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Division of Applied Mathematics

An Introduction to Modern Pricing of Interest Rate Derivatives

Master Thesis in Financial Engineering

Author: Hossein Nohrouzian

Mälardalen University

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Outline

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Risky Asset vs Risk-Less Asset

- Does exist two kind of investments?

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Risky Asset vs Risk-Less Asset

- Does exist two kind of investments?
- An example is pension salary vs inflation.

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Risky Asset vs Risk-Less Asset

- Does exist two kind of investments?
- An example is pension salary vs inflation.
- NASDQ value increased by almost 150% in 5 years.

Figure: Price behavior of the NASDAQ from 2010 to 2015

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Interest Rate and Economics Factors

- Interest rate and monetary policy.
- Interest rate and international trading.
- Interest rate and economic growth.

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Interest Rate and Economics Factors

- Interest rate and monetary policy.
- Interest rate and international trading.
- Interest rate and economic growth.
- 0.0%, -0.1% and -0.25%

Figure: The exchange rate between USD and SEK

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Jump Diffusion

- On 15th of January 2015 SNB unexpectedly scrapped its cap on the Euro value of the Franc.

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Jump Diffusion

- On 15th of January 2015 SNB unexpectedly scrapped its cap on the Euro value of the Franc.
- The result was 27.5% change in USD vs CHF and shake in stock prices.

Figure: Exchange rate between USD and CHF

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Banks vs Market Rates

- Banks offered rates to individuals and companies.
- Interest rates in the market.

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Banks vs Market Rates

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  - From and after the economic crisis in 2007 and 2008:
    - Collateral rate:

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        - Collateral rate:
        - is used in the collateral agreement or CSA,
        - calculated daily on the overnight index swaps.

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    - Collateral rate:
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    - calculated daily on the overnight index swaps.
- Swap rates:
  - Fixed rates are calculated from forward rates,
  - floating rates are calculated from OIS rates.

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Risk-Neutral Evaluation

- Risk-Neutral world
  1. The expected return on a stock (or any other investment) is the risk-free rate,
  2. The discount rate used for the expected payoff on an option (or any other investment) is the risk-free rate.

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- Under Risk-neutral P^* equivalent to the P
  1. The discounted price of a derivative is martingale,
  2. The discounted expected value under the P^* or Q of a derivative, gives its no-arbitrage price.

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Money Market Account as a Numériare

- Money Market Account

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Money Market Account as a Numériare

- Money Market Account
  - Constant Interest Rates

$$B(t) = \left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n\right]^t = e^{rt}, \qquad t \geq 0.$$

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Money Market Account as a Numériare

- Money Market Account
  - Constant Interest Rates

$$B(t) = \left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n\right]^t = e^{rt}, \qquad t \geq 0.$$

  - Stochastic Interest Rates

$$B(t) = \exp\left\{\int_0^t r(u)du\right\}, \qquad t \geq 0,$$

$r(t)$ is time-t instantaneous interest rate.

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Discount Bond as a Numériare

- Forward Rates

Discount Bond as a Numériare

- Forward Rates
  - Instantaneous forward rate

$$f(t,T) = -\frac{\partial}{\partial T} \ln v(t,T), \qquad t \leq T.$$

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Discount Bond as a Numériare

- Forward Rates
  - Instantaneous forward rate

$$f(t,T) = -\frac{\partial}{\partial T} \ln v(t,T), \qquad t \leq T.$$

  - Default-free discount bond

$$v(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}, \qquad t \leq T.$$

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$$v(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}, \qquad t \le T.$$

  - $r(T) = f(t,T)$, $t \le T$. If $r(t)$ is deterministic

$$v(t,T) = \exp\left\{-\int_t^T r(s)ds\right\} = \frac{B(t)}{B(T)}, \qquad t \le T.$$

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Pricing under Risk-Neutral Method

- Price of European Option under Q

$$\pi(t) = B(t)\mathrm{E}^{\mathrm{Q}}\left[\frac{h(S(T))}{B(T)}\middle|\mathcal{F}_t\right], \qquad 0 \leq t \leq T.$$

where $\pi(T) = h(S(T))$.

