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An Introduction to Modern Pricing of Interest Rate Derivatives

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Hossein Nohrouzian

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MÄLARDALEN UNIVERSITY

SE-721 23 VÄSTERÅS, SWEDEN

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Authors:

Hossein Nohrouzian

Supervisors:

Jan Röman

Anatoliy Malyarenko

External Reviewer:

Daniel Andrén

Examiner:

Linus Carlsson

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Dedicate to my father Javad and my mother Ashraf.

Abstract

This thesis studies interest rates (even negative), interest rate derivatives and term structure of interest rates. We review the different types of interest rates and go through the evaluation of a derivative using risk-neutral and forward-neutral methods. Moreover, the construction of interest rate models (term-structure models), pricing of bonds and interest rate derivatives, using both equilibrium and no-arbitrage approaches are discussed, compared and contrasted. Further, we look at the HJM framework and the LMM model to evaluate and simulate forward curves and find the forward rates as the discount factors. Finally, the new framework (after financial crisis in 2008), under the collateral agreement (CSA) has been taken into consideration.

Keywords: Interest Rates, Negative Interest Rates, Market Model, Martingale, Security Market Model, Term Structure Model, Risk-Neutral Measure, Forward-Neutral Measure, LIBOR, HJM, Collateral, Swap, Tenor, Interest Rate Derivatives, CSA Agreement, Bachelier.

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Chapter 1

Introduction

At the late of previous century and the beginning of current century, the return on the investments has been an important discussion in almost everyone's daily life. In economics perspective of view, a rational person would prefer more rather than less and people would like to increase the amount of their capital by entering to the safe investments. There exists a variety of investments, like starting a company to produce a product or starting a business to provide some services, but not all people have the corresponding specialties and capital to do so or they would not take such a big risk. Now, let's see what the terms "return" and "risk" really mean?

It is quite often to say that, there exists two kinds of investments, riskless and risky ones. The riskless investment can be seen as an investment without the risk of losses. A good example of a riskless investment is putting the money in the bank and receiving the interest rate (in the case that the interest rate is positive). On the other hand, the risky investment contains the risk of losses and at the same time the investor might be awarded by more profit because of taking such risks.

It is not the whole story yet. Even before introducing the negative interest rates in the market, some economists, market specialists, portfolio managers and hedge funds managers have been claiming that the riskless return on the investment is not worthy. In my opinion, the main reasons of such a claim are firstly comparing the average return of risky investments, like return on the stocks and average rate of return on the riskless investments, like getting interest rate from a bank, governments bonds and treasury bills. As an example, the growth in NASDQ value has been increased by almost 150% from the beginning of 2010 till beginning of 2015. This is depicted in Figure 1.1 (source Reuters). Additionally, taking the inflation into account and see the average return on a riskless investment would recover the inflation rate. Finally, keeping money in the account does not lead any value-added to capital, while buying an apartment or buying some shares of a developing and successful company may have some positive return and value-added. On the contrary, some specialist say that an exponential growth in the return of risky investment is impossible and this kind of returns will collapsed and will not continue in long term. Let's discuss the role of interest rates in more details.

Figure 1.1: Price behavior of the NASDAQ from 2010 to 2015 (source Reuters)

In the financial market, interest rates are the key parameters to evaluate the prices of derivatives and in the theory it is common to say that the average return on the risky asset is nothing but the interest rates. Further, if everyone is going to be awarded with just profits and no losses, then it means that the market provides the opportunity of free lunch or arbitrage for the investors. Such story in macroeconomics points of view, means less and less productions, services, new jobs, social welfare and so on. But, since the financial market contains risks, most investors, especially small individuals loose a fraction of their initial investments or gain very little. Besides, due to the huge amount of money, the large number of transactions and the benefits from the international marketing, the financial market provides a society with lots of new job opportunities and guarantees the governments with more taxes and an active and alive economy. The question that may arise is, what is the role of interest rate as a powerful object in the hand of governments?

The rate of interest is a powerful key in the government's monetary policy. Governments can control consumption by increasing and decreasing the interest rates. If the governments set the interest rate high (sometimes by selling treasury bills) it means that they would like the society to consume less and save more. On the contrary, decreasing the interest rate can happen when the governments would like the society consumes more and save less. In the international trading point of view, the interest rate can be an instrument to control the exchange rate between currencies. In the last few years, some countries have started reducing their interest rate to make their exported products cheaper and to compete easier in the international market. At the same time, lower exchange rate would encourage people to consume national products instead of imported goods and services. In this manner, the society will be more active in its own economy and industry. For example and as we can see in Figure 1.2 (source Reuters) the exchange rate of Swedish Krona against US dollar has been changed from 6.5 in the beginning of year 2014 up to almost 9.0 in the beginning of year 2015[1].

