Valuation of cancelable interest rate swaps via Hull-White trinomial tree model

by

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Abstract

The thesis studies a problem of cancelable interest rate swap pricing via Hull-White trinomial tree model. Obtained valuation results are compared to Swedbank internal valuation and are found to be consistent. Peculiarities of cancelable interest rate swap contracts, their replication and valuation via Hull-White trinomial tree model are investigated during the study. Obtained mathematical results are implemented in Object Pascal.
Acknowledgements

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Chapter 1

Introduction

The primary goal of this thesis is to valuate a cancelable interest rate swap contract, which is included into one of Swedbank’s portfolios. Results are compared to internal bank valuation. This implies the following subproblems:

- Construct a model;
- Calibrate the model;
- Develop an application.

Hull-White trinomial tree model was selected for valuation purposes. Levenberg-Marquardt algorithm was used for optimization problem. The application is developed in Object Pascal using Embracadero Delphi XE as integrated development environment. Valuation of cancelable interest rate swap contracts is a non-trivial problem of financial engineering. Roughly speaking, the problem of cancelable interest rate swap valuation reduces to Bermudan style option pricing. Due to complexity of the stated problem only limited number of models are capable of pricing such type of contracts. The problem of cancelable interest rate swaps valuation is not very popular in the world of academia. We do not know many works on particularly cancelable interest rate swaps subject. It is explained by the fact that cancelable swaps valuation is a derivative problem to a more general one — interest rate modeling and bermudan options.

The remainder of the thesis is organized as follows. Chapter 2 presents main concepts of interest rate swap and swaption contracts, peculiarities of cancelable swap contracts and current state of interest rate derivatives contracts market. Chapter 3 contains description of Black 76 model and explicit presentation of Hull-White trinomial tree model where we show step-by-step analytical derivation of lattice representation. In addition, yield curve construction method is also presented in Chapter 3. Chapter 4 discusses obtained results of model implementation while Chapter 5 sums up all achieved results and makes conclusions with suggestions for further research.
Chapter 2

Short Note on Interest Rate Swaps and Swaptions

This chapter deals with interest rate swaps and swaptions contract’s concepts, discusses in details idea of cancelable swap contract.

2.1 Main concepts

The following two definitions are taken from [7].

Definition (Swap). “An agreement to exchange cash flows in the future according to a prearranged formula.”

Generally speaking, cash flows can be defined in a wide variety of ways. A cash flow can be linked to an interest rate, currency price, commodity price or even to a trigger on some event — default of the obligation, for example. The most widespread type of swap contract is a, so called, plain vanilla swap. With this type of contract one side cash flows are based on a fixed rate on a notional principal amount, while, on the other side, cash flows are linked to some index\(^1\) which evolves as the time flows. If a company agrees to pay cash flows based on a fixed rate in exchange for floating payments this is a payer swap. On the contrary, receiver swap implies paying at a floating rate and receiving fixed interest.

Definition (Swap option). “An option to enter into an interest rate swap where a specified fixed rate is exchanged for floating.”

\(^1\)It can be LIBOR, STIBOR, EURIBOR, etc.
Swap options, or swaptions, give the holder the right, but not the obligation, to enter into a certain interest rate swap at a certain time in the future.

From now we will refer to swap instead of interest rate swap and to swaption instead of swap option.

Usually, following types of swaptions are distinguished.

1 Depending on execution type:
   - European — executable on a specific date;
   - American — executable at any time during the lifetime of the contract;
   - Bermuda — executable during specific period of time.

2 Depending on underlying swap type:
   - Payer — gives right to enter payer swap contract;
   - Receiver — gives right to enter receiver swap contract.

We will examine in more details last two types of swaptions, as, sometimes, it might be tricky to understand, which type of contract is actually behind the definition.

Common practice is to understand a call swaption as an option on a payer swap, thus the buyer of the call swaption buys the right to enter into a interest rate swap as fixed rate payer. However, if one enters a query “call swaption definition” in Google, directly contrasting results can be found among first ten ones.

For example, website www.investorwords.com gives the following common definition:

“A swaption in which the buyer has the right to enter into a swap as a fixed-rate payer.”

At the same time website www.invest.yourdictionary.com gives the opposite definition:

“A swaption is an option to enter into an interest rate swap at a specified future date with the call giving the purchaser the right, but not the obligation, to receive a fixed interest rate.”

The problem is that they are both correct. The source of discrepancy lies in the convention, which right underlies the option — to pay fixed rate or to receive it. If option is written on the right to make fixed rate payments,
then call means payer. But if option is written on the right to receive fixed rate payments call means receiver.

To summarize stated above, the following table will enable the reader to determine type of the contract correctly.

<table>
<thead>
<tr>
<th>Call</th>
<th>Fixed</th>
<th>Floating</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payer</td>
<td>Receiver</td>
<td>Payer</td>
</tr>
</tbody>
</table>

Table 2.1: Types of swaption contracts

2.2 Cancelable swaps

The following definition cites [8].

