_1 to t_2 that made today, however the forward rate may or may not equal to the real interest rate⁵. In fact, the forward rate provides an estimation of investors to grab the chance of making profits. How to get the best forward rate curve is always be the issue in the financial market.

Present Value, Discounting rate

The today's value of the future cash flow is called present value. Discount rate is the return of risk free within investment period. For instance, a year later is \$1.05, and its present value is \$1, the discounting rate of a year later is: $\frac{\text{Present Value}}{\text{Future Value}} = \frac{1}{1.05} = 95.24\%$. The Discounting rate is use for measure the present value of the future cash flows.

2.2 Instruments

There are lots of financial instruments that on the different market, some instruments used on the interest rate market are introduced. The Characteristics of the instruments affect the accuracy of the interest rate curve. In general, the most popular and well trade instrument in the competitive market is the best one to be chosen for constructs the interest rate curve.

Euro Dollar Future

Eurodollars are time deposits denominated in U.S. dollars at banks outside the United States, and thus are not under the jurisdiction of the Federal Reserve.⁶ A Eurodollar future is similar to a forward rate agreement to borrow or lend U.S. dollars for three months starting on the contract settlement date, actually it reflect the relatively London

⁵ The real interest rate: the exact interest rate in the real financial market. However, when the real interest rate is get, it is not useful for investors to get the opportunity for benefits, because the time is gone.

⁶ <http://en.wikipedia.org/wiki/Eurodollar>

Inter Bank Offer Rate⁷. The Euro Dollar future is one of the most popular and well traded interest rate products in the international financial market.

Deposits (Depos)

Cash Deposits are Over-The-Counter (OTC) contracts⁸ which start at reference date t_0 (today or spot) and paid out by banks or financial institutions in a specified fixed interest rate at the maturity date. Standard plain vanilla Deposits quote in the market are often starting at spot date and bears a various maturities up to one year.

Forward Rate Agreement (FRAs)

Forward Rate Agreement (FRA) is a forward starting contract between parties that fixes the interest rate to be paid or received on at a pre-determined future start date. The type of rates to be used, the notional value⁹ along with the maturity date are also be specified in the contract.

For example, Company A has entered into a 3X6 FRA¹⁰ with Bank B, assume in the Contract Company A agrees to receive a fixed rate of 6% in six month on a principle of 1 million SEK and Bank B agrees to receive the STIBOR rate on the same principle amount. After six month, if the STIBOR rate is higher than 6% (which is the fixed rate too), let us say 6.5%, then Bank B will pay Company A the differences which is 1 million * 0.5% =5,000 SEK; if the STIBOR is less than the fixed rate, the Company A is going to pay Bank B the differences.

Bond

Bond is the simplest fixed-income securities. When a bond is issued, the investor is

⁷ The Abbreviation is LIBOR is one of the Inter Bank rates that are decided by a number of investment banks on daily basis. In Sweden, the Inter Bank rate is STIBOR.

⁸ OTC contract is a bilateral contracts, it is between two parties that agree on how a trade or agreement to settle in the future.

⁹ The national value is the nominal or face amount that be used to calculate the payments made on financial instruments.

¹⁰ 3X6 FRA means that the contract is a six month agreement starting three month forward.

the essentially lending money to the issuer in exchange of interest payments and the promise of repayment of the principal at a future date. There are various bonds, such as Callable bonds, Put able bonds, zero-coupon bond, and bullet bonds and so on. The maturity of the bond is up to 30 years.

Interest Rate Swap

Briefly, a swap is contractual agreement between two parties in which they agree to make periodic payments to each other according to two different indices¹¹, typically a floating interest rate cash flow against a fixed rate cash flow.

Vanilla Swaps and Basis Swaps are the common swaps, the main difference between them is: the Vanilla Swaps are floating vs. fixed swaps admitting underlying rates with different tenors; the Basis Swaps are floating vs. floating swaps admitting underlying rates. The fixed - rate side of the Basis Swaps that can be seen as the sum of zero-coupon bonds, because the fixed swap pays the fixed interest payments.

2.3 Day Count Convention

In daily life, the details of calculating interest rate tenor seems trivial, however in financial area, especially in the field of fix-income finance and interest rate curve construction, all these details of date calculating is vital. Day count convention must be taken into consideration. What is mean by “Day count convention”? What kind situations that will happen? Let see the following parts.

Day count convention, is a complex subject, is a user-defined, instrument-specific parameter, and is must be used when valuing that particular instrument. If an investor saves some amount money at 30th of April, saving for one month, when is the maturity

¹¹Interest rate indices

date? 30th May or 31st May? Faced with problem, the rule of the end month date will be applied; the maturity date certainly is 31st May.

How many days in a year will be the same everywhere, however the “Business day¹²” is differ due to the various holidays in different nations. Which countries calendar should be used is significant of getting the right result. In habit, the initial of the city that the financial center of the county is located is used to represent the country. For instance, NYC (New York City) means the calendar of American, LDN (London) represents the calendar of Britain. Sometimes, it may use the combination of two calendars, NYC+LDN, this means will choose the business day for both countries.

In the trading of financial instruments, due to some practical issues, it may not use the real tenor to calculate interest, instead to use Day counting¹³. Day counting is a fraction, the numerator represent how to calculate days, the denominator represent how to set the days in a year. Some alternative day count conventions are: 30/360, 30E/360, Act/360, Act/365, NL/365 and Act/Act¹⁴ etc. How to choose the day count conventions depend on instruments and the personal judgment of trader’s.

In the real trading process, there are more rules required to following. Firstly, the maturity date and the pay date may not on the same day, the maturity date do not have to be on the business day, but the pay date has to be on the business day, if the date is not match, Day Shift Convention will be used to make the right pay date. Normally, Day Shift Convention will make the next business day to be the pay date. And Trade date and Spot date may not be the same date; the reason is that traders need time to

¹² It means the national working day.

¹³ Sometimes it is called basis.

¹⁴ 360 and 365 mean no matter the real days in a year, it is based on 360 or 365 days in a year. Act and NL represent act number of the calendar days in a year. 30E means account every month to be 30 days. 30 means account every month to be 30 days, except the later date is the last day of February, that month is considered to have its actual number of days; when the later date of the period is the 31st and the first day is not the 30th or the 31th, the month that includes the later date is considered to have its actual number of days.

make the exchange. Furthermore, the rate start day¹⁵ and the rate end day¹⁶ of the financial product, and the rate start day and the rate cut-off day of the index interest rate may not be the same. In the construction of the interest rate curve, the rate start day of the financial product and the index interest rate is the same, however, for instance, the forward rate curve has only, the rate start day and tenor, the end date is just a logical method in order to make it easier to be understood. Besides, the Reset Day, Reset Lead Days and Rate cut-off are also important in the financial trading¹⁷. In order to get the most accurate price for the financial products, the date matters in every single details of the trading process. In section 4, day count convention that used in the paper will be present precisely.

2.4 Curve Construction of Interest Rates

The interest rate curve provides the information to estimate the value of the financial instruments. The index curve and the discounting curve are two important kinds of the interest rate curve. The index curve can be the zero rate curve that the curve describes the relationship between the zero rates and the maturity, or the forward rate curve that represents the relationship between the forward rate and the maturity; it used to estimate the cash flow¹⁸. The Discounting curve is the relationship between discount factor and the maturity; it used to get the present value of the cash flow.

All the points on the interest rate curve are the estimation of the interest rate that is in the future standing on the today's point of view. Traders use their own interest rate curve to get the index curve, then to estimate the intrinsic value of products. That's why that the quality of the interest rate curve will directly affect the performance of the investment bank and traders. However, the question of "How to construct the

¹⁵ The data on that interest rate period starts.

¹⁶ The data on that interest rate period ends.

¹⁷ In the real financial market, these definitions are significant for getting the right rate from the interest rate curve. This paper is not going to talk about those in details.

¹⁸ Cash flow = rate from index curve * Tenor

interest rate curve?” and “how to judge the curve is good or not?” will be explained in the following parts.

Constructing the interest rate curves actually is a process of curve fitting, because the market prices of instruments only provide some information of the interest rate that is with specific time¹⁹. For example, if the Eurodollar future chosen to be the input of constructing the interest rate curve, then, within a year, the information that provides only four points exactly. Between these points, any curve could be made; no doubt, these is no standard answer exists for the curve fitting.

In the financial market, the most significant incident is the credit and liquidity crisis in 2008. Even now, the some countries still stuff the damage that was brought by the crisis. Before the crisis, non-zero basis swap spreads were already quoted and understood, however their magnitude was traditionally neglected because of their small value. After the crisis, due to the credit risk influence, the financial markets need to find a better methodology to price and hedge interest rate instruments, cause the instruments price depends on the present value of the future interest rate-linked cash flows, and the old method cannot reflect the intrinsic value of the financial products.