[1]To see the difference, let's have an imaginary scenario. Assume Miss Lucky has 130,000 SEK in the beginning of year 2014. Further, there is no transaction costs and there is no spread to exchange currencies. Miss

The main reasons of USD strength against SEK are the positive and successful growth in the US economy and decreasing interest rate in Sweden (to 0%, -0.1% and -0.25%). It is worth to mention that, in the most cases governments are willing to achieve a desirable inflation rate. Such an inflation rate can easily make the profits from interest rate (in the case that interest rate is positive) close to zero. The reason is that, if everyone postpones buying, consuming or investing only because of increasing their purchasing power in the future, then the economy of a society will not work properly. Now let's consider the market and see how the market sets and evaluates the interest rates.

Figure 1.2: The exchange rate between USD and SEK (source Reuters)

Before the economic crisis in 2007 and 2008 the XIBOR was used as the risk-free market interest rates. Here X stands for the capital city. For example, LIBOR is London Interbank Offered Rate and SIBOR stands for Stockholm Interbank Offered Rate. The LIBOR can be described as a reference of interest rate for loans in the international financial market [18]. Further, in a swap contract we have both fixed (which can be obtained from forward rates or forward curve) and floating rates (usually evaluate on the overnight indexed swaps). These rates are evaluated every working day in the market. It is easy to see that this evaluation cannot give a constant interest rate and in fact the interest rate is stochastic and the volatility in the market can affect such interest rates. Because of the existing volatilities in the market the overnight rate has a key role in the evaluation of interest rates. Stock prices and currencies

Lucky, changed her 130,000 SEK to 20,000 USD and bought the NASDAQ stocks. In a year the NASDAQ value has gone up by 25% (which means a good profit) and she sold her stocks for 25,000. Directly after selling her stocks, she exchanged her USD to SEK by exchange rate of 9. So, she could successfully increase her 130,000 SEK to 225,000 which means $(225-130)/130 = 0.73$ or 73% profit in a year. If Miss Lucky did the reverse, she would end up with making her capital less than a half.

exchange rates are really sensitive to news and correlated to some factors. For example the exchange rate of NOK (Norwegian Krona) is highly correlated to oil's price. News about the economic growth, unemployment rates, GDP (Gross Domestic Product) and GNP (Gross National Product) can affect the prices of stocks and exchange rates between currencies. According to the Reuters, on 15th of January 2015 the Swiss National Bank (SNB) unexpectedly scrapped its cap on the Euro value of the Franc and consequently the CHF (Switzerland Franc) became very strong against the basket of major currencies. As we can see in Figure 1.3 the exchange rate between USD and CHF had the highest value of 1.02 and lowest value of 0.74. This means $(1.02-0.74)/1.02 = 0.2745 = 27.5\%$ change in the exchange rate. Such examples shows the vital role of daily evaluation of interest rates.

Figure 1.3: Exchange rate between USD and CHF (source Reuters)

From and after economic crisis in 2007 and 2008 the LIBOR rate has been replaced by collateral rate. The collateral rate is used in the collateral agreement or CSA (Credit Support Annex) and this rate is also calculated daily on the overnight index swaps. The best advantage of collateral agreement and collateral rate over the LIBOR rate are their safeties against the credit defaults and their strength to reduce the possibility of huge losses due to the credit defaults of other counterparties. Further, when the collateral agreement is valid in more than one currency, the collateral rate is set in a currency which gives the highest rate, i.e. cheapest to deliver. The valid currency which gives the collateral rate can be changed very often. Let's close the background and introduction here and start presenting our objectives at this work.

Main Objectives in Modern Pricing of Interest Rate Derivatives

As the main objective in this thesis, we are willing to write in a way that all readers from different groups of people can easily communicate and follow our text, steps and explanations. Therefore, we will try to explain the market structure and present the basic ideas and backgrounds in economics, finance and financial mathematics related to our work. Although in more complicated level of math, i.e., stochastic processes, we have tried to explain every details, but still it seems that the reader should have at least some basic knowledge and background in financial mathematics and stochastic calculus. Moreover, we have used LaTeX to type this thesis and for convenience we constructed index and glossary parts at the end of this work and the PDF version of report has the capability to guide the readers directly to referred chapters, sections, equations, definitions, theorems, formulas, figures and references. As the topic of this thesis states, we are going to introduce the modern pricing of interest rate derivatives. So, we mainly deal with three key words, i.e., interest rate, derivative and pricing. We have considered following objectives and we have taken following steps in this work.