**Definition** (Cancelable swap). “A plain vanilla swap where the one of the counterparties has the right but not the obligation to terminate the swap on one or more predetermined dates during the life of the swap.”

Cancelable swaps can be divided into callable and puttable. In a callable swap the party which pays fixed rate has the right to terminate the contract. In a puttable one the fixed rate receiving party has a possibility to terminate the contract. In both cases termination dates are specified by the contract and, in most cases, are the same as reset dates.

For purposes of valuation cancelable swap is replicated by plain vanilla interest rate swap (swap body) and option on an opposite swap (cancellation premium). Indeed, a callable swap is a combination of plain-vanilla swap and a receiver swaption, while a puttable swap is a plain-vanilla one in combination with a payer swaption. While plain vanilla interest rate swap is a well-known interest rate instrument, cancellation premium needs more detailed discussion.

Cancellation premium is determined as follows. Canceling a swap contract is equal to entering an opposite one with the same parameters. For example, receiver interest rate swap can be cancelled by setting up a payer one, providing all other parameters like notional amounts, maturity, fixed rate and floating leg index are the same. So, we see cancellation of the payer/receiver swap with fixed rate \( x\% \) as a Bermuda style variable American option on a receiver/payer swap with strike \( x\% \). But what does “variable” mean? It means that the length of the underlying swap is dependent on time at which the option is exercised. To illustrate it let us consider an example.
Parties A and B enter a cancelable swap contract according to which party A in exchange for Libor + 25 basis points makes payments basing on a fixed at $x\%$ interest rate. The contract starts in 2013 and lasts for 15 years till 2028. Starting from 2017 Part A has a right to terminate the contract on any reset date. So, whenever Party A exercises it’s right for cancellation, the underlying swap’s maturity is fixed — it is 2028. Thus, the lifetime of the underlying swap is in inverse proportion to the time option is held. Stated above is illustrated on the figure 2.1.

Figure 2.1: Cancelable swap mechanism.

Cancellation premium can be expressed in two different, but still equivalent, forms:

- Single payment, as swaption price;
- Incorporation into swap fixed rate.

Usually swap contract is set up in such a way that it’s present value is 0. So, common market practice is to incorporate cancellation premium into swap fixed rate. If cancellation right is provided to the fixed rate payer party then the fixed rate will be higher comparing to regular IRS. In case fixed rate receiver party owns the right of cancellation — fixed rate is lower comparing to regular IRS.

### 2.3 Swap and swaption market

The swap market is considered to began in 1981. On that year IBM and the World Bank agreed to swap obligations on loans in the Swiss Franc and the US dollar. The Bank for International Settlements (BIS) publishes statistics on the notional amounts outstanding in the OTC derivatives market. According to the latest figures, volume of interest rate contracts is estimated to be approximately 550 trillion US dollars. Interest rate swaps constitute the biggest share of interest rate contracts’ value. Moreover, swaps experienced
21% growth in volume during first half of 2011 [9].

<table>
<thead>
<tr>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Forward rate agreements</td>
<td>46,812</td>
<td>51,779</td>
<td>56,242</td>
<td>51,587</td>
<td>55,842</td>
</tr>
<tr>
<td>Interest rate swaps</td>
<td>341,903</td>
<td>349,288</td>
<td>347,508</td>
<td>364,377</td>
<td>441,615</td>
</tr>
<tr>
<td>Options</td>
<td>48,513</td>
<td>48,808</td>
<td>48,081</td>
<td>49,295</td>
<td>56,423</td>
</tr>
<tr>
<td>Interest rate contracts (Total)</td>
<td>437,228</td>
<td>449,875</td>
<td>451,831</td>
<td>465,260</td>
<td>553,880</td>
</tr>
</tbody>
</table>

Table 2.2: Interest rate contracts market volume dynamics.

Figure 2.2: Interest rate contracts market volume dynamics.

Table 2.2 and figure 2.2 illustrate structure and dynamics of interest rate contracts market volume since Jun 2009. A natural question one can ask is how only interest rate contracts market volume can be approximately 9 times greater than entire world GDP? The total value of assets can exceed the net value because of leverage and effects of risk exposure. In case of an interest rate swap each leg has its notional amount. Even though notional amounts are usually not exchanged and net cash flow is determined as a difference on corresponding legs’ payments, statistics sums notional amounts to contract’s value. That is how figures of hundreds of trillions of dollars are obtained.
Chapter 3

Theoretical framework

3.1 Black model

The Black model (also known as the Black-76 model) is commonly used by traders to valuate European swaptions. The Black model owes its popularity to high-speed performance which is critical for option traders in their continuous decision-making process. Mathematical description of the model presented in this section follows [10], in which the model was first presented by Fischer Black.

The Black formula for swaption is similar to the Black-Scholes formula for valuing stock options except that the spot price of the underlying is replaced by a forward swap rate.