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- Samuelson price process

$$\begin{cases} dS(t) = \mu S(t)dt + \sigma S(t)dW, \\ S(0) = S_0. \end{cases}$$

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- Samuelson price process

$$\begin{cases} dS(t) = \mu S(t)dt + \sigma S(t)dW, \\ S(0) = S_0. \end{cases}$$

- Black–Scholes-Merton Lognormal Price

$$S_T = S_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W_T - W_t\right)\right\}.$$

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Pricing under Forward-Neutral Method

- Price process of discount bond under Q

$$\frac{dv(t,T)}{v(t,T)} = r(t)dt + \sigma_v(t)dW^*, \qquad 0 \le t \le T,$$

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Pricing under Forward-Neutral Method

- Price process of discount bond under Q

$$\frac{dv(t,T)}{v(t,T)} = r(t)dt + \sigma_v(t)dW^*, \qquad 0 \le t \le T,$$

- Price process of security under Q

$$\frac{dS}{S} = r(t)dt + \sigma(t)dW^*, \qquad 0 \le t \le T,$$

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Pricing under Forward-Neutral Method

- Price process of discount bond under Q

$$\frac{dv(t,T)}{v(t,T)} = r(t)dt + \sigma_v(t)dW^*, \qquad 0 \le t \le T,$$

- Price process of security under Q

$$\frac{dS}{S} = r(t)dt + \sigma(t)dW^*, \qquad 0 \le t \le T,$$

- Price of European claim under Q^T

$$\pi_C(t) = v(t,T)\mathrm{E}^{Q^T}\left[h(S(T))\middle|\mathcal{F}_t\right], \qquad 0 \le t \le T.$$

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Term-Structure Models

- Spot-Rate (Equilibrium) Models

$$dr = a(m - r)dt + \sigma r^{\gamma} dW, \quad t \geq 0,$$

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Term-Structure Models

- Spot-Rate (Equilibrium) Models

$$dr = a(m - r)dt + \sigma r^{\gamma} dW, \quad t \geq 0,$$

  - Rendleman–Bartter Model, (+) Rates,
  - Vasicek Model, (-) Rates,
  - Cox–Ingersoll–Ross (CIR) Model, (+) Rates,
  - Longstaff–Schwartz Stochastic Volatility Model, (-) Rates,

Term-Structure Models

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  - Rendleman–Bartter Model, (+) Rates,
  - Vasicek Model, (-) Rates,
  - Cox–Ingersoll–Ross (CIR) Model, (+) Rates,
  - Longstaff–Schwartz Stochastic Volatility Model, (-) Rates,
  - Problem: Do not fit today's term structure of interest rate.

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Term-Structure Models

- Spot-Rate (No Arbitrage) Models

Term-Structure Models

- Spot-Rate (No Arbitrage) Models
  - Ho–Lee Model, Developed from Lattice approximation (Binomial Tree),
  - Hull–White (One-Factor) Model, Application in pricing American option via trinomial tree,
  - Black–Derman–Toy Model, Developed from binomial tree model for lognormal spot rate, Identical to Lognormal version of Ho–Lee Model,
  - Black–Karasinski Model, Extension of Black–Derman–Toy Model,
  - Hull–White (Two-Factor) Model.

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Pricing Discount Bond via Vasicek Model

- Market price of risk $\lambda(t) = \lambda$ and SDE under Q

$$dr = a(\bar{r} - r)dt + \sigma dW^*,$$

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Pricing Discount Bond via Vasicek Model

- Market price of risk $\lambda(t) = \lambda$ and SDE under Q
$$dr = a(\bar{r} - r)dt + \sigma dW^*,$$
- risk-adjusted (r.a.) mean reverting level
$$\bar{r} = m - \frac{\sigma}{a}\lambda,$$
- r.a. drift $m(r, t) = a\bar{r} - ar$ & diffusion $\sigma(r, t) = \sigma$.