What are interest rates? The answer to this question can be found in Chapter 2, where we present the most commonly used interest rates and their usage in the market. We introduce briefly the role of constant interest rate in the price evaluation of financial derivatives. Further, we explain the basic idea of evaluation of forward rates via bootstrapping. Finally, we end this chapter by introducing the money market account and interest rate derivatives. Still, some big questions might be remained and these questions are, how the interest rates can be used in the evaluation of security prices? What is price process? Is interest rate always a constant and positive? To answer these questions we construct individual chapters.

How to price derivative securities and interest rate derivatives? After introducing interest rates, we need to know the pricing procedure of interest rate derivatives. To do so, we need to be familiar with price processes and pricing derivatives. The necessary conditions for finding a price is to be familiar with stochastic calculus. In Chapter 3 we state the most necessary definitions and theorem one needs to know for evaluating the price of a derivative. There, we present some related economics term in mathematical language like asset pricing theorem and no-arbitrage models. Further, we consider two main pricing models, namely risk-neutral and forward-neutral evaluations. In risk-neutral evaluation, we present the usage of constant interest rate and how we can discount our price by risk-free interest rates. We look at Black–Scholes–Merton Model as an example related to risk-neutral method as well. After that, we look at some stochastic volatility models under risk-neutral evaluation framework which can somehow describe the sudden movements in the prices like what we have seen in Figure 1.3. Finally, we state the forward-neutral model which is suitable to deal with stochastic interest rate. At the end of Chapter 3, we have the forward-neutral method and forward rates as a tool to discount the price of interest rate derivatives. But, how we can price the interest rate derivative? The answer to this question is coming in Chapter 5 and before that we have to be familiar with the stochastic interest rate models and their corresponding stochastic differential

equations (SDE), i.e., term-structure models which we will look at them in Chapter 4.

What are the dynamics of stochastic interest rate processes? Now, is the time to deal with the dynamics of stochastic interest rates. In Chapter 4, we look at interest rate models or term-structure models. There, we present some models in two main categories, namely equilibrium models and no-arbitrage models. Further, we compare and contrast these two models and we will see that there already exists some models which can give us negative interest rates. Finally, we will present some approaches to price a discount bond in this chapter. Still, we have not presented the evaluation of interest rate derivatives? What are the tools to do so?

What are the most commonly used tools to price an interest rate derivative? In Chapter 5, we will see how one can evaluate the price of an interest rate derivative. We introduce three main commonly used tools in the price evaluation of interest rate derivatives in the market. They are caps, floors and swaptions. In this procedure we will look at bond options, forward LIBOR and Black's formula as well. We state Black's formula and explain how this formula has been used in the market. After that, we will go through the problems with the characteristic of Black's formula and negative interest rate. That is the price process in Black's model is lognormally distributed and guarantees the positive prices. After that we look at the Bachelier model where the price process is normally distributed and can give us negative prices. The question may arise is, how to estimate our discount factor in the interest rate models using forward rates?

How to evaluate the forward rates and use them to find the discount factor? To evaluate forward rates and use them to find the discount factor, we need to have some proper models and we have to use computer programs. Here, we have stochastic interest rates and we might use implied volatility, constant volatility or different volatilities during the time. Therefore, we need to have a model and using such a model simulate our forward rate for several million times and take the average of our result and estimate the forward rates. After we obtain the estimated forward rates, we can find our discount factor and price a desirable interest rate derivative. In Chapter 6 we will look at the Heath, Jarrow and Morton (HJM) framework and the LIBOR Market Model (LMM) to see how we can evaluate the forward rates and forward-LIBOR rates in the long term (usually up to 30 years). In the HJM framework, we go through the evaluation of the forward rates under risk-neutral method and forward-neutral method and in the LMM model, we consider spot measure. We will present some algorithm to implement these models in computer programming languages and finally we discuss about the volatility in these models. We omit to simulate and bootstrap forward curves with real data, because it can be really time consuming and it is beyond the time scope of this work. However, I personally have planned to do some proper simulation in future works. After this chapter, we will update ourselves with the new framework which is used in today's market.