Values of a payer swaption $V_p$ and receiver swaption $V_r$ are:

$$V_p = \frac{L}{m} \sum_{i=1}^{m-n} P(0,T_i) \left[ S_0 N(d_1) - S_k N(d_2) \right]$$  \hspace{1cm} \text{(3.1)}

$$V_r = \frac{L}{m} \sum_{i=1}^{m-n} P(0,T_i) \left[ S_k N(-d_2) - S_0 N(-d_1) \right]$$  \hspace{1cm} \text{(3.2)}

Where:

$$d_1 = \frac{\ln \left( \frac{S_0}{S_k} \right) + \sigma^2 T}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$
And

- $N()$ — cumulative distribution function of standard normal distribution
- $L$ — notional amount
- $m$ — number of swap resets per year
- $n$ — lifetime of the underlying swap in years
- $T_i$ — cash flow date, expressed in years from the valuation date
- $T$ — the time in years until the expiration of the option
- $P(0, T_i)$ — discount factor corresponding to $T_i$ years maturity
- $S_0$ — forward swap rate calculated at the valuation date
- $S_k$ — strike rate, or, i.e., fixed rate in the underlying swap contract
- $\sigma$ — volatility of the forward swap rate

While majority of parameters are defined by the swaption contract specification, two the most crucial for the valuation ones must be determined by the valuator. These are forward swap rate $S_0$ and forward swap rate volatility $\sigma$.

Forward swap rate is a fixed rate, that makes value of the underlying swap contract equal to zero at some time moment in future.

Forward swap rate is calculated as follows:

$$S_0 = \frac{P(0, T) - P(0, T_n)}{\sum_{0}^{n-1} (T_{i+1} - T_i) P(0, T_{i+1})}$$

And,

$$P(0, T_i) = e^{-r_i T_i}$$

Where, $r_i$ — instantaneous spot rate applicable at time $T_i$ (expressed in years from now). $r_i$ is interpolated from the yield curve. Yield curve is a basis for forward swap rate determining. So, it is very important to set up a proper interest rate curve for correct valuation of swaptions. We discuss in details yield curve construction in section 3.3.

However, the most contradictory parameter is the forward swap rate volatility. Volatility parameter is an attempt to quantify an uncertainty of the return realized on an asset.
The volatility is taken from a two dimensional grid of at-the-money\(^1\) volatilities as observed from prices in the Interbank swaption market. Interbank quotes for swaptions are in implied volatilities, i.e., it is a number, that should be inputted in the Black formula as volatility in order to determine price of the contract.

![Figure 3.1: Grid of at-the-money volatilities. Row contain time to expiration of the swaption, columns — length of the underlying swap. Thus, 6M option on 7Y at-the-money swap volatility is 23.70%.](image)

### 3.2 Hull-White trinomial tree model

The Hull-White is a model of future interest rates. It was first introduced in 1990 in [3]. The Hull-White interest rate model is a generalization of the Vasicek model with time dependent parameters [11]. A distinctive feature of the Hull White interest rate model is a possibility to transform the mathematical description of the interest rate process to tree or lattice. Treelike model allows valuation of bermudan and american options, which is exactly the problem formulated in section 2.2. Recall that cancelable swap is replicated by plain vanilla swap and bermudan option on the opposite vanilla swap with variable length. So, treelike model is a natural choice for the presented valuation problem. Mathematical description of the model discussed in this chapter follows [12] and [6].

\(^1\)“At-the-money” option is an option whose strike price equals the price of the underlying asset. \((S_0 = S_k)\)
3.2.1 Interest rate equation

Consider a mathematical representation of the model. Interest rate evolves according to the equation:

\[ dr = (\Theta (t) - \alpha(t) r) dt + \sigma(t) dV(t) \]  

(3.3)

Where \( \Theta (t) \) is a deterministic function of time. The function \( \Theta (t) \) is used for adjustment of the Hull-White interest rate model to the current term structure. In other words, \( \Theta (t) \) is calibrated to prices of zero-coupon bonds that are equivalent to pure discount factors. In turn, parameters \( \alpha \) and \( \sigma \) enable model to fit market volatility structure. This implies calibrating the model against market prices of relevant instruments (depending on valuation purposes of the model) by exhaustion of parameters \( \alpha \) and \( \sigma \).

Let us solve for \( r(t) \). Differentiate 3.3:

\[ d(e^{at}r) = ae^{at}r dt + e^{at}dr \]  

(3.4)

Insert 3.3 to the rhs of 3.4:

\[ ae^{at}r dt + e^{at}dr = ae^{at}r dt + e^{at}\Theta(t) dt - ae^{at}r dt + e^{at}\sigma dW \]

Integrate.

\[ e^{at}r(t) - r(0) = \int_0^t e^{as}\Theta(s) ds + \sigma \int_0^t e^{as} dW(s), \]

\[ r(t) = e^{-at}r(0) + \int_0^t e^{a(t-s)}\Theta(s) ds + \sigma \int_0^t e^{a(t-s)} dW(s), \]

\[ r(t) = \psi(t) + \sigma \int_0^t e^{a(t-s)} dW(s), \text{ where } \psi(t) = e^{-at}r(0) + \int_0^t e^{a(t-s)}\Theta(s) ds \]