Pricing Discount Bond via Vasicek Model

- Market price of risk $\lambda(t) = \lambda$ and SDE under Q

$$dr = a(\bar{r} - r)dt + \sigma dW^*,$$

- risk-adjusted (r.a.) mean reverting level

$$\bar{r} = m - \frac{\sigma}{a}\lambda,$$

- r.a. drift $m(r,t) = a\bar{r} - ar$ & diffusion $\sigma(r,t) = \sigma$.
- default-free discount bond price

$$v(t,T) = H_1(T-t)e^{-H_2(T-t)r(t)}, \qquad 0 \le t \le T,$$

$$H_2(t) = \frac{1 - e^{-at}}{a},$$

$$H_1(t) = \exp\left\{ \frac{(H_2(t) - t)(a^2\bar{r} - \sigma^2/2)}{a^2} - \frac{\sigma^2 H_2^2(t)}{4a} \right\}.$$

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Pricing Discount Bond via CIR Model

- Market price of risk $\lambda(t) = \frac{a(m-\bar{r})}{\sigma\sqrt{r(t)}}$, and SDE under Q

$$dr = a(\bar{r} - r)dt + \sigma\sqrt{r(t)}dW^*, \qquad 0 \le t \le T,$$

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- r.a. drift $m(r,t) = a\bar{r} - ar$ & diffusion $\sigma(r,t) = \sigma^2 r$.
- Let$\gamma = \sqrt{a^2 + 2\sigma^2}$, then price of d.f.d.b. is

$$v(t,T) = H_1(T-t)e^{-H_2(T-t)r(t)}, \qquad 0 \leq t \leq T,$$

$$H_1(t) = \left(\frac{2\gamma e^{(a+\gamma)t/2}}{(a+\gamma)(e^{\gamma t}-1)+2\gamma}\right)^{2a\bar{r}/\sigma^2},$$

$$H_2(t) = \frac{2(e^{\gamma t}-1)}{(a+\gamma)(e^{\gamma t}-1)+2\gamma}.$$

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Forward LIBOR and Black's Formula

- T_i-forward LIBOR $L_i(t)$ under $\mathrm{Q}^{T_{i+1}}$ is a martingale

$$L_i(t) = \mathrm{E}^{\mathrm{Q}^{T_{i+1}}}\left[L_i(\tau)\middle|\mathcal{F}_t\right], \quad t \le \tau \le T,$$

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- SDE T_i-forward LIBOR under $\mathbb{Q}^{T_{i+1}}$

$$\frac{dL_i}{L_i} = \sigma_i(t)dW^{T_{i+1}}, \quad 0 \le t \le T_i,$$

$\{W^{T_{i+1}}(t)\}$ is a standard Brownian motion under $\mathbb{Q}^{T_{i+1}}$.

Cap and Caplets

- Caplet price

$$\mathrm{Cpl}_i(t) = \delta_i v(t, T_{i+1}) \left[L_i(t)\Phi(d_i) - K\Phi(d_i - \varsigma_i) \right], \quad (1)$$

where δ_i are interval between tenor dates and

$$d_i = \frac{\ln(L_i(t)/K)}{\varsigma_i} + \frac{\varsigma_i}{2}, \qquad \varsigma_i > 0.$$

and $\varsigma_i^2 = \int_t^{T_i} \sigma_i^2(s) ds$ is accumulated variance.

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- Cap (portfolio of caplets) price

$$\mathrm{Cap}(t) = \sum_{i=0}^{n-1} \delta_i v(t, T_{i+1}) \left[L_i(t)\Phi(d_i) - K\Phi(d_i - \varsigma_i) \right], \ t < T_0$$

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Cap and Caplets

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- Same procedure for floor and floorlets. If $\delta_i = 1$, then (1) is identical to Black's formula.