2.4.1 Characters of Interest Rate Curve

In short, the interest rate curves are relationship between rate and maturity. The discounting curve reflects the change of the discounting factor; today’s discounting factor must be “1”. The value of the discounting factor is decreasing as the time goes, so that the discounting curve tends to be decreasing as the maturity becomes larger. Traders can choose the relative discounting factor to estimate the value of the instruments. Normally, the bank use 3 month LIBOR rate curve to be the discounting

¹⁹ Some interest rate instruments have the standard maturities, such LIBOR rate.

curve.

The discounting factor can derivative only one the forward rate, however the forward rate cannot get the only one discounting factor. Because the discounting factor calculates the daily rate, the forward rate is an accumulation of the rate with the tenor. Obviously, it can deduct from parts to the sum; but cannot deduct from the total to get the only part without additional information.

Any one of the zero curve and the discounting curve is known, the other curves can be found, let us see the formulas. Define the function of the discounting factor²⁰: $df = e^{-r \frac{\text{days}}{365}}$, then the function of the forward rate²¹: $R = \left(\frac{1}{t}\right) \ln\left(\frac{df1}{df2}\right)$, and the formula of the zero rate²² will be

$$r = - \frac{\ln (df)}{\frac{\text{days}}{365}}$$

The discounting curve is defined everywhere, because no matter what the date is, interest has to be paid. The index curve is not defined everywhere, it based on the calendar and business day, due to it provides the rate of the interest rate. From the interest rate curve, the forward rate that implies with different tenor can be read.

2.4.2 The criteria for Judging a Curve Construction

Although the construction of the interest rate curve has no standard answer, there still are some basic standards to judge the curves.

- The interest rate curve must reflect the market price of the interest rate

²⁰ r: is the zero rate, days: is the actual days between valuation and maturity. Formula is in continuous compounding rate

²¹ df1: the discounting factor for t_1 , df2: the discounting factor for time t_2 , t: the time interval $t_2 - t_1$, and $t_2 \geq t_1$. R is calculated in continuous compounding rate.

²² r: is the zero rate, days: is the actual days between valuation and maturity. Formula is in continuous compounding rate

instruments. Apparently, if the curve that constructed cannot match the data of marker price, it must be the wrong curve. However, the curve also can not 100% reflect the market price; on some level, it is not necessary either²³. Overall, the interest rate curve should have consistency with the price of the active trading instruments, such as Eurodollar or Swaps.

- The curve must be smooth and continuous. To be clear, the smooth and continuous here is not defined in mathematically, but visually. Due to the interest rate change of market price will not have the significantly frequent fluctuation within a short time, the curve must be smooth. In fact, the interest rate maybe jumps, because the interest rate effect by the decision of the national financial institution, such as, the Federal Open Market Committee²⁴ (FOMC) can adjust the interest rate by the unit of 25 basis points²⁵. Therefore, the curve can have some jumps, but it still has the certain tendency, namely, smooth and continuous visually.
- The interest rate curve must be relatively stable, and it cannot be too complex. Market price of instruments is ever-changing, and the data of the price is used to be the parameters for constructing the interest rate. Indeed, the curve changes if the parameters change. In spite of that, the curve still cannot fluctuate too much if there is only the little change of market prices or relevant parameters. Moreover, if the market price changes, the new curve has to be constructed and displayed immediately. That is the reason why the method for curve construction is better to be simple. If there are two interest curves that constructed by different methods, and the competitive advantage is not obvious, the curve with relatively simple method will be chosen. Because there is no prove that the complex method can

²³ Some special instruments are affected by the some random factors, it is not necessary to make the curve fit 100%.

²⁴ It is a committee within the Federal Reserve System. It is the Federal Reserve committee that makes keys decisions about interest rates.

²⁵ It denotes an rate change in financial instruments, 1% change = 100 basis points (bp).

get the better curve to reflect the market price than the simple one.

Apart from the three judgments, the interest rates curves have to be analysis specifically combine with the methods for rates' algorithms, the instruments' character and the specific market situation.

2.4.3 Methodology for the Curve Construction

There is no standard answer for the curve construction, so that there is no standard shape for the interest rate curve. The data of the instruments market price provide some points on the curve directly or some series points of the summary information. Such as Swaps, the rates of 10years or 20years are given, but no rate given is between 10 to 20 years.

In order to fitting the curve, the interpolation²⁶ function has to be chosen. The issue of "How to chosen the interpolation method?" needs some more information and conditions: the summary information between the node points; the actual situation for the market; consideration of the specific implementation.

In mathematically, the interest rate curves can be generated into some functions, the discounting curve is a function²⁷ $df = f(t), t \geq 0$, and the index curve²⁸ is $rate = f(t), t \geq 0$ (or $f(t_1, t_2, T)$). Practically, the function of the interest curve is an interpolation or a combination of the series interpolation function.

There are two popular ways to construct the interest curve: price difference minimum

²⁶ Interpolation provides a method in the mathematical field of numerical analysis. Interpolation is applied to construct new data points within the range of a discrete set of known data points. Any function that satisfied the certain constrains can be used as an interpolation function.

²⁷ A function depends on the variable t, t means time

²⁸ The index curve is the function that depends on the variable t, or sometime it depends on more than one variable, such as t_1, t_2, T . In the section 3, the details of curve function is presentation in details.

and Bootstrapping. The former one is using some parameters and functions to deduct the price of the interest rate instruments, and then compare the results with the known market price of these instruments, by changing the value of the parameters to get the minimum difference between those two prices; the latter one is the bootstrapping method that mentioned in section 2.1.

However, no matter which method is used, there are some important procedures. Firstly, the instrument has to be decided; secondly, the interpolation function has to be chosen; after that, the market data and some parameters have to be set carefully. Any step would bring a lot of influence on the curve's performance. This paper focuses on the Bootstrapping method that will be explained in next section.

2.4.4 Bootstrapping

The method of solve the unknown variables gradually is called Bootstrapping, as it mentioned in section 2.1 for the deduction of the forward rate. However, the specific steps of Bootstrapping for the different instruments are not the same. Owing to using data of swap in section 4, the bootstrap model for the swap is presented here.

Constructing the swap curve by the bootstrapping, the data of the swaps with different maturities will be taken. Sometimes the par swap rate can be read directly from the market data; otherwise interpolation method has to be applied.

Suppose the swaps with maturity time T_i , where $i = 1, 2, 3 \dots n$. By using the formula below to find the discounting factor curve,

$$r^{\text{par}}(T_n) = \frac{1 - df(T_n)}{\sum_{i=1}^n df(T_i)}$$

$r^{\text{par}}(T_n)$ is the par swap rate with maturity T_n , df is the discounting factor.

If the par swap rate is given, the discounting factor is calculated by using the par swap

rate, for the first discount factor, is:

$$df(T_1) = \frac{1}{1 + r^{\text{par}}(T_1)}$$

After gets the first n-1 discounting factor, the nth one is:

$$df(T_n) = \frac{1 - r^{\text{par}}(T_n) \sum_{i=1}^{n-1} df(T_i)}{1 + r^{\text{par}}(T_n)}$$

Follow the process, all the rates that are needed for the curve construction can be got.

When the par swap rate is not given, the interpolation has to be used. In section 3, the interpolations is presented and discussed.

2.5 Details and Practical Problems

Although the general idea for curve construction is clear, this is not enough for using in real financial world, let us check some practical problems.

Actually, the interest rate in the market is not the same for every day in a year, because the bank is not willing to lend money during same season or special holiday, Certainly, the interest rate will increase in these periods. However, the curve of interest rate only has the general information, and then the spread²⁹ will be proposed to fill the gap of interest difference. Sometime the spread adjustment would change the performance of the whole interest curve.

As we all know, the well traded (no arbitrage) instrument in the financial market will be the best choice for curve construction. If, in some immature financial market, there is no arbitrage free instrument that can be chosen to construct curve, what can we do? In fact, the discounting curve will be chosen to be the main curve for value the price,

²⁹ The difference between the bid and ask price of the instrument.

because this curve do not affected by the market price, even the different financial institutions have their own way to make the discounting curve.

Theoretically, the construction of interest rate curve is simple; however the hard part is that this has no standard answer and lots of unsure problems in practice. The financial market is an increasable innovation industry, new problems come endless; all these bring more trouble to get a good interest rate curve. No matter how hard, the construction of the interest curve is still based on the choice of instruments, methods, parameters, and market data. In the next section, some methods of Interpolation will be listed and presented; they will bring more and different ideas for construction curves.

3. Interpolation

To construct the interest rate curve, the method and the function of the curve have to be decided. The numerical calculation cannot be avoided due to the complex and lengthy process. There are interpolation and extrapolation methods, the main research of this paper is the interpolation because almost the all the interest curves sustain the interpolation. The several interpolation methods will be introduced in following parts. Some of those interpolations should be intimately connected to the bootstrapping. In general, we gather the interpolation into two classes, the linear interpolation that the classical one and the cubic interpolation that is the complex one. Certainly, there are more interpolations, such as the Step interpolation, the Nonlinear interpolations and so on. But they will not be mentioned more, the reason is that these interpolations are not match the goal of the paper,

3.1 Linear Interpolation

The linear method can be applied to discount factor, zero rate, logarithm of the discount function or logarithm of zero rate, it depends on the preference of the user. The most popular character of the linear interpolation is easy and simple, and the function only depends on two surrounding points that defined the locality. There are several linear interpolation methods; the linear method discussed here is only the linear interpolation on discounting factors, and then based on the discounting curve to estimate $r(t)$: the function of interest rate r depends on time t .