Add \( \Delta t \).

\[ r(t + \Delta t) = \psi(t + \Delta t) + \sigma \int_0^{t+\Delta t} e^{a(t-s-\Delta t)} dW(s), \]

\[ r(t + \Delta t) - e^{-a\Delta t}r(t) = \psi(t + \Delta t) - e^{-a\Delta t}\psi(t) + \sigma \int_0^{t+\Delta t} e^{a(t-s-\Delta t)} dW(s), \]  

(3.5)

The Itô stochastic integral is defined as \( Y_t = \int_0^t H_s dX_s \), where \( X \) is a Brownian motion. This is exactly the third term of the rhs of 3.5 — \( \sigma \int_0^{t+\Delta t} e^{a(t-s-\Delta t)} dW(s) \). Thus, by the property of Itô stochastic integral it
is a normally distributed stochastic variable with mean zero and variance equal to:

\[ \sigma^2 \int_0^{t+\Delta t} e^{2\alpha(s-t-\Delta t)} dW(s) = \frac{\sigma^2}{2\alpha} [1 - e^{2\alpha\Delta t}] \]

Hence, \[ r(t + \Delta t) = e^{\alpha\Delta t} r(t) + h(t) Z_t, \]
where \( h^2(t) = \frac{\sigma^2}{2\alpha} \left[ 1 - e^{2\alpha\Delta t} \right] \)

\[ r(t + \Delta t) = r(t) + \left( e^{-\alpha\Delta t} - 1 \right) r(t) + h(t) Z_t. \] (3.6)

\[ \text{Solving the systems of equations gives:} \]
\[ a = -\sqrt{3}, \ b = 0, \ c = \sqrt{3}, \ P_a = 1/6, \ P_b = 2/3, \ P_c = 1/6. \]

In order to obtain a recombining tree\(^2\) one must put the following constraints on \( r(t + \Delta t) - r(t) \):

\[ Mr(t) + h(t) Z_t = \begin{cases} -h(t)\sqrt{3} \\ 0 \\ h(t)\sqrt{3} \end{cases} \]

\(^2\)A recombining tree is a tree in which a down move followed by an up one is equivalent to an up move followed by a down one.
i.e.:

\[ Z_t = \begin{cases} 
\sqrt{3} + q(t) \\
q(t) \\
-\sqrt{3} + q(t) 
\end{cases} \]

Where \( q(t) = -\frac{Mr(t)}{h(t)} \). The up-, middle- and down-branching probabilities solve the following system of equations:

\[
\begin{cases}
P_u + P_m + P_d = 1 \\
E[z] = (\sqrt{3} + q(t)) P_u + qP_m + (-\sqrt{3} + q(t)) P_d = 0 \\
E[z^2] = (\sqrt{3} + q(t))^2 P_u + q^2 P_m + (-\sqrt{3} + q(t))^2 P_d = 1
\end{cases}
\]

Solving the systems of equations yields into branching probabilities:

\[
P_u = 1/6 + \frac{q^2 - \sqrt{3}}{q}6 \\
P_m = 2/3 - \frac{q^2}{3} \\
P_d = 1/6 + \frac{q^2 - \sqrt{3}}{q}6
\]

Note that \( P_m \) is negative if \( q > \sqrt{2} \). This is avoided by introducing alternative branching at the edges of the tree. Modified dynamics of \( Z_t \) and corresponding probabilities at the top edge:

\[
Z_t = \begin{cases} 
q(t) \\
-\sqrt{3} + q(t) \\
-2\sqrt{3} + q(t) 
\end{cases} \\
P_u = 7/6 + \frac{q^2 - 3\sqrt{3}}{q}6 \\
P_m = -1/3 - \frac{2\sqrt{3}q - q^2}{3} \\
P_d = 1/6 + \frac{q^2 - \sqrt{3}}{q}6
\]

Bottom edge case:

\[
Z_t = \begin{cases} 
2\sqrt{3} + q(t) \\
\sqrt{3} + q(t) \\
q(t) 
\end{cases} \\
P_u = \frac{2\sqrt{3} + q(t)}{q}6 \\
P_m = \frac{2\sqrt{3} - q^2}{3} \\
P_d = 1/6 + \frac{q^2 - \sqrt{3}}{q}6
\]
\[
P_u = 1/6 + \frac{q^2 + \sqrt{3}}{6} \\
P_m = -1/3 - \frac{2\sqrt{3}q + q^2}{3} \\
P_d = 7/6 + \frac{q^2 + 3\sqrt{3}}{6}
\]

3.2.3 Tree consistency with market term structure

An adjustment procedure was first introduced by Hull and White in [6]. They suggested adding the function \( g(t) \) to the \( r(t) \) process at each node. Recall that \( g(t) \) is a function of \( \theta(t) \) and selection of \( \theta(t) \) enables the model to fit the term structure. In other words, nodes of the tree are adjusted in order to correctly price zero-coupon bonds at all maturities, or, what is equivalent, provide correct discount factors.