What is the most commonly used framework in today's market? In today's market, the most commonly used and popular framework is called collateral agreement or credit support annex (CSA). In Chapter 7 we will look at this new secure framework and we will compare it with unsecure framework before economic crisis in 2007 and 2008. Again, we will present both risk-neutral and forward-neutral measures to evaluate the price of a derivative under collateral agreement. We will see the role of collateral rate and we present briefly the role of multiple currencies and exchange rate between currencies in this chapter as well. Further, we review the forward curve construction using three major swap rates, i.e. overnight indexed swap (OIS), interest rate swap (IRS) and tenor swap (TS). Finally, we look at HJM framework to construct the forward curves under collateral agreements. After that, we will close this work by our conclusion.

Chapter 2

Interest Rates

One of the most important factors to evaluate the price of a derivative is interest rate. To begin with, the reader who wants a deeper knowledge of the concepts and definitions in Chapter 2 is referred to the standard textbooks in the subject, i.e. [18] and [22]. However, this chapter briefly introduces the measurements and analyzes the different type of interest rates. Further, we will cover the definition of some financial terms like zero rates, par yields, yield curves and bond pricing. We will also discuss the procedure to calculate zero-coupon interest rates, i.e., bootstrapping. Finally, we close this chapter by introducing derivative securities and money market account.

2.1 Type of Rates

An interest rate can be seen as the amount of money a borrower promises to pay the lender. This is also true even for a given currency with specific type of rate. The rate of interest rate depends on the credit risk, i.e., when a borrower faces a default.

There are different kind of rates such as treasury rate[1], LIBOR, repo rate and risk-free rate. The risk-free interest rate has extensively been used to evaluate the price of derivatives. Because of the key roles of risk-free interest rates, we will review them separately in the next section. We close this discussion with following remarks.

Remark 2.1.1.

- Although treasury bills and bonds are counted as a risk-free, they do not imply the risk-free rate. Because they give an artificially low level of rate due to the tax and regulatory issues.

[1]Treasury rates are counted as a risk-free rates, because a government does not go to a default.

- LIBOR rates were traditionally used by investors as a risk-free rate, but the rates are not totally risk-free.
- After financial crisis in 2007 and 2008 many dealers started using OIS (Overnight Indexed Swap) rates as the risk-free rate. See page 77 in [18].

2.1.1 Zero Rates

The n-year zero-coupon interest rate is an interest which can be earned on an investment which starts today and lasts for n-years such that no intermediate payment will be occurred and the interest amount and principal will be realized at maturity.

Remark 2.1.2. Some times, zero-coupon interest rate is referred to the n-year spot rate, the n-year zero rate, or just n-year zero.

2.1.2 Bonds

Unlike zero-coupon interest rates, almost all bonds provide some payments to their holder and these payments are based on predetermined periods. The bond's principal [2] like zero-coupon interest rate is paid back at the end of bond's life.

Remark 2.1.3. We skip the evaluation of pricing a bond just now, but when we find the theoretical price of a bond we can find a single discount rate, i.e., bond yield. Moreover, it is possible to find a single coupon rate par yield which causes bond price to be equal to its face value [18].

2.1.3 Forward Rates

The rates of interest which are implied by today's zero-coupon interest rates for a period of time in the future, are called forward rates.

2.2 Risk-Free Rate

Although it is common to say there exist no risk-free rate [29], the term "risk-free rate" or risk-free interest rate has been used in lots of literature and has a key role in the price evaluation of security derivatives [18]. The risk-free rate is mostly used as a discount factor and is a component in deterministic part of price processes when we are dealing with models which

[2] Is also called *notional*, *par value* and *face value.*

assume the interest rate is a constant. Through, this work we will use the term risk-free rate in such models to distinguish the constant interest rate and stochastic interest rate. As we mentioned in Remark 2.1.1, the market and its traders have their own procedure to evaluate such a risk-free rate and it can be different with the interest rate the central banks or individual banks set for their costumers. For example, before economic crisis in 2007 and 2008 the LIBOR rate was commonly being used as a risk-free rate and now a days the OIS rate is used as a proxy for the risk-free rate in the market.

2.2.1 Pricing Financial Derivatives

Talking about the financial market and its derivatives contains some important terms such as price, risk and expected return. Simply, a rational investor, invests in some assets on the market to get some positive return on a portfolio. Additionally, the investments can be categorized in two main fields, riskless and risky ones. The riskless investments have predetermined returns and contain no risks, such as investing in banks or buying government bonds for a specific level of returns. On the other hand, the risky investments such as buying derivatives or options can have either positive or negative level of returns. In general, investors may invest in the risky market and take the existing risk of losses, when they know that they would be awarded by some higher level of returns than returns on investments with lower risks. Simply, taking the higher level of risk, demands the higher level of expected return.