Given time $t \in (T_1, T_n)$, and assume that $T_i \leq t \leq T_{i+1}$, suppose the discounting factor $df_1, df_2 \dots \dots, df_n$ are known and $df_i = df(T_i)$. The interpolation formula of this method is:

$$df(t) = df_i + \frac{t - T_i}{T_{i+1} - T_i} (df_{i+1} - df_i) \quad (1)$$

This method is quite popular because of its simplicity; however it has been proved as poor method by many users due to the graph of the interest curve is not smooth by applying in the linear interpolation.

3.2 Cubic Spline Interpolation

The fundamental idea behind cubic spline interpolation is based on the engineer's tool used to draw smooth curves through a number of points³⁰ – numerical data. The coefficients on the cubic polynomial used to interpolate the data. And these coefficients can 'bend' the line so that it passed through each of the data points without any erratic behavior or breaks in continuity.³¹ There are three cubic splines, in the paper, Clamped Cubic Spine, Monotone Preserving Cubic Spine, and Monotone Preserving Cubic Spine. These three Cubic Splines are typical and interesting we think;

³⁰ <http://online.redwoods.cc.ca.us/instruct/darnold/laproj/fall98/skymeg/proj.pdf>

³¹ <http://online.redwoods.cc.ca.us/instruct/darnold/laproj/fall98/skymeg/proj.pdf>

they maybe help us to get the better curve.

The same as the linear interpolation, given time $t \in (T_1, T_n)$, and assume that $T_i \leq t \leq T_{i+1}$, assume T_1, T_2, \dots, T_n and r_1, r_2, \dots, r_n are known, and r is the zero rate. Here for the cubic spline, the coefficients (a_i, b_i, c_i, d_i) is defined, where $1 \leq i \leq n - 1$. In order to find the function value at time “t”, the cubic polynomial in below needs to be satisfied,

$$S_i(t) = a_i + b_i(t - T_i) + c_i(t - T_i)^2 + d_i(t - T_i)^3, \quad T_i \leq t \leq T_{i+1} \quad (2)$$

for $i = 1, 2, 3, \dots, n - 1$

And, define $r(t)$ is the functional form of the polynomial, then,

$$r(t) = a_i + b_i(t - T_i) + c_i(t - T_i)^2 + d_i(t - T_i)^3, \quad T_i \leq t \leq T_{i+1}$$

for $i = 1, 2, 3, \dots, n - 1$

To Notice, each equation will has 4 unknowns, and for $n - 1$ equations, there are $4n - 4$ unknowns in total. In order to get the coefficients (a_i, b_i, c_i, d_i) , some more constraints are needed.

The first and second derivatives of the polynomial are the frequently used in the process, and given by,

$$S_i'(t) = b_i + 2c_i(t - T_i) + 3d_i(t - T_i)^2, \quad T_i < t < T_{i+1}$$

$$S_i''(t) = 2c_i + 6d_i(t - T_i), \quad T_i < t < T_{i+1}$$

$$S_i'''(t) = 6d_i, \quad T_i < t < T_{i+1}$$

for $i = 1, 2, 3, \dots, n - 1$

There are four common constraints for cubic methods in below:

1) The spline can interpolate all the data and therefore, when $T = T_i$

$$S_i(t = T_i) = r(T_i) = a_i, \quad \text{for } i = 1, 2, 3, \dots, n - 1$$

2) The entire spline must be continuous on the interval $[T_1, T_n]$

$$S_i(T_i) = S_{i+1}(T_i) , \quad \text{for } i = 1, 2, 3 \dots n - 2$$

3) Ensure the first derivative of the spline is continuous on the interval $[T_1, T_n]$

$$S'_i(T_i) = S'_{i+1}(T_i) , \quad \text{for } i = 1, 2, 3 \dots n - 2$$

4) Ensure the entire spline is twice differentiable on the interval $[T_1, T_n]$

$$S''_i(T_i) = S''_{i+1}(T_i) , \quad \text{for } i = 1, 2, 3 \dots n - 2$$

The constraint 1) gives n constraints, and from 2) to 4) provide $3(n-2)$ constraints. Thus, there still have $4n-6$ constraints with $4n-4$ unknowns, leaving 2 degrees of freedom for choosing the specific cubic method.

3.2.1 Clamped Cubic Spline

Clamped cubic method defined the two extra constraints are:

$$S'_1(t = T_1) = r'(T_1)$$

$$S'_{n-1}(t = T_n) = r'(T_n)$$

They are also called fixed boundary conditions.

Other choice of extra constraints are possible, it all depends on the cubic interpolation method the user choose, for instance, the Natural cubic spline is the same as the clamped cubic method , except the 2 extra constraints become the second derivatives at each end point equal 0 .

3.2.2 Monotone Preserving Cubic Spline

This method is first introduced by James M. Hyman in 1983, based on the method described by Fritsch & Butland in 1984.

As defined, T_1, T_2, \dots, T_n and r_1, r_2, \dots, r_n are known,

$$h_i = T_{i+1} - T_i \quad (1 \leq i \leq n - 1)$$

$$m_i = \frac{a_{i+1} - a_i}{h_i} \quad (1 < i < n - 1)$$

This method determines b_i , where $1 < i < n$, in a specific way, from Hyman 1983, we know,

$$b_1 = \frac{(2h_1 + h_2)m_1 - h_1m_2}{h_1 + h_2}$$

$$b_n = \frac{(2h_{n-1} + h_{n-2})m_{n-1} - h_{n-1}m_{n-2}}{h_{n-1} + h_{n-2}}$$

We now define b_i , where $1 < i < n$, When the data are locally monotone, that is, if $m_{i-1}m_i \geq 0$, we define

$$b_i = \frac{3m_{i-1}m_i}{\max(m_{i-1}, m_i) + 2\min(m_{i-1}, m_i)} \quad (1 < i < n)$$

Where the Hyman adjustment (Hyman in 1983) also need to be introduced as following,

$$b_i = \begin{cases} \min(\max(0, b_i), 3\min(m_{i-1}, m_i)) & \text{if } \min(m_{i-1}, m_i) > 0 \\ \max(\min(0, b_i), 3\max(m_{i-1}, m_i)) & \text{if } \max(m_{i-1}, m_i) < 0 \\ 0 & \text{if } m_{i-1}m_i < 0 \end{cases}$$

When the data fail to be locally monotone, a turning point exists. In order to retain piecewise monotonicity, define that,

$$b_i = 0$$

The algorithm for calculating c_i and d_i is as following,

$$r'(t) = b_i + 2c_i(t - T_i) + 3d_i(t - T_i)^2$$

If let $t = T_i$, then get

$$r'(T_i) = b_i$$

If let $t = T_{i+1}$, then get

$$r'(T_{i+1}) = b_i + 2c_i h_i + 3d_i h_i^2 = b_{i+1} \quad (3)$$

c_i and d_i can easily be solved from (3), as for each i , there are two equations with two unknowns. Therefore, for each i , following relations are holed,

$$c_i = \frac{3m_i - b_{i+1} - 2b_i}{h_i} \quad (4)$$

$$d_i = \frac{b_{i+1} + b_i - 2m_i}{h_i} \quad (5)$$

Above relations (4) and (5) are held for any kind of cubic interpolation method

3.2.3 Monotone Preserving Cubic on Logarithm of Zero Rates

This method is a development of the monotonic preserving cubic method introduced in the last section. The only difference between this method and monotonic preserving cubic method is that the cubic spline is applied not to $r(t)$ but to the function $\log r(t)$. As before, T_1, T_2, \dots, T_n and r_1, r_2, \dots, r_n are known, so the interpolating function becomes,

$$\log r(t) = a_i + b_i(t - T_i) + c_i(t - T_i)^2 + d_i(t - T_i)^3 \quad T_i \leq t \leq T_{i+1}$$

$$r(t) = e^{a_i + b_i(t - T_i) + c_i(t - T_i)^2 + d_i(t - T_i)^3} \quad T_i \leq t \leq T_{i+1}$$

for $i = 1, 2, 3, \dots, n - 1$

The way to calculate coefficients (a_i, b_i, c_i, d_i) is definitely the same as monotonic preserving cubic method.

3.3 Summary

Obviously, the theory of the linear interpolation is much easier than the theory of the cubic spline interpolation, and the algorithms of cubic spline are complex because of the coefficients and constrains. However, the cubic spline provides the various methods for getting the smooth curve; they may be practical in real world. In order to evaluate the interpolation methods, the Excel VBA is applied to programming the different methods by using the data from the real financial world.

4. Application in Excel VBA

Excel VBA (Visual Basic for Applications) is an implementation built into Microsoft Office applications, and it enables users to build own defined functions and automate processes. According to the requirement of using plenty of market data, programming complex methods, and showing the various graphs of curves, VBA (in Excel) is one of the best applications software for us to work on this thesis.