Firstly, prices of benchmark securities are determined. In this particular case zero-coupon bond is the benchmark. For every time moment at which tree nodes are placed zero-coupon price is calculated, which is, essentially, the same as a discount factor. Discount factors are computed from the discounting yield curve according to simple discounting formulas. It is \( D = e^{-rt} \) in case of continuous compounding, for example. This is one part of adjusting equation.

Another one is a price of the same security given by the tree model. Note, that adjusting process \( g(t) \) added to the tree at time node \( t \) will affect all subsequent cash flows. Thus, a recursive expression that allows to express the price of security that pays 1 money unit at time \( t_i \) depending on the price of the same security at time \( t_{i-1} \) is needed. For these purposes concept of Arrow-Debreu prices is used. The concept was introduced in [13] and since [14] is used in financial economics. The Arrow-Debreu price is a price of security that pays 1 money unit if a specific condition is fulfilled and 0 otherwise. In our case, the condition is reaching by the interest rate process a specific node of the tree, node \( hk \) for instance. Moreover, in order to keep recursiveness we incorporate condition on previous state of the interest rate process, i.e. a price of the security at node \( hk \) on condition that node \( ij \) was reached on the previous step. Let us denote Arrow-Dbreu price as \( Q \), value of the security as \( V \) and cash flow as \( C \). Then:

\[
V(h,k) = \sum_{i>k} \sum_j Q(i,j|h,k)C_{i,j}
\]
where,
\[ Q(i, j|h, k) = p(i, j|i - 1, k) e^{-r_{i-1,k}(t_i-t_{i-1})} \]
and \( p(i, j|i - 1, k) \) denotes a conditional probability of reaching node \( j \) on step \( i \) being at node \( k \) on step \( i - 1 \). The summation is taken over all time moments \( i \) which follow time moment \( k \) and all nodes at particular time step \( i \). Hence, the price of zero-coupon bond, or a discount factor can be expressed as following:
\[ P_{i+1} = \sum_i Q_{i,j} e^{-f(r_{i,j}+g_i)(t_{i+1}-t_i)} \]
where:
\[ g_i = \frac{\ln \left( \sum_j Q_{i,j} e^{-f(r_{i,j})\Delta t_i} \right) - \ln (P_{i+1})}{\Delta t_i} \]
which is the adjustment process at time moment \( i \). It remains only to note, that since \( Q_{0,0} = 1 \) the value of \( g_0 \) equals the initial \( \Delta t_0 \) interest rate.
Finally, the recursive process is continued for every \( t_i \) and computed adjustment process \( g(t) \) is added to the interest rate process \( r(t) \).

On completion of adjustment process constructed tree is consistent with respective term structure and branching process produces correct discount factors.

However, it is time to mention one of the main HW trinomial tree models pitfalls. Unfortunately, the HW model can lead to negative rates. Adjustment process has no built-in protection against negative interest rates. In order to guarantee mean-reversion and recombining property we must accept the fact that negative interest rates can appear in the tree branching process. The risk of negative rates is even higher when market implies low interest rates as nowadays (2012).

### 3.2.4 Calibration

The final stage of tree construction is calibration. Like adjustment, calibration refines model parameters in such a way that the model produces results consistent with real life phenomena. But in contrast to adjustment, calibration process is not so definite. There is no all-purpose instrument in compliance with which volatility structure is calibrated. Theory, market practice and intuition suggest to use instruments as similar as possible to the one is being priced. In addition, target instruments should be sufficiently liquid to represent current market conditions.

In this paper the tree is calibrated to European swaptions volatility grid, which was described in section 3.1. There is no instrument available on the
market that is more close to cancelable swap and in the same time has quotes in volatility then European swaption.

Calibration is a problem of optimization. Levenberg-Marquardt algorithm is used to find set of volatilities that minimizes the sum of the squares of the difference between the model price and market price of European swaptions. It is worth to mention, that the same method is used for built-in trees calibration routines in Matlab.

3.3 Constructing a yield curve

The first step in valuation of any kind of financial contract is construction of a yield curve. The yield or discounting curve is constructed from relevant instruments quotes currently available on the market. The quote rates curve is a raw material from which a zero coupon yield curve is derived usually using a method called “bootstrapping”. This involves deriving each new point on the curve from previously determined zero coupon points. Beyond a shadow of a doubt, problem of yield curve construction is worth another thesis. This section presents methodology according to which zero-coupon yield curve for purposes of this paper was constructed.

To begin with, one must determine a set of instruments from which the curve will be constructed. In case of pricing cancelable swaps, zero-coupon curve is used for discounting of swap cash flows. Therefore, it was decided to use the following set of instruments:

- Overnight deposit as the most liquid and riskless instrument currently available on the market.
- STIBOR Index as the most common floating leg index on the Swedish market.
- Floating rate notes (FRN) as allied to swaps instruments. Swap contracts are usually replicated by a portfolio of FRN.
- Plain vanilla interest rate swaps in SEK.