Swap Rate and Swaptions

- Swap rate

$$S(t) = \frac{V_{FL}}{V_{FIX}} = \frac{v(t, T_0) - v(t, T_n)}{\delta \sum_{i=1}^{n} v(t, T_i)}, \qquad 0 \leq t \leq T_0.$$

Swap rates can be used as an underlying asset for an option so called swaptions.

Swap Rate and Swaptions

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- Swaption's SDE

$$\frac{dS}{S} = \sigma_s(t) dW^{Q^{T_i+1}}, \quad 0 \le t \le \tau$$

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  Swap rates can be used as an underlying asset for an option so called swaptions.

- Swaption's SDE

$$\frac{dS}{S} = \sigma_s(t) dW^{\mathbb{Q}^{T_i+1}}, \quad 0 \leq t \leq \tau$$

- Swaption price is approximated by Black's formula.

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Black's Volatility

- Dynamic of forward rates (cap/floor/swap rate)
- Lognormally distributed, i.e. Black's Model

$$df = \sigma_B f dW$$

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Black's Volatility

- Dynamic of forward rates (cap/floor/swap rate)
- Lognormally distributed, i.e. Black's Model

$$df = \sigma_B f dW$$

- Let $\pi_{C_N}(t) = \pi_{C_B}(t)$

$$\sigma_N = \frac{\sigma_B (f_0 - K)}{\ln(f_0/K)\left[1 + \frac{1}{24}\left(1 - \frac{1}{120}[\ln(f_0/K)]^2\right)\sigma_B^2\tau + \frac{1}{5760}\sigma_B^4\tau^2\right]}, \quad \frac{f_0}{K} > 0, \quad f_0 \neq K.$$

τ exercise date in years.

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τ exercise date in years.

- The alternative formula

$$\sigma_N = \frac{\sigma_B\sqrt{f_0 K}\left(1 + \frac{1}{24}[\ln(f_0/K)]^2\right)}{1 + \frac{1}{24}\sigma_B^2\tau + \frac{1}{5760}\sigma_B^4\tau^2}, \quad \text{for} \quad \left|\frac{f_0 - K}{K}\right| < 0.001.$$

Numerical methods (Newton-Raphson method) to get σ_B knowing σ_N.

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Bachelier's Model

- Bachelier's SDE

$$dS(\tau) = S(t)\sigma dW(\tau), \qquad 0 \leq t \leq \tau \leq T.$$

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Bachelier's Model

- Bachelier's SDE

$$dS(\tau) = S(t)\sigma dW(\tau), \qquad 0 \leq t \leq \tau \leq T.$$

- Normal price process

$$S(T) = S(t)\left[1 + \sigma\left(W(T) - W(t)\right)\right].$$

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- Normal price process

$$S(T) = S(t)\left[1 + \sigma\left(W(T) - W(t)\right)\right].$$

- Bachelier's price formula

$$\pi_C(t) = [S(t) - K]N(d) + S(t)\sigma\sqrt{(T-t)}\phi(d),$$
$$\pi_P(t) = [K - S(t)]N(-d) - S(t)\sigma\sqrt{(T-t)}\phi(-d),$$
$$d = \frac{S(t) - K}{S(t)\sigma\sqrt{(T-t)}}.$$

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$$\pi_P(t) = [K - S(t)]N(-d) - S(t)\sigma\sqrt{(T-t)}\phi(-d),$$
$$d = \frac{S(t) - K}{S(t)\sigma\sqrt{(T-t)}}.$$

- ATM $S(t) = K$ and implied volatility

$$\pi_C(t) = S(t)\sigma\sqrt{\frac{(T-t)}{2\pi}}, \quad \sigma = \frac{\pi_C(t)}{S(t)}\sqrt{\frac{2\pi}{(T-t)}}.$$

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Black's Model vs Normal Model

- Black's model

- Normal model

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Black's Model vs Normal Model