In the previous chapters, the procedure of curve construction has been introduced; let us follow the procedure to make the construction step by step: the instrument of Swap in Swedish has been chosen; the parameter and data are got from the Swedish financial market; interpolation methods that were mentioned in section 4 is programmed in VBA. The following subsections present the reasons and details for the important procedure.

4.1 Instruments Selection

A very important step in bootstrapping is to select the reasonable instruments. Interest rate market nowadays is very complex; different interest rate markets have different internal dynamics, liquidity, associated with different underlying tenors can tell very different story of the distinct market. There are many instruments can be chosen to bootstrapping a curve, but not all the instruments has different maturities , and usually they can only cover parts of bootstrapping time interval. Therefore we sometimes choose several similar instruments on the same underlying to bootstrapping one curve that can present all the necessary information.

In our thesis, we only focus on the Swedish market, because it is convenient to get the market data. Therefore the currency will be Swedish kroner (SEK) throughout in the

programming. Referring to bootstrap a zero-coupon curve, the most liquid instruments in the current market is used, because instruments are more liquid, the curves are more accurate to reflect overview of the distinct market.

4.2 Parameters and Data

In order to construct the curve that has the rate information up to 30 years, we choose an over-night rate (O/N), a tomorrow-next rate (T/N), cash deposit rate for one week and one to three month maturities for the short term of the curve; and then use some Forward Rate Agreements (FRA) for the medium term because FRA are more liquid than the short term swaps. Finally, we use the swaps form 4 to 30 years to bootstrap the long term for the curve. The set up data in our program is showing in the tables below:

Table 1 Value of Parameters

Parameter	Value
Calendar	SWE
Days per Year ³²	360/365
Interpolation	Linear/Cubic
Discounting	Continuous
Spot Days	2
ForwardCalc	Continuous
Start day	2012-04-03

The business days and holidays are all following Swedish calendar in our program, we choose to use 365 per year and the interpolation focus on linear and cubic interpolation methods. All the rate calculations are under the continuous interest rate condition throughout. The start day of the market data in our program is 2012-04-03, and the spot days are settled to be 2 days.

³² According to some quoted data, 360 are applied. In this paper, 365 are applied.

Table 2 Cash deposits with maturities O/N, T/N, 1W, 1M, 2M, 3M

Deposit		
Cash		
Maturity	Bid (%)	Ask (%)
O/N	1,4000	1,6000
T/N	1,6280	1,6280
1W	1,7450	1,7450
1M	1,9900	1,9900
2M	2,1000	2,1000
3M	2,2550	2,2550

Table 3 OMX STIBOR FRA 3M with maturities on the IMM days³³

Short Fut		
OMX STIBOR FRA 3M		
Maturity	Bid (%)	Ask (%)
jun-12	2,1200	2,1400
sep-12	2,0075	2,0275
dec-12	2,0113	2,0313
mar-13	2,0350	2,0550
jun-13	2,0575	2,0875
sep-13	2,0988	2,1288
dec-13	2,1438	2,1738
mar-14	2,1988	2,2288
jun-14	2,2483	2,2883
sep-14	2,3067	2,3467
dec-14	2,3617	2,4017
mar-15	2,4167	2,4567

³³IMM days: International money market days, which are the third Wednesday in March, June September and December.

Table 4 Swap rates from SEK STIBOR A 3M

Swap SEK STIBOR A 3M		
Maturity	Bid (%)	Ask (%)
4Y	2,2625	2,3125
5Y	2,3500	2,4000
6Y	2,4350	2,4850
7Y	2,5050	2,5550
8Y	2,5575	2,6075
9Y	2,5975	2,6475
10Y	2,6350	2,6850
12Y	2,6900	2,7500
15Y	2,7550	2,8150
20Y	2,7250	2,8050
25Y	2,6750	2,7550
30Y	2,6250	2,7250

The bid and ask prices list in the table 2, table 3 and table 4 with maturities equal or under 3months, between 6 months to 3years, and between 4 to 30 years. According to the different tenors differ in the algorithms; there are presented in details in the next section.

4.3 Method in VBA

The interpolations are the main methods for this paper, and they are clearly introduced in the section 3. However, it is no enough that to get the result if only the interpolations are programmed in VBA. There are some methods for solving the detail problems that seems trivial but vital. The methods in the following paragraphs bring a

lot of influence for getting the accurate results.

In the Swedish market all the instruments are quoted in yield, so the market quotes are the same as the rates and the quotes (par rates) are calculated as the 1/2 of the sum of the Bid and Ask rates,

$$r^{\text{par}} = \frac{\text{Bid rate} + \text{Ask rate}}{2}$$

For the cash deposits, when calculate the discounting factors and the zero rate, we start by using the O/N rate, therefore the cash deposits $D_{O/N}$ is,

$$D_{O/N} = \frac{1}{1 + r_{O/N}^{\text{par}} * \frac{d_{O/N}}{360}}$$

Here “d” represents actual days between valuation and maturity.

The zero rate $Z_{O/N}$ with continuous compounding is given by,

$$Z_{O/N} = -100 * \frac{\ln(D_{O/N})}{\frac{d_{O/N}}{365}}$$

For the T/N rate,

$$D_{T/N} = \frac{D_{T/N}}{1 + r_{T/N}^{\text{par}} * \frac{d_{T/N}}{360}}$$

$$Z_{T/N} = -100 * \frac{\ln(D_{T/N})}{\frac{d_{T/N}}{365}}$$

With other cash deposits,

$$D_i = \frac{D_{T/N}}{1 + r_i^{\text{par}} * \frac{d_i}{360}}$$

$$Z_i = -100 * \frac{\ln(D_i)}{\frac{d_i}{365}}$$

Where $i=1$ week, 1M, 2M, 3M.

Now let us see the FRA rates. All the FRA rates are the 3M forward rates, so the yield quoted are forward rates, then we need a value called “Stub” - this rate just serve as a

yield of a instrument that have a maturity the same as the start date of the FRA we have in our data, therefore we can get the stub rate by using linear interpolation:

$$r_{\text{stub}}(t) = r(T_1) + \frac{r(T_n) - r(T_1)}{T_n - T_1} (t - T_1)$$

$$D_{\text{stub}} = e^{-Z_{\text{stub}} * \frac{T_{\text{stub}}}{365}}$$

Here, T_1 stands the start date and T_n stands the end point date in our data.

The other formulas for FRA :

$$D_i^{\text{FRA}} = \frac{D_{i-1}^{\text{FRA}}}{1 + D_i^{\text{FRA}} * \frac{d_i^{\text{FRA}}}{360}}$$

$$Z_i^{\text{FRA}} = -100 * \frac{\ln(D_i^{\text{FRA}})}{\frac{d_i^{\text{FRA}}}{365}}$$

Where $D_0^{\text{FRA}} = D_{\text{stub}}$ and i = maturity of each FRA contract, they are {jun-12, sep-12, dec-12, mar-13, jun-13, sep-13, dec-13, mar-14, jun-14, sep-14, dec-14, mar-15} in our case.

In order to complete the bootstrapping process, we need to deal with the swap rates, however the swaps start from 4th year in our data, therefore we have to approximate the swap rate for 1-3 year by interpolation firstly, see table 5.

Table 5 Approximation of the swap rate with Maturity 1-3 years

Year	Maturity	DF	Zero Rate
1	2013-04-05	0,979087105	2,101949163
2	2014-04-07	0,958355231	2,115247862
3	2015-04-07	0,936241479	2,188068553

For calculating the discounting factor from the 4th year,

$$D_i^{\text{Swap}} = \frac{D_{T/N} - r_i^{\text{swap}} \sum_{t=1}^{i-1} D_t^{\text{FRA}}}{1 + r_i^{\text{swap}}}$$

Here, r_i^{swap} is the calculated par rate of the i th swaps, and $i = \{4y, 5y, 6y, 7y, 8y, 9y, 10y, 12y, 15y, 20y, 25y, 30y\}$.

The Zero rates is also given by the formula as before,

$$Z_i^{\text{swap}} = -100 * \frac{\ln(D_i^{\text{swap}})}{\frac{d_i^{\text{swap}}}{365}}$$

Where $i = \{4y, 5y, 6y, 7y, 8y, 9y, 10y, 12y, 15y, 20y, 25y, 30y\}$.

In our data, the market swap rates are missing in 11, 13, 14, 16-19, 21-24, 26-29 year, So for the year 11, 13, 14, 16-19, 21-24, 26-29, we need to use interpolation methods to find the Zero Rate or Discounting factor and then calculate the other ones.

The interpolation methods we used in our program are Linear on discounting factor , Clamped cubic , Monotonic Preserving Cubic , Monotonic Cubic on log zero rates, and the algorithms of these interpolation methods has been introduced very detail in the previous chapter.

4.4 Results

Based on the data of the Swedish market with various tenors between O/N to 30 years (from 2012-04-03 to 2042-04-05), the zero curve, the forward curve and the discounting curve are shown in the following figures.