General algorithm of curve derivation:

1. First calculate the discount factor, $DF$, using par rates, $pr$, from short term instruments (overnight deposit and STIBOR) using the formula $DF = \frac{1}{1+(pr/100)\times N_{days}}$.

2. Using $DF$ calculate zero coupon rate, $r_z$, according to formula $r_z = \frac{-100ln(DF)}{N_{days}/365}$.
3 In order to proceed with FRN interpolate zero-coupon “virtual” stub with maturity equal to starting date of the first FRN. Interpolate between rates calculated at stage 2.

4 Calculate $DF$ for stub using value from 3 in formula $\exp\left[\frac{-r}{100 \times \frac{N_{\text{days}}}{365}}\right]$

5 Calculate $DF$ for FRN as $DF_i = \left(\frac{1}{1+\frac{pr}{100} \times \frac{N_{\text{days}}}{365}}\right)DF_{i-1}$, where $DF_0$ is taken from step 4.

6 Proceed with swaps. Note, that since swap quotes are not available for every year, additional interpolation for swaps is applied

$$\sum_i P \times r_{\text{fixed}} \times yf_i \times DF_i = \sum_i P \times r_{\text{floating}} \times yf_i \times DF_i$$

$$\sum_i r_{\text{fixed}} \times yf_i \times DF_i = \sum_i \left(\frac{DF_{i-1}}{DF_i} - 1\right) \times \frac{1}{yf_i} \times yf_i \times DF_i$$

$$DF_{sd} - DF_{ed} - \sum_{i=sd+1}^{ed} r_{\text{fixed}} \times yf_i \times DF_i = 0$$

$$DF_{ed} = \frac{DF_{sd} - \sum_{i=sd+1}^{ed-1} r_{\text{fixed}} \times yf_i \times DF_i}{r_{\text{fixed}} \times yf_{ed} + 1}$$
Chapter 4

Model implementation

Overview of the developed application and valuation results are presented in this section.

4.1 Overview of the developed application

The application has been developed on the basis of Hull-White trinomial tree model. IDE is Embracadero Delphi, programming language — Object Pascal.

Main window consists of four group boxes:

- Yield curve specification box. The yield curve is imported from external .txt of .csv files.
• Output tab panel. This panel contains charts of yield curve and calibration output information. Tree viewer tab presents an interest rate lattice in matrix form.

• Valuation section is a tab panel which contains controls for specification of different instruments.

• Console panel. Various additional information is represented in the console (for example error and warning messages).

Subproducts of cancelable swaps solutions can be used for valuation of less complex instruments. Indeed, calculation of discount factor is equal to valuating a zero coupon bond. Tree calibration routine involves fitting the model to market prices of European style swaptions. Cancelable swap is replicated with plain-vanilla swap and bermuda style variable swaption. So, application developed for purposes of this paper allows user to valuate following types of instruments:

• Bonds:
  Zero-coupon bond;
  Coupon bond;
  Callable/Putable bond.

• Bond option:
  European;
  American.

• Swaption:
  European;
  American.

• Cancelable Swap.

4.2 Valuation

In this section valuation of existing cancelable swap contract is presented. Contract details were kindly provided by Swedbank. The presented interest rate swap was set up between Swedbank and another major European bank in May 2011.
Contract details:
Start date : 2011 31 May
Maturity date : 2026 31 May
Type : payer
Notional : 20 000 000.00
Currency : SEK
Index : STIBOR
Spread : 0
Fixed rate : 1.45%
Payments per year : 4
Cancellable from : 2013 31 May
Cancellation dates : as payment dates

The valuation date is 2011 Nov 09.

4.2.1 Constructing the tree

We construct the tree in such a way that nodes are placed at payment dates. As each leg of the contract under consideration has 4 resets per year, nodes of the tree are evenly spaced each quarter (except of the first one). Time in the model is represented as a fraction of year. Year fraction between Nov 09 2011 and Feb 29 2012 is 0.0575, so first node is placed 0.0575 years from now. Each following node is placed 0.25 years from the preceding one. The year fraction between valuation date and maturity date is 14.5575 years which is exactly the same as 0.0575 + 0.25 · 58 = 14.5575 Next step is to load a

Figure 4.2: Tree construction dialog.
yield curve. Zero-coupon yield curve was constructed from raw market data according to the methodology described in section 3.3. After specifying time moment at which nodes are placed and loading a yield curve trinomial tree is constructed and ready to be used. Adjustment procedure according to algorithm discussed in section 3.2.3 is called automatically after loading the yield curve.