- Black's model
  - Black's SDE

$$df = \sigma_n f dW, \qquad 0 \leq t \leq T.$$

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$$f_T = f_t \exp\left\{\sigma_n(W_T - W_t)\right\},$$

equivalently

$$\ln\left(\frac{f_T}{f_t}\right) = \sigma_n(W_T - W_t), \quad \frac{f_T}{f_t} > 0, \quad f_t \neq 0.$$

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$$f_T = f_t + \sigma_n (W_T - W_t).$$

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Generating Sample Paths

- Heath–Jarrow–Morton (HJM) Framework

$$df(t,T) = \mu(t,T)dt + \boldsymbol{\sigma}(f,t,T)^\top d\boldsymbol{W}(t).$$

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$$df(t, T) = \left(\boldsymbol{\sigma}(f, t, T)^{\top} \int_t^T \boldsymbol{\sigma}(f, t, u)du\right) dt + \sigma(f, t, T)^{\top} d\boldsymbol{W}(t),$$

None of forward rates become martingale.

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  - Forward-neutral valuation under Q^{T_F}

$$df(t, T) = -\boldsymbol{\sigma}(f, t, T)^\top \left(\int_T^{T_F} \boldsymbol{\sigma}(f, t, u)du\right) dt + \boldsymbol{\sigma}(t, T)^\top d\boldsymbol{W}^{T_F}(t), \qquad t \leq T \leq T_F.$$

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- LIBOR Market Model (LMM)
  - Forward-LIBOR SDE (Spot measure)

$$\frac{dL_n(t)}{L_n(t)} = \sum_{j=\eta(t)}^{n} \frac{\delta_j L_j(t)\boldsymbol{\sigma}_n(t)^\top \boldsymbol{\sigma}_j(t)}{1+\delta_j L_j(t)} dt + \boldsymbol{\sigma}_n(t)^\top d\boldsymbol{W}(t), \quad 0 \le t \le T_n, \quad n = 1, \ldots, M.$$

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  - $L_n = (v_n - v_{n+1})/(\delta_n v_{n+1})$, bond price is martingale when it is deflated (rather discounted) by the numériare asset.

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Lehman Brothers Bankruptcy

Biggest bankruptcy in the US history, Sep 15,2008

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- Main reasons
  1. Liquidity problem (Lender refused to roll over funding),
  2. High leverage (ratio 31:1),
  3. Risky investments (Large positions in mortgage derivatives).

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  1. Create systemic risk,
  2. OTC transaction cost can lead others to go to default,
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- Credit default swap (CDS)
  1. $400 billions of CDS contracts,
  2. $155 billions out standing dept,
  3. Payout to buyers of CDS was 91.375% of principle.

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Unsecure vs Secure Trade

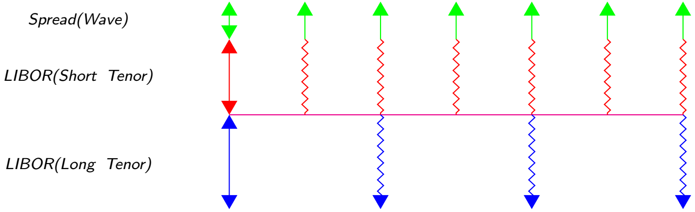

Figure: A 3-month floating against a 6-month floating rate

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Unsecure vs Secure Trade

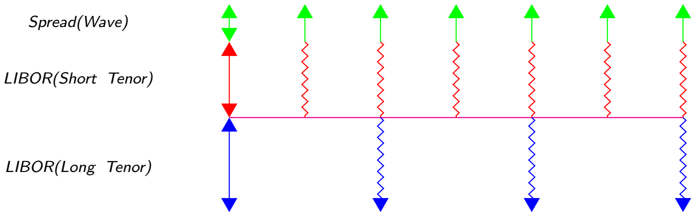

Figure: A 3-month floating against a 6-month floating rate

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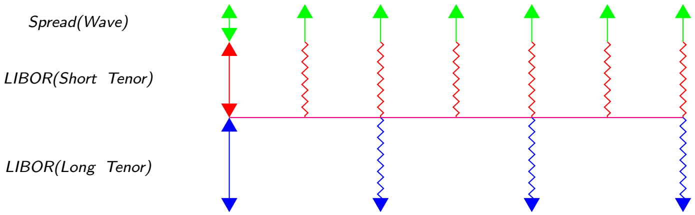

Figure: A 3-month floating against a 6-month floating rate

- Before crisis, spread (wave) considered to be zero/close to zero,
- After, it represents the difference in risk levels and it can be quite significant.