As shown on figure 1, figure 2, figure3 and figure 4, no matter which interpolation is applied, the zero curves are much smoother than the forward curves, and the forward curves are above the zero curves around the first 15 years, then the forward curves start to dropping and meet the zero curves in the middle years, after that the forward curve is below the zero curve.

Figure 2 Linear interpolation on logarithm of zero rates

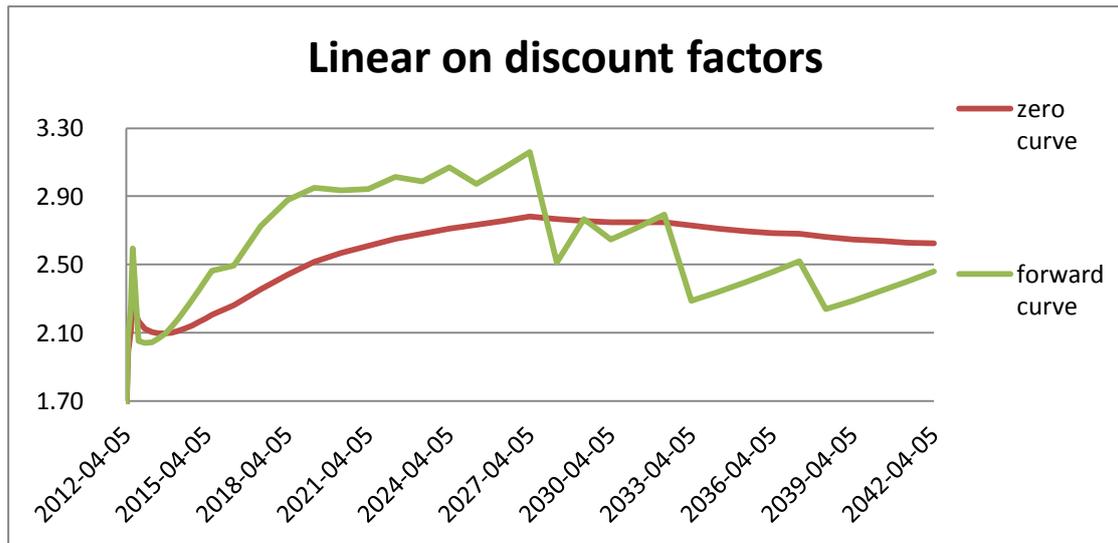


Figure 3 Clamped cubic spline interpolation

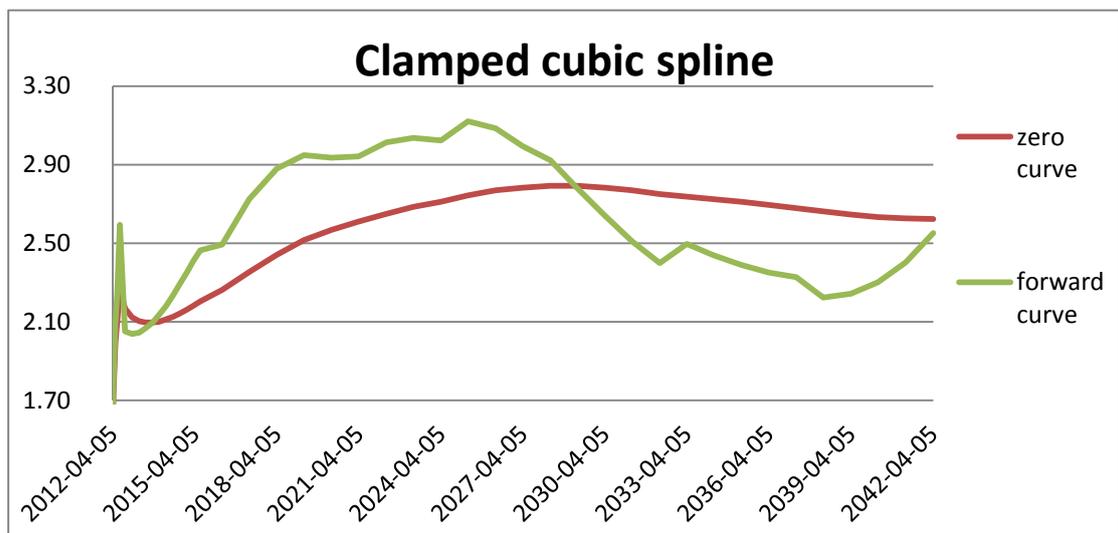


Figure 4 Monotonic preserving cubic interpolation

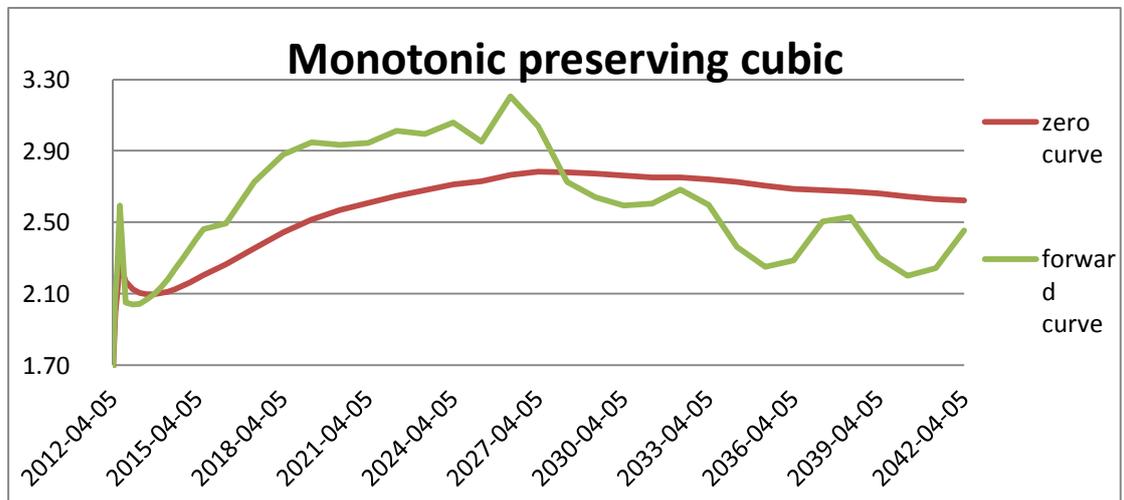
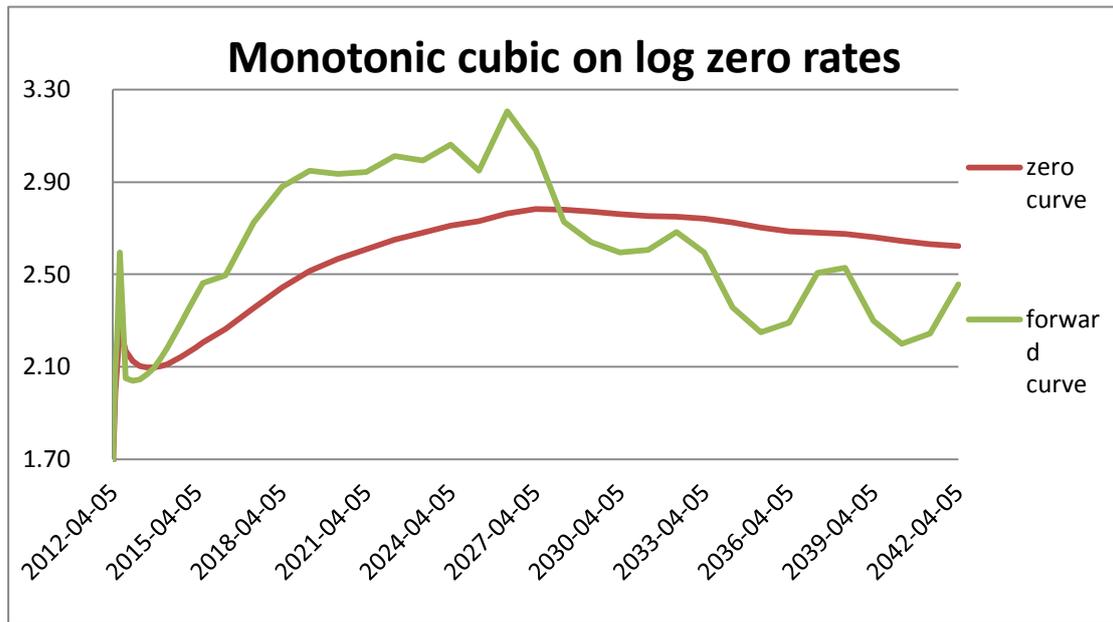
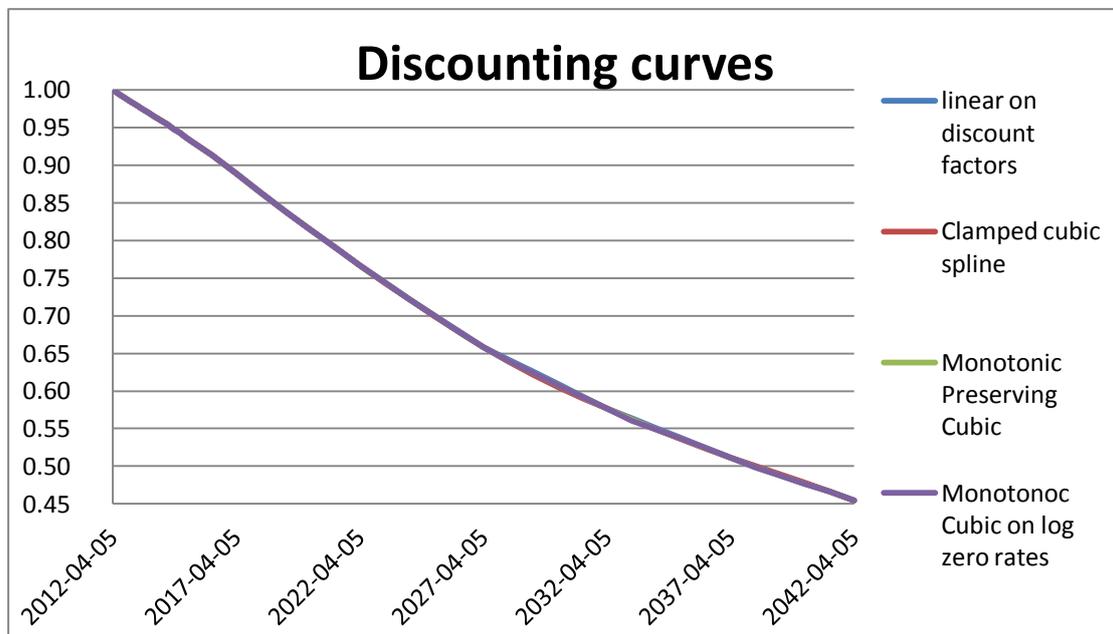