In simple financial mathematics texts and elementary courses, we can see that the price of a financial derivative is set to be equal to its discounted expected payoff. But, how to define and measure the discount factor? To do so, we need to be familiar with a very important and fundamental principle in the pricing derivatives known as risk-neutral valuation. Let's discuss risk-neutral valuation in the following section and after that we will go through the expected payoff.

2.2.2 Risk-Neutral Valuation

Risk-neutral valuation assumes that, in valuing a derivative all investors are risk-neutral. This assumption states that investors do not increase the expected return they require from an investment to compensate for increased risk. This world where all investors are risk-neutral is called a risk-neutral world. The risk-neutral world has contradictions to the world we are living in, which is true. As we said before, in the real world the higher risk demands the higher expected return. However, this assumption gives us the fair price of a derivative and the right measurement for discount factor in the real life. The reason is simply because of the risk aversion. The more risky investments make investors more risk averse. See Chapter 12.2 in [18].

Now, let's introduce two important features of risk-neutral world in pricing derivatives

1. The expected return on a stock (or any other investment) is the risk-free rate,

2. The discount rate used for the expected payoff on an option (or any other investment) is the risk-free rate.

In more mathematical words and under some assumptions, there exists a unique risk-neutral probability measure P^* equivalent to the real probability measure P such that under this probability P^* [8]:

1. The discounted price of a derivative is martingale[3] ,
2. The discounted expected value under the risk-neutral probability measure[4] P^* of a derivative, gives its no-arbitrage price[5].

Now, we know how to discount an expected payoff, i.e., the expected return to calculate the price of a derivative. But, how do the prices over the life time of derivatives change? What will be the expected payoff? We will discuss this in more details in following sections.

2.2.3 Expected Payoff

To begin with, denote the time-t price of a derivative by $\pi(t)$, where we discount the expected payoff with continuously compounded interest rate r for the derivative's life time T. Further, define the payoff for the contingent claim X by $h(X)$. Then, we have:

$$\pi(t) = e^{-r(T-t)}\mathsf{E}\left[h(X)\right]$$

As we know, the payoff depends on which financial instrument we are using. For example, the payoff to European call option is simply $h(X) = \max\{S_T - K, 0\}$, where S_T represent the price of stock at its maturity time T and K is strike price.

2.3 Introduction to Determine Treasury Zero Rates and LIBOR Forward Rates Via Bootstrapping

There are several different ways to determine the zero rates such as *yield on strips*, *Treasury bills* and *coupon bearing bonds*. However, the most popular approach is known as bootstrap method. See Chapter 4.5 in[18]. Our objective is to calculate all necessary coordinates for a zero-coupon yield curve using the market data. This curve is continuous in a specific time interval, but the market data are usually provided for different time interval, i.e., $\Delta T_1, \Delta T_2, \ldots, \Delta T_n$. The bootstrap method and bootstrapping have key roles in our work. So, at this point we will give an example from page 82 in [18] to illustrate the nature of this method. Later at this work, we will deal with more complicated and realistic approaches for bootstrapping forward curves. In other words, we will look at some models where the interest rates as

[3]The detailed definition of martingale is presented in Section 3.2.5.
[4]The mathematical meaning of risk-neutral probability measure is given in Section 3.2.6.
[5]The meaning of no-arbitrage price will be presented in Theorem 3.2.5 and Section 3.2.7.

well as forward rates are stochastic and volatility plays more important role in the evaluation of a forward curve.

Example: The prices of five bonds are given in Table 2.1. Using a bootstrap method to find the continuously compounded zero rates and draw the zero-coupon yield curve. Note that coupons are assumed to be paid every 6 months.

Bond principal ($)	Time to maturity (years)	Annual coupon ($)	Bond Price ($)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

Table 2.1: Data for bootstrap method

Solution The first row on the table tells us that a \$97.5 investment will turn out to \$100 after $\Delta T_1 = 0.25$ years or 3 months respectively. We can easily calculate the corresponding zero rates of this investment with continuous compounding as follow:

$$100 = 97.5e^{r_1 \times 0.25} \Rightarrow \ln\left(\frac{100}{97.5}\right) = \ln\left(e^{r_1 \times 0.25}\right)$$
$$r_1 = 4 \times [\ln(100) - \ln(97.5)] = 0.10127 = 10.127\%$$

Similarly, we can calculate r_2 and r_3 for second and third bonds and their respective times $\Delta T_2 = 0.50$ and $\Delta T_1 = 1.0$ year. Which will give us $r_2 = 10.496$ and $r_3 = 10.536$.