4.2.2 Calibration

The tree is calibrated against market quotes of at-the-money European swaptions, as was discussed in section 3.2.4 Ideal case would be to fit the volatility at each tree node. But, tree constructed for valuation of cancelable swap maturing in almost 25 years with quarterly resets has 60 time moments. Fitting volatility in each of them is extremely time-consuming. Moreover, Market quotes are not available for each combination of option and swap maturities and calibrating to interpolated values is unlikely to increase model accuracy. So, it was decided to use piecewise constant volatility function and calibrate the tree to maturities and volatilities presented in table 4.2.2. Unfortunately, the model was not able to match all market prices and failed to price swaptions with both — short and long maturities. Results of calibration routine
Figure 4.4: Volatility grid as calibration input

Table 4.1: Swaption maturities and corresponding volatilities: inverse relation.

<table>
<thead>
<tr>
<th>Option length, years</th>
<th>Swap length, years</th>
<th>Volatility, %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.558</td>
<td>13</td>
<td>0.3350</td>
</tr>
<tr>
<td>2.558</td>
<td>12</td>
<td>0.3165</td>
</tr>
<tr>
<td>3.558</td>
<td>11</td>
<td>0.2915</td>
</tr>
<tr>
<td>4.558</td>
<td>10</td>
<td>0.2885</td>
</tr>
<tr>
<td>5.558</td>
<td>9</td>
<td>0.2820</td>
</tr>
<tr>
<td>6.558</td>
<td>8</td>
<td>0.2780</td>
</tr>
<tr>
<td>7.558</td>
<td>7</td>
<td>0.2800</td>
</tr>
<tr>
<td>8.558</td>
<td>6</td>
<td>0.2770</td>
</tr>
<tr>
<td>9.558</td>
<td>5</td>
<td>0.2770</td>
</tr>
<tr>
<td>10.558</td>
<td>4</td>
<td>0.2770</td>
</tr>
<tr>
<td>11.558</td>
<td>3</td>
<td>0.2780</td>
</tr>
<tr>
<td>12.558</td>
<td>2</td>
<td>0.2800</td>
</tr>
<tr>
<td>13.558</td>
<td>1</td>
<td>0.2860</td>
</tr>
</tbody>
</table>
are displayed on Figure 4.5

According to the figure 4.5, the model failed to fit all market volatilities. Prices of swaptions with maturities from 6 to 10 years were fitted with great accuracy. But, the model was unable to fit both short and long term volatilities. There can be several explanations of such outcome. The easiest way is to blame the calibration method, but it is not a case in this paper. Levenberg-Marquardt non-linear least square method is a trustworthy one.

Figure 4.5: Calibration output.

Figure 4.6: Sigma volatility parameter
which proved itself to be a good optimization tool. It would be worth to mention that Levenberg-Marquardt routine is used for calibration in Financial Derivatives toolbox in Matlab. So, let us discuss benchmarks for calibration in more details. As has already been mentioned several times, benchmark for calibration is a set of European style options with maturity being in inverse proportion to underlying swap length. So, the model failed to capture inverse relation between option and underlying swap lifetime. Secondly, market quotes are not available for every combination of option and underlying swap lifetime. So, intensive usage of interpolation leads to additional inaccuracy.

Let us check calibration performance in case of directly proportional relationship between option and underlying swap maturity and calibrate model to the volatility structure presented in 4.2.2.

As displayed on figure 4.8, calibration accuracy is considerably greater compared to previous case. So, further research should be done concerning ability of trinomial tree model to capture inverse option-swap maturity relation. However, we continue with valuation of cancelable swap contract.

### 4.2.3 Results

After adjusting the tree to market volatilities we can proceed to the final stage and call cancelable swap valuation function via corresponding tab in the valuation tab. The output consists of the following information:
<table>
<thead>
<tr>
<th>Option length, years</th>
<th>Swap length, years</th>
<th>Volatility, %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,558</td>
<td>1</td>
<td>0.3350</td>
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<td>0.2800</td>
</tr>
<tr>
<td>13,558</td>
<td>13</td>
<td>0.2860</td>
</tr>
</tbody>
</table>

Table 4.2: Swaption maturities and corresponding volatilities.

Receiving leg present value (PV) 6 173 205 SEK
Paying leg PV 6 709 045 SEK
Swap value - 535 840 SEK

(2.6792% of notional amount)
Swap zero rate 2.4838%
Cancelation premium 2,2583% of nominal

(451 660 SEK)
Obtained results were compared to valuation kindly provided by Swedbank which is illustrated on figure 4.9. As can be observed on the graph:

- Present value of floating leg cash flows are perfectly correlated with correlation coefficient $\approx 0.99$;

- There is constant spread between two floating cash flows series with average $\approx 3000$ SEK per cashflow and standard deviation of 1500 SEK.

We would like to propose the following explanation of the nature of this spread. As recent financial crisis has shown, even such big financial institutions as major international investment banks can go into default during a short period of time. So perception of what is risk-free has changed drastically. This resulted in evolution of risk-spread models. Yield curve which has been constructed for purposes of this paper does not account for counterparty risk and uses no risk adjustments spread. This facts results in parallel shift between model presented in this paper and Swedbank valuation model. Counterparty risk and risk-spread adjustment is a topic sufficient for another bunch of papers and is beyond a scope of this paper. We treat the yield curve as input information for the Hull-White trinomial tree model, and there can be numerous ways of input information definition according to particular circumstances.