Figure 5 Monotonic cubic on logarithm interpolation



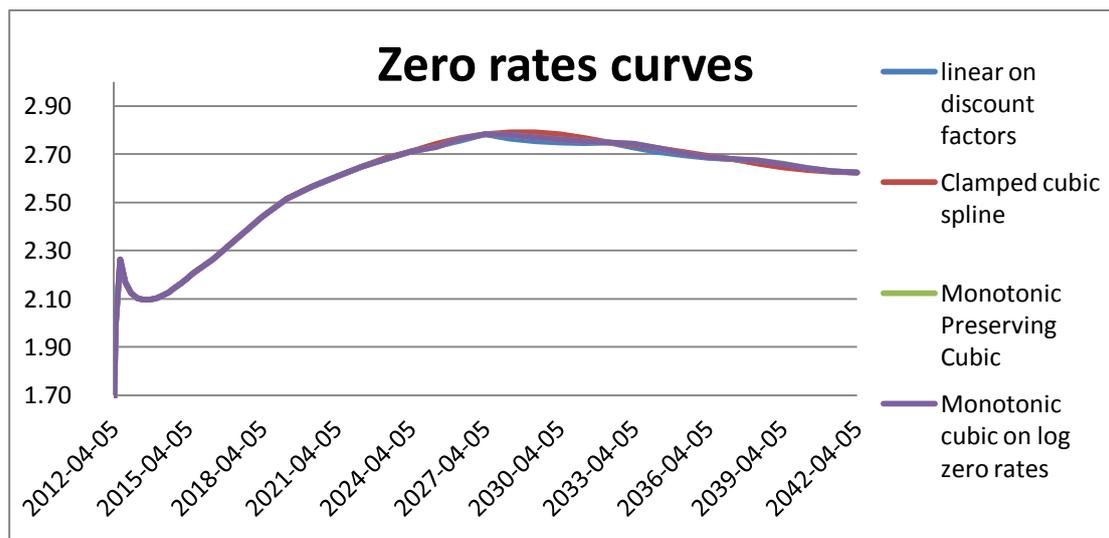
It is clearly to see in figure 5, the discounting curves almost completely overlap, although they are calculated by different interpolations.

Figure 6 Discounting curves from four different interpolations



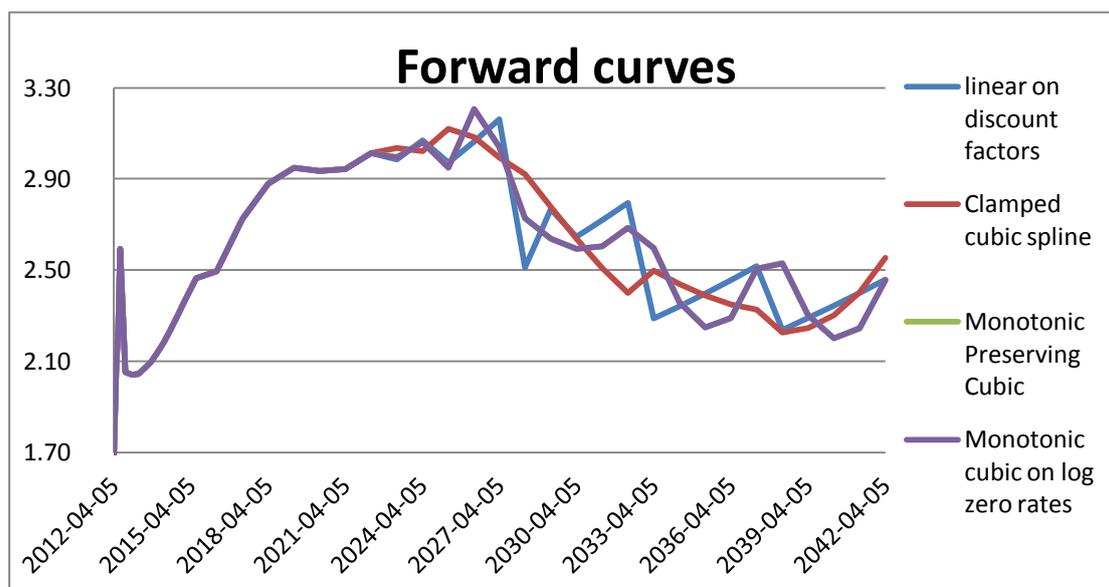
Comparing the zero curve by different interpolations, there is nearly no difference in the first 15 years, but a little differ in the last 15 years, especially, the zero curve is constructed by the clamped cubic spline.

Figure 7 Zero rate curves from four different interpolations



Obviously, the forward curves differ from the different interpolation methods; the difference starts around year 2022, and becomes more complex in the following years. To notice that, the forward curves form the monotonic preserving cubic and the monotonic cubic on logarithm nearly overlap, see figure 7. However they are not exactly same if checking the results from the Excel file “Curve Construction, xlsm”.

Figure 8 Forward curves from four different interpolations



The figures only show the general information of the curves, the analysis is needed to judge that which curve is the best-fit curve and which method is the best interpolation under this situation.

5. Analysis

5.1 Comparison of Results

All the curves from section 4 are span of 30 years, from 2012 to 2042. However, the interpolations are applied to the last 20 years, due to the market quotes are started to missing in the 11th year, namely the year 2023. That's why the obvious difference of the curves starts to display around the year 2023. When applying the methods, zero rates or discount factors are the data that we used to apply in interpolation methods to complete missing information of the market. Obviously, zero curves or discount curves are the first curves we got from different interpolation methods, but the zero rate curves only have slightly difference by using different methods to interpolate them.

The reason for the discounting curves are very similar is: the lacking market data of discounting rates are all calculated from the formula³⁴: $df = e^{r \cdot \frac{\text{days}}{365}}$. Based on the similar interpolated zero rates from the cubic spline methods, the discounting rates become more similar according to the property of the exponent function.

Forward curves are fluctuating a lot from years 2023 to 2042. The reasons for these fluctuations are the algorithm³⁵: $R = \text{forward}(t_1, t_2) = \frac{r_2 \cdot t_2 - r_1 \cdot t_1}{t_2 - t_1}$. For instance, the zero rates of year 2028 and 2029 given by linear interpolation is 2.7660% and 2.7550%, and the zero rate in the same year interpolated by clamped cubic are 2.7916% and 2.7908%, there is only small difference between (0.025% and 0.036%) the two years zero rates. However, when put these two rates in the forward formula, the difference of the forward rate given by these two methods become 0.246%. It shows

³⁴ From the section 2.4.1

³⁵ From the section 2.1

that the small difference in the zero rates leads to a bigger difference in the forward rates.

In addition, the zero curves and forward curves given by the two monotonic cubic splines are almost overlapped. As the name suggests – Monotonic cubic method and Monotonic cubic on logarithm of zero rates, they share the same specific conditions on the cubic spline, the only difference between them is where to apply the cubic spline, one is directly on zero rates and the other is applied on logarithm of zero rates. Due the similarity in the algorithm of the two monotonic cubic, the forward curves look similar.