For forth and fifth bonds, we have to consider coupon payments as well. So, for the forth bond, the bond holder will get \$4 after 6 months and another \$4 after a year. Finally, the holder of the bond will get \$104 after 1.5 years, i.e., the bond principal and its coupon payment. The corresponding cash flows are graphically shown in Figure 2.1.

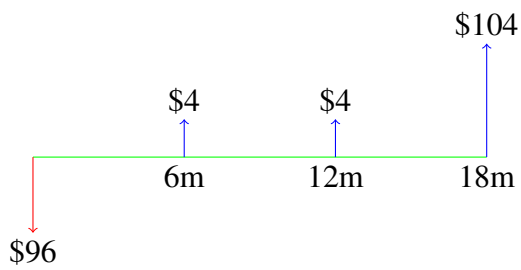

Figure 2.1: The cash flows of 4th bond

Then, we will have:

$$4e^{-r_2\times\Delta_{T_2}} + 4e^{-r_3\times\Delta_{T_3}} + (100+4)e^{-r_4\times\Delta_{T_4}} = 96 \Rightarrow r_4 = 10.681\%$$

Note: We did not use the interest rate for 3 months, because coupons are paid every six months.

Similarly, we can calculate $r_5 = 10.808\%$. Now, we have all zero rates in Table 2.2 for creating our yield curve, as shown in Figure 2.2.

Maturity (years)	Zero rates (continuously compounding)
0.25	10.127
0.50	10.469
1.00	10.536
1.50	10.681
2.00	10.808

Table 2.2: Continuously compounded zero rates determined from data in Table 2.1

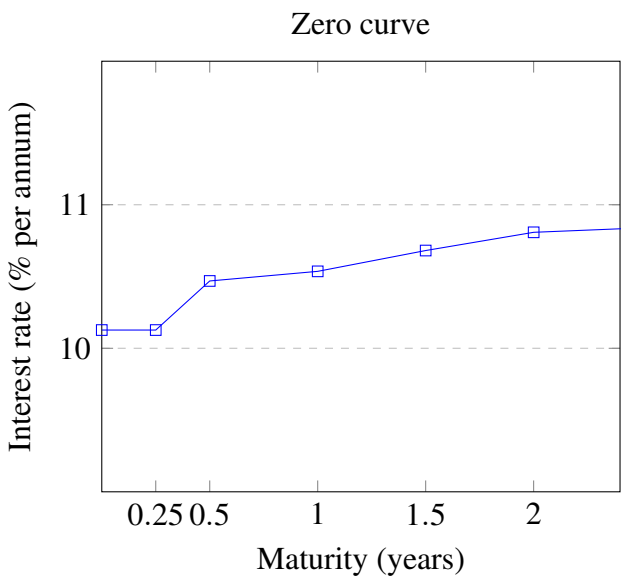

Figure 2.2: Zero rates given by the bootstrap method

□

Remark 2.3.1.

- For simplicity, we assume that the zero curve is linear between the points determined via bootstrapping, i.e., we use linear interpolation to find the zero rate at time 1.25. Although, we can use extrapolation, or polynomial approximation to approximate our zero curve but for some technical reasons, linear interpolation is the most commonly used method.
- We use the same approach to draw the LIBOR forward curve.

2.4 Interest Rate Swap

In early 1980s, the first swap contracts were used and now a days interest rate swaps are in the core of derivatives market. A swap can be described as an over-the-counter (OTC) agreement between to parties to exchange cash flows in future. The swap agreement contains the dates of cash flows and the way of calculating these cash flows. The calculation of these cash flows can involve the market variables such as the future value of an interest rate, an exchange rate and so on. There exists different kind of swaps such as plain vanilla interest rate swaps, fixed-for-fixed currency swaps, compounding swaps, cross currency swaps and asset swaps. In this section, an attempt is made to illustrate the most commonly used swap which is plain vanilla interest rate swap. This swap is an agreement between two counter parties in which counter party A pays cash flows equal to an interest at a predetermined fixed rate on a notional principal for a predetermined number of years to the B. On the other hand, counter party B pays interest rate of the floating rate on the same notional principal for the same lifetime agreement to the A.