Figure 4.9: Present value of swap cash flows
4.2.4 Cancellation premium

As was mentioned in section 2.2, cancellation premium can be expressed in two equivalent forms: either as a single cash flow or being incorporated into fixed rate. In this section in the previous subsection cancellation premium has been determined as a single cashflow as the contract under consideration has been already started with all parameters specified already. But, if parties are about to enter a new cancelable swap contract then cancellation premium is incorporated into a fixed rate, making it higher or lowering depending on which party has a right of cancellation. Let us discuss how cancellation premium adjusts fixed rate in more details.

On the initiation date interest rate swap is set in such a way, that it’s present value is 0. This condition means that present value of both legs are equal:

\[ N r_f \sum_{i=1}^{n} \left( t_i \frac{D(t_i)}{T_i} \right) = N \sum_{j=1}^{m} \left( f_j t_j \frac{D(t_j)}{T_j} \right) \]  \hspace{1cm} (4.1)

Where \( N \) denotes notional amount (can be canceled), \( n \) and \( m \) — number of cash flows on fixed and floating legs respectively, \( r_f \) — fixed rate and \( D(t) \) is a discount actor which corresponds to time moment \( t \), \( f_j \) — forward rate and \( T_i \) is the time basis selected according to the specific day count convention.

In addition, let us denote premium for cancellation as \( P_c \). Thereafter, fixed rate is easily solved from 4.1:

\[ r_f = \frac{\sum_{j=1}^{m} \left( f_j t_j \frac{D(t_j)}{T_j} \right)}{\sum_{i=1}^{n} \left( t_i \frac{D(t_i)}{T_i} \right)} \]

Now, consider two cases:

1 Fixed rate payer party receives a right to terminate the contract at some predefined date and \( P_c \) (expressed in % of \( N \)) has already been calculated. Equation 4.1 must still hold, but now with \( P_c \) taken into account:

\[ r_f \sum_{i=1}^{n} \left( t_i \frac{D(t_i)}{T_i} \right) = \sum_{j=1}^{m} \left( f_j t_j \frac{D(t_j)}{T_j} \right) + P_c \]

\[ r_f = \frac{\sum_{j=1}^{m} \left( f_j t_j \frac{D(t_j)}{T_j} \right) + P_c}{\sum_{i=1}^{n} \left( t_i \frac{D(t_i)}{T_i} \right)} \]

Where \( P_c \geq 0 \) by definition. Thus, for fixed rate payer party the possibility to terminate the contract results in paying higher fixed rate.
2 An opposite case: floating rate payer party can annul the swap. Expression 4.1 becomes:

\[ r_f \sum_{i=1}^{n} \left( \frac{t_i}{T_i} D(t_i) \right) + P_c = \sum_{j=1}^{m} \left( f_j \frac{t_j}{T_j} D(t_j) \right) \]

\[ r_f = \frac{\sum_{j=1}^{m} \left( f_j \frac{t_j}{T_j} D(t_j) \right) - P_c}{\sum_{i=1}^{n} \left( \frac{t_i}{T_i} D(t_i) \right)} \]

what results in receiving lower fixed interest rate in comparison to plain-vanilla swap.
Chapter 5
Conclusions

In this paper the problem of valuation of cancelable swaps via Hull-White trinomial tree model has been investigated. Results of this study can be summarized as follows:

• Valuation of cancelable swap contract performed in this paper is consistent with Swedbank internal one. Correlation between cash flow series obtained in this paper and the one provided by Swedbank is 0.99. Mean difference is \( \approx 3000 \) SEK per cashflow, and standard deviation \( \approx 1500 \) SEK. Premium for cancellation has been determined to be 2,2583 % of notional amount.

• During the calibration the model failed to capture inverse relationship between maturities of swaption and underlying swap, fitting only about a half of benchmark prices. However, calibration with directly proportional relationship worked fine. So, we suggest to perform more tests on calibration problem trying different optimization methods.

• The constant shift between cash flows obtained in this study and provided by Swedbank can be explained by adding risk spread to the discounting yield curve. The yield curve adjustment is dependent on particular counterparty and this question is as well of great interest. We suggest it as another area of further research.

• One of Hull-White trinomial tree model pitfalls is generating of negative interest rates on the bottom nodes of the tree. This phenomena occurs during extremely volatile and low interest rate market, which is observed nowadays: interest rates are low and uncertainty about solvency of financial institutes and even states is high. In order to fulfill Hull-White trinomial tree model assumptions and properties we have
to accept negative interest rates phenomena. More studies should be performed on compensation or adjustment of negative interest rates algorithms.

- Developed application provides the user with comfortable GUI for working not only with cancelable swaps, but also with swaptions of european and american style, bonds and bond options.
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