5.2 Best-fit Curve

Based on the choice of Instruments, the four interpolations are applied to work the data of Swedish swap. As the curves that got, “How to choose the Best-fit curve?” is the key issue. The discounting curve and the zero rate curve share the similar shapes and the values do not change so much, while the forward rate curves have obvious differences when comes to different interpolation methods. That is why when analysis the best-fit curve, we mainly focus on the forward rate curve. Following sections will check the smoothness, continuity, stability, locality and some practical perspectives of this four interpolation methods.

5.2.1 Smoothness and Continuity

That is reviewing some criterions in section 2.4.2, most curves from the graph are smooth and continuous visually; the results display the smoothest forward curve is the curve from the Clamped cubic spline. As described in section 3.2, the algorithm of cubic spline interpolation is that time interval for interpolation has been divided into

many small subintervals. Each subinterval is interpolated by using a third-order polynomial, of which the coefficients are having certain constraints. Certainly, different cubic interpolation methods have some different constraints as introduced in section 3.2, but the general conditions of cubic spline are function continuity and passing through all given data points. According to the algorithm, all the cubic methods will guarantee the continuity and certain smoothness of the curves, while the linear interpolation method as a simple method has no such algorithm to ensure its smoothness and continuity; therefore it suffers big difficulties on smoothness and continuity.

5.2.2 Forward Stability

To check the stability, let us change the bid price of swap with maturity 10 year from 2.653% to 2.735%, 0.1% which is 10 basic points ,the comparison of the value changing showing in the table below.

Table 6 Forward rate (%) stability comparison of different interpolations

Interpolation methods	Linear on discount factors	Clamped cubic	Monotonic preserving cubic	Monotonic cubic on zero rates
Maximum changed value	0,297708595	0,503756031	0,318698937	0,320727382
Minimum changed value	4,93897E-05	2,51284E-05	0,000187807	0,000199342
Average changed value	0,029171485	0,041636736	0,029416298	0,029416249

As shown in the table 6, the maximum changed value when the market price has changed 10 basis points is 0,503756031 given by clamped cubic interpolation, and

also the averaged changed value of clamped cubic interpolation is the maximum of the four interpolation methods. Although the clamped cubic interpolation gives the smoothest curve, it also has the poorest forward stability. Other three methods present the similar changes; besides, the linear interpolation on discount factor has the best forward stability compare to other three methods.

5.2.3 Locality

In order to find the locality of these methods, let us change the market price at point $t = t_i$, and then define the time interval (t_{i-j}, t_{i+k}) which will be influenced when market price of point $t = t_i$ is changing. The method has the influenced time interval as small as possible.

Table7 Locality of different interpolation methods

Interpolation method	Influenced Time interval
Linear on discount factors	(t_{i-1}, t_{i+1})
Clamped cubic spline	(t_1, t_n)
Monotonic preserving cubic	(t_{i-2}, t_{i+1})
Monotonic cubic on log rates	(t_{i-2}, t_{i+1})

Among these three cubic methods, the clamped cubic method is not attractive at all concerning to the locality; the Monotonic preserving cubic and the Monotonic cubic on log rates influence in the same interval when change the market price at point t_i . Present, they also have relatively good locality. The simplest method - linear on discount factors has the best locality.

5.2.4 Practical Issues

The last several sections have examined the interpolation method directly from the numerical results. However, these are not enough to judge the best-fit curve, when it comes to the real world trading. Here some practical issues have to be considered.

Actually, the time cost for constructing the curves really matters in the real financial world; otherwise traders cannot grab the opportunity to get the best benefits, even if the interest rate curves are perfect. Concerning smoothness and continuity, no doubt the cubic spline interpolations can handle data better than linear interpolation. The two monotonic cubic methods present similar forward stability and locality, and better than the linear interpolation. However the simple method may take less time to make the curve, such as the linear interpolation can be done by simply by hands. Although it may display discontinuity and some mispricing sometimes, so far due to its convenience, linear interpolation is quite popular in the trading process.

5.2.5 Summary

In general, the curve that constructed by clamped cubic spline is the smoothest one, but not attractive when concerning to locality and forward stability. The two monotonic cubic methods can present good forward stability and locality with the entire forward curve continuity and relatively smooth. The linear interpolation is the simplest method and most practical one in the real trading process, it gives the forward curve which lack the important continuity and smoothness.

In order to choose the best interpolation for the instrument, it is still required fit the special situations. Well understand the interest rate market and familiar with the instruments will help the traders to choose the best method for get the best-fit curve.

5.3 Perspective

Nowadays, there is still no standard answer for the best-fit curve and its method for construct the curves. Interpolation, no matter what, cannot say it completely reflect the real world data. Research on “How to get the best-fit curve” is keep going by financial professionals.

Today, based the Overnight-Index-Swap (OIS) discounting, integrated Credit Value Adjustment (CVA), or some spreads, some new interest rate modeling framework is evolving. Due to the increasing of the counterparty credit risk, the CVA takes into account the likelihood the counterparty may default, along with expected exposures, volatility of these expected exposures, and wrong way risk. No matter how many the uncertainties in the real financial market, transactions will never stop. There are always some other techniques or new agreements starts to be made and become popular.

6. Conclusion

To conclude, the linear interpolation is the simplest method but lack of continuity and smoothness. The curve constructed by clamped cubic spline is the smoothest one, but it has poor locality and stability. The two monotonic cubic methods present relatively good stability and locality with the continuous and relatively smooth curve. Anyway, eyes of the beholder, and wise see wisdom, there is no standard answer for curve construction. The research on the best-fit curve is still continuing, some new techniques maybe found in the future.

7. References

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8. Appendixes

Appendix 1

Let's see a simple example below using Bootstrapping method for deduct the forward rate.

At 1st February, the bank is made two actions:

1. Saving \$1 million with $r_{\text{annual}} = 6\%$, $T = 2$ month;
2. Lending \$1 million with $r_{\text{annual}} = 12\%$, $T = 5$ month.

The net cash flows at 1st of February are zero. After 2 month (at 1st April), the first action is at maturity, the bank gets income $1.01\text{million} = 1\text{million} * (1 + 6\% * \frac{2}{12})$.

After 5 month (at 1st July), the bank pays $1.05\text{million} = 1\text{million} * (1 + 12\% * \frac{5}{12})$.

We think in the way that the bank borrows 1.01million at 1st April and pays 105 at 1st July. The rate for this 3months loan will be $15.84\% = \frac{40000}{1.01\text{million}} / \frac{3}{12}$. However, the rate is calculated at 1st February, we can say that 15.84% is the forward rate of the month loan that starts after 2 months. This way that we call "Bootstrapping": using the present zero rate to get the forward rate.

Appendix 2

Code for Interpolations

' interpolation och extrapolation

Function IPOLr(x As Date, pX As Range, pY As Range, method As Integer) As Double

Dim j, i, N As Integer

Dim IX() As Date

Dim IY() As Double

N = Application.Count(pX) ' Length of the ranges

ReDim IX(N) As Date

ReDim IY(N) As Double

For i = 1 To N

IX(i) = pX.Cells(i)

IY(i) = pY.Cells(i)

Next i

For i = 2 To N

If (x = IX(i - 1)) Then

IPOLr = IY(i - 1)

Exit Function

Elseif (x = IX(i)) Then

IPOLr = IY(i)

Exit Function

End If

If (x > IX(i - 1) And x < IX(i)) Then

IPOLr = IY(i - 1) + (x - IX(i - 1)) * (IY(i) - IY(i - 1)) / (IX(i) - IX(i - 1))

Exit Function

End If

Next i

If (x > IX(N - 1)) Then

If (method > 0) Then

IPOLr = IY(N) + (x - IX(N)) * (IY(N) - IY(N - 1)) / (IX(N) - IX(N - 1))

Else

IPOLr = IY(N)

End If

End If

End Function

```
Private Sub tridiagonal(N As Integer, ByRef c As Variant, ByRef a As Variant, _  
                        ByRef b As Variant, ByRef r As Variant)
```

```
    Dim i As Integer
```

```
    For i = 1 To N - 1
```

```
        b(i) = b(i) + a(i)
```

```
        a(i + 1) = a(i + 1) - c(i) * b(i)
```

```
    Next i
```

```
    r(1) = r(1) / a(1)
```

```
    For i = 2 To N
```

```
        r(i) = (r(i) - c(i - 1) * r(i - 1)) / a(i)
```

```
    Next i
```

```
    For i = N - 2 To 1 Step -1
```

```
        r(i) = r(i) - r(i + 1) * b(i)
```

```
    Next i
```

```
End Sub
```

```
Private Function spline_eval(pX As Range, pY As Range, ByRef b As Variant, ByRef c As Variant,  
                             ByRef d As Variant, _  
                             t As Double) As Double
```

```
    Dim i, N As Integer, found As Integer
```

```
    Dim IX() As Date
```

```
    Dim IY() As Double
```

```
    N = Application.Count(pX) ' Length of the ranges
```

```
    ReDim IX(N) As Date
```

```
    ReDim IY(N) As Double
```

```
    For i = 1 To N
```

```
        IX(i) = pX.Cells(i)
```

```
        IY(i) = pY.Cells(i)
```