2.4.1 LIBOR

In most interest rate swap agreement, LIBOR is the floating rate. In other words, LIBOR can be seen as a reference rate of interest rate for loans in the international financial market. To illustrate the idea, let us look at following example.

Example

Here, we present a shorter version of an example at page 149 in [18]. Assume a 2-years swap which is initiated on 5th of March 2014 between two financial firms, Red and Blue on a \$100 million. The Blue firm (*fixed-rate payer*) has agreed to pay an interest of 5% on the agreed principal and in return the Red firm (*floating-rate payer*) has promised to pay the Blue firm the 6-month LIBOR rate on the same principal. This procedure is given in Figure 2.3. Assume

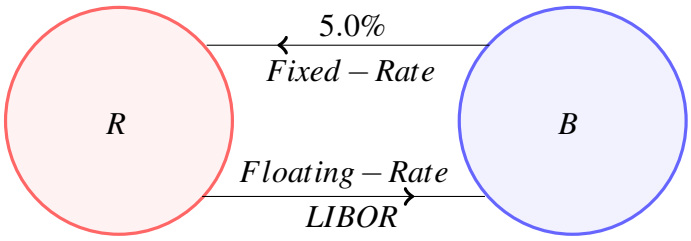

Figure 2.3: Interest rate swap between Red and Blue firms.

that the interest rate is quoted with semi-annual compounding. The first exchange of payments would occur after six month, i.e., 5th of September 2014. The Blue firm would pay the fix amount of interest which is $5\% \times 0.5 \times 100 \times 10^6 = \$2.5$ million to the Red firm. As we can

see on the forth column of Table 2.3 this amount will be fixed during the whole period of contract. On the other hand, the Red firm has an obligation to pay the Blue firm the interest at the 6-month LIBOR rate for last six month, say 4.2%. This means the Red firm has to pay the Blue firm $4.2\% \times 0.5 \times 100^6 = \$2.1$ million. Suppose we are in the March 2016, so we know 6-month LIBOR rates at all payment's time. These LIBOR rates are given in the second column of Table 2.3. Looking at Table 2.3 we can see the net cash flows for the Blue firm. In our example, the Blue firm has a negative net cash flow of $100,000.

Date D/M/Y	6-month LIBOR rate (%)	Floating cash flow (received)	Fixed cash flow (paid)	Net cash flow
5/3/2014	4.20			
5/9/2014	4.80	+2.10	-2.50	-0.40
5/3/2015	5.30	+2.40	-2.50	-0.10
5/9/2015	5.50	+2.65	-2.50	+0.15
5/3/2016		+2.75	-2.50	+0.25

Table 2.3: Cash flows in millions of dollar to the Blue firm.

2.4.2 Using Swap to Transform a Liability

Suppose the Blue firm wants to transform a floating-rate loan into a fixed-rate loan. For this purpose, the Blue firm has already borrowed $100 million at LIBOR rate plus 10 basis points (One basis point is one-hundredth of 1% , i.e., $0.01 \times 1/100 = 1 \times 10^{-4}$). So, the Blue firm will pay LIBOR+0.1% for the money has borrowed. On the other hand, the Red firm wants to transform a fixed-rate loan into a floating-rate loan. For this purpose, the Red firm has got a 2-year $100 million loan with a fixed rate at 5.2% [18]. In this case, after entering to the contract the cash flows (paying and receiving amount) for the Blue and the Red firms is illustrated in Figure 2.4.

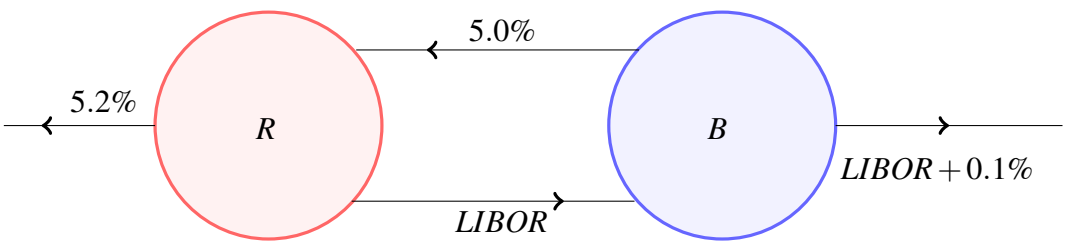

Figure 2.4: Red and Blue firms use the swap to transform liability.