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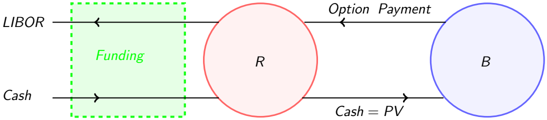

Figure: Unsecured trade with external funding.

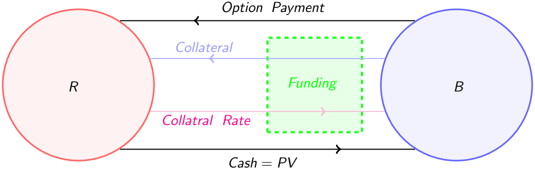

Figure: Secured trade with external funding.

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Collateral Agreement (CSA)

Base Currency	USD
Eligible Currency	USD, EUR, GBP
Independent Amount	5 Million
Haircuts	[Schedule]
Threshold	50 Million
Minimum Transfer Amount	500,000
Rounding	Nearest 100,000 USD
Valuation Agent	Red Firm
Valuation Date	Daily, New York Business Day
Notification Time	2:00 PM, New York Business Day
Interest Rate	OIS, EONIA, SONIA
Day Count	Act/360

Figure: Data in a collateral agreement.

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Multiple Currency Bootstrapping

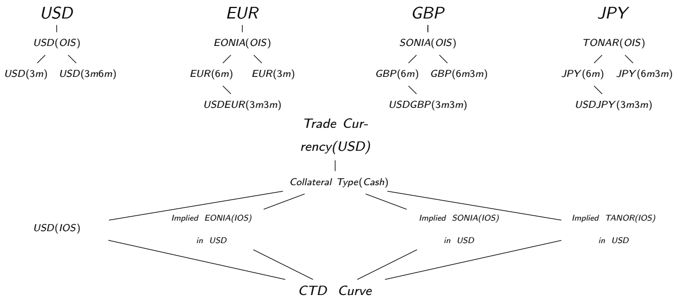

Figure: An Example of Multiple Currencies Bootstrapping Amounts

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Pricing under CSA

- Assumptions
  1. Full collateralization (zero threshold) by cash,
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- R instantaneous return (cost if it is negative)

$$R^{(f)}(t) = r^{(f)}(t) - c^{(f)}(t).$$

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$$\begin{aligned}\pi^{(d)}(t) &= \mathrm{E}^{\mathrm{Q}(d)}\left[\exp\left\{-\int_t^T r^{(d)}(u)du + \int_t^T R^{(f)}(u)du\right\}\pi^{(d)}(T)\Big|\mathcal{F}_t\right] \\ &= v^{(d)}(t,T)\mathrm{E}^{\mathrm{Q}^T_{(d)}}\left[\exp\left\{\int_t^T R^{(d,f)}(u)du\right\}\pi^{(d)}(T)\Big|\mathcal{F}_t\right], \qquad 0 \le t \le T.\end{aligned}$$

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Pricing Derivatives Under CSA

- Curve construction in single currency
  1. Choose the calibration instrument to adjust the starting point of simulation,
  2. Bootstrap a forward curve,
  3. Find the discount factor.

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Pricing Derivatives Under CSA

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- Calibration instruments
  1. Overnight indexed swap (OIS),
  2. Interest rate swap (IRS),
  3. Tenor swap and basis spread.

Conclusion

- Deterministic and stochastic interest rates,
- Risk and forward neutral probability measure,
- Term-structure model and negative interest rate,
- Pricing interest rate derivatives,
- Creating sample paths,
- New framework under CSA.

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