Next i

i = 2

found = 0

Do While (found = 0 And (i < N))

 If (t < IX(i)) Then

 found = 1

 Else

 i = i + 1

 End If

Loop

t = IY(i - 1) + (t - IX(i - 1)) * (b(i - 1) + (t - IX(i - 1)) * (c(i - 1) + (t - IX(i - 1)) * d(i - 1)))

spline_eval = t

End Function

```
'-----  
' Determine the coefficients for the monotonic cubic spline  
' inputs:  n           the number of interpolating points  
'          x           interpolating points  
'          f           function values to be interpolated  
'-----
```

Public Function cubic_ipolate(t As Double, pX As Range, pY As Range) As Double

 Dim h(1 To 100) As Double

 Dim dl(1 To 100) As Double

 Dim dd(1 To 100) As Double

 Dim du(1 To 100) As Double

 Dim b(1 To 100) As Double

 Dim c(1 To 100) As Double

 Dim d(1 To 100) As Double

 Dim fpa As Double, fpb As Double, y As Double

 Dim i, N As Integer

 'cubic_ipolate(pointDate, TimeRange, ZeroRateRange)

 Dim IX() As Date

 Dim IY() As Double

 N = Application.Count(pX) ' Length of the ranges

```

ReDim IX(N) As Date
ReDim IY(N) As Double

```

```

For i = 1 To N
    IX(i) = pX.Cells(i)
    IY(i) = pY.Cells(i)
Next i

```

```

For i = 1 To N - 1
    h(i) = IX(i + 1) - IX(i)
    dl(i) = du(i) = h(i)
Next i

```

```

dd(1) = 2 * h(1)
dd(N) = 2 * h(N - 1)
c(1) = (3 / h(1)) * (IY(2) - IY(1)) - 3 * fpa
c(N) = 3 * fpb - (3 / h(N - 1)) * (IY(N) - IY(N - 1))

```

```

For i = 1 To N - 2
    dd(i + 1) = 2 * (h(i) + h(i + 1))
    c(i + 1) = (3 / h(i + 1)) * (IY(i + 2) - IY(i + 1)) - (3 / h(i)) * (IY(i + 1) - IY(i))
Next i

```

```

Call tridiagonal(N, dl, dd, du, c)

```

```

For i = 1 To N - 1
    d(i) = (c(i + 1) - c(i)) / (3 * h(i)) ' condition : the entire function is twice differentiable
    b(i) = (IY(i + 1) - IY(i)) / h(i) - h(i) * (c(i + 1) + 2 * c(i)) / 3
Next i

```

```

y = spline_eval(pX, pY, b, c, d, t)

```

```

    cubic_ipolate = y
End Function

```

```

Public Function forwardRate(t1 As Double, r_t1 As Double, t2 As Double, r_t2 As Double) As Double

```

```

    forwardRate = (r_t2 * t2 - r_t1 * t1) / (t2 - t1)
End Function

```

```

' -----

```

```

' Determine the coefficients for the monotonic cubic spline
' inputs:  n          the number of interpolating points
'          x          interpolating points
'          f          function values to be interpolated
'-----

```

```

Public Function monotone_cubic(t As Double, pX As Range, pY As Range) As Double
    Dim h(1 To 100) As Double
    Dim dl(1 To 100) As Double
    Dim dd(1 To 100) As Double
    Dim du(1 To 100) As Double
    Dim bb(1 To 100) As Double
    Dim b(1 To 100) As Double
    Dim c(1 To 100) As Double
    Dim d(1 To 100) As Double
    Dim m(1 To 100) As Double
    Dim bmin As Double, bmax As Double, y As Double, min As Double, max As Double
    Dim i, N As Integer

```

```

'cubic_ipolate(pointDate, TimeRange, ZeroRateRange)

```

```

Dim IX() As Date
Dim IY() As Double

```

```

N = Application.Count(pX) ' Length of the ranges
ReDim IX(N) As Date
ReDim IY(N) As Double

```

```

For i = 1 To N
    IX(i) = pX.Cells(i)
    IY(i) = pY.Cells(i)
Next i

```

```

For i = 1 To N - 1
    h(i) = IX(i + 1) - IX(i)
    m(i) = (IY(i + 1) - IY(i)) / h(i)
    dl(i) = du(i) = h(i)
Next i

```

```

dd(1) = 2 * h(1)
dd(N) = 2 * h(N - 1)
b(1) = ((2 * h(1) + h(2)) * m(1) - h(1) * m(2)) / (h(1) + h(2))

```

$$b(N) = ((2 * h(N - 1) + h(N - 2)) * m(N - 1) - h(N - 1) * m(N - 2)) / (h(N - 1) + h(N - 2))$$

For i = 1 To N - 2

```

    dd(i + 1) = 2 * (h(i) + h(i + 1))
    min = Application.min(m(i), m(i + 1))
    max = Application.max(m(i), m(i + 1))
    bb(i + 1) = (3 * m(i) * m(i + 1)) / (max + 2 * min)
    bmin = Application.min(0, bb(i + 1))
    bmax = Application.max(0, bb(i + 1))

```

'Hyman Adjustment

If (m(i + 1) * m(i) <= 0) Then

 b(i + 1) = 0

End If

If (m(i + 1) * m(i) > 0) Then

 If (min > 0) Then

 b(i + 1) = Application.min(bmax, (3 * min))

 Elseif (max < 0) Then

 b(i + 1) = Application.max(bmin, (3 * max))

 End If

End If

Next i

Call tridiagonal(N, dl, dd, du, b)

For i = 1 To N - 1

 d(i) = (b(i + 1) + b(i) - 2 * m(i)) / (h(i) * h(i))

 c(i) = (3 * m(i) - b(i + 1) - 2 * b(i)) / h(i)

Next i

y = spline_eval(pX, pY, b, c, d, t)

monotone_cubic = y

End Function

```

'-----
'   Determine the coefficients for the monotonic cubic spline on log zero rates
'   inputs:  n           the number of interpolating points
'            x           interpolating points
'            f           function values to be interpolated
'-----

```

```

Public Function monotone_log(t As Double, pX As Range, pY As Range) As Double
    Dim h(1 To 100) As Double
    Dim dl(1 To 100) As Double
    Dim dd(1 To 100) As Double
    Dim du(1 To 100) As Double
    Dim bb(1 To 100) As Double
    Dim b(1 To 100) As Double
    Dim c(1 To 100) As Double
    Dim d(1 To 100) As Double
    Dim m(1 To 100) As Double
    Dim bmin As Double, bmax As Double, y As Double, min As Double, max As Double
    Dim i, N As Integer

```

```

    Dim IX() As Date
    Dim IY() As Double

```

```

    N = Application.Count(pX) ' Length of the ranges

```

```

    ReDim IX(N) As Date

```

```

    ReDim IY(N) As Double

```

```

    For i = 1 To N

```

```

        IX(i) = pX.Cells(i)

```

```

        IY(i) = pY.Cells(i)

```

```

    Next i

```

```

    For i = 1 To N - 1

```

```

        h(i) = IX(i + 1) - IX(i)

```

```

        m(i) = (IY(i + 1) - IY(i)) / h(i)

```

```

        dl(i) = du(i) = h(i)

```

```

    Next i

```

```

    dd(1) = 2 * h(1)

```

```

    dd(N) = 2 * h(N - 1)

```

```

    b(1) = ((2 * h(1) + h(2)) * m(1) - h(1) * m(2)) / (h(1) + h(2))

```

```

    b(N) = ((2 * h(N - 1) + h(N - 2)) * m(N - 1) - h(N - 1) * m(N - 2)) / (h(N - 1) + h(N - 2))

```

```

    For i = 1 To N - 2

```

```

        dd(i + 1) = 2 * (h(i) + h(i + 1))

```

```

        min = Application.min(m(i), m(i + 1))

```

```
max = Application.max(m(i), m(i + 1))
bb(i + 1) = (3 * m(i) * m(i + 1)) / (max + 2 * min)
bmin = Application.min(0, bb(i + 1))
bmax = Application.max(0, bb(i + 1))
```

```
'Hyman Adjustment
```

```
If (m(i + 1) * m(i) <= 0) Then
```

```
    b(i + 1) = 0
```

```
End If
```

```
If (m(i + 1) * m(i) > 0) Then
```

```
    If (min > 0) Then
```

```
        b(i + 1) = Application.min(bmax, (3 * min))
```

```
    Elseif (max < 0) Then
```

```
        b(i + 1) = Application.max(bmin, (3 * max))
```

```
    End If
```

```
End If
```

```
Next i
```

```
Call tridiagonal(N, dl, dd, du, b)
```

```
For i = 1 To N - 1
```

```
    d(i) = (b(i + 1) + b(i) - 2 * m(i)) / (h(i) * h(i))
```

```
    c(i) = (3 * m(i) - b(i + 1) - 2 * b(i)) / h(i)
```

```
Next i
```

```
y = spline_eval(pX, pY, b, c, d, t)
```

```
monotone_log = Exp(y)
```

```
End Function
```