2.4.3 Using Swap to Transform an Asset

Now, suppose the Blue firm is willing to transform a nature of an asset. In this case, a swap can be seen as transformation of an asset which earning a fixed interest rate to an asset which earn a floating rate of interest. Suppose that for next two years, the ownership of $100 million bonds provide 4.7% of yearly income in terms of interest to the Blue firm. On the other hand, the Red firm has the opportunity to transform its asset earning a floating-rate of interest to a fixed-rate of interest. Let's assume that, the Red firm has a source of income by LIBOR minus 20 basis point due to its $100 million investment. In this case, after entering to the contract the cash flows (paying and receiving amount) for the Blue and the Red firms is illustrated in Figure 2.5.

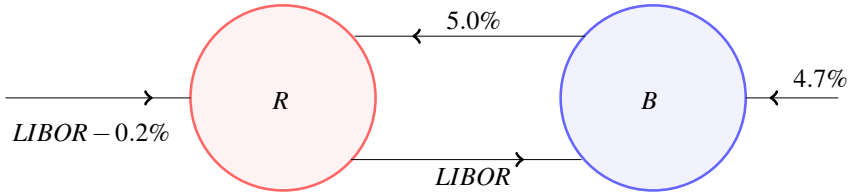

Figure 2.5: Red and Blue firms use the swap to transform an asset.

2.4.4 Role of Financial Intermediary

Suppose, the Blue and Red firms are non financial firms. In this case, these firms make a separate deal with a financial intermediary (Let's call it the Green firm) like a bank or financial institution. "Plain vanilla" fixed-for-floating swaps on US interest rate are setting in such a way that the intermediary party earn 3 or 4 basis points, which means 0.03% or 0.04% respectively on a pair of offsetting transactions [18]. If we consider 3 basis points, then our Figure 2.4 and Figure 2.5 will turn into Figure 2.6 and Figure 2.7.

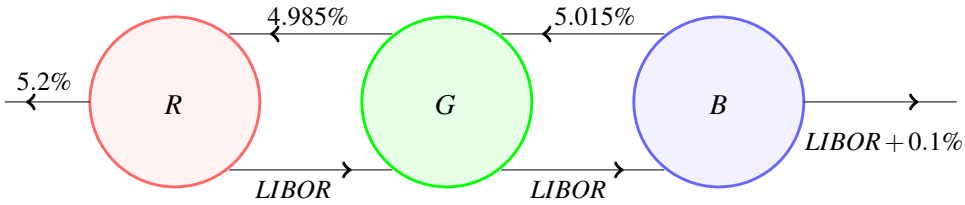

Figure 2.6: Swap as a liability in the presence of financial intermediary.

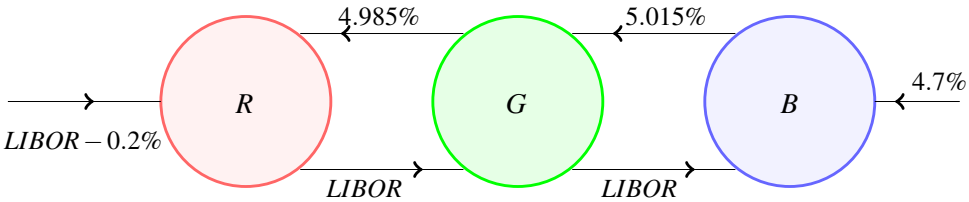

Figure 2.7: Swap as an asset in the presence of financial intermediary.

2.4.5 Currency Swap

Currency swap in its simplest form is exchanging principal and interest rate in one currency for principal and interest payment in another currency. The principal in most cases is exchanged in the beginning and end of the life of contract [18]. The currency swap can be transformed to a liability or an asset. Further, the currency swap agreement can be made in a presence of a financial intermediary as well. The graphical illustration are fairly the same as the graphs in previous part[6].

2.5 Interest Rate Cap

Interest rate cap is a popular interest rate option which is offering by financial institution in the over-the-counter market. Interest rate cap can be described by a floating-rate note whose interest rate is supposed to be reset in equal time period to LIBOR. The time interval between reset dates is called tenor. For example, if the tenor is a 3-month tenor, then a 3-month LIBOR rate will be applied on the note.

An interest rate cap, can be seen as an insurance against dramatic increase in the interest rate on the floating side. A certain level is set to determine the maximum amount of increasing. This maximum amount is called cap rate. The cap payoff diagram is shown in Figure 2.8, and Figure 2.9 represents a cap payoff with 6-month tenor basis. Note that, if the payoff is applicable (i.e., if is positive), then the payment would be occurred on the next reset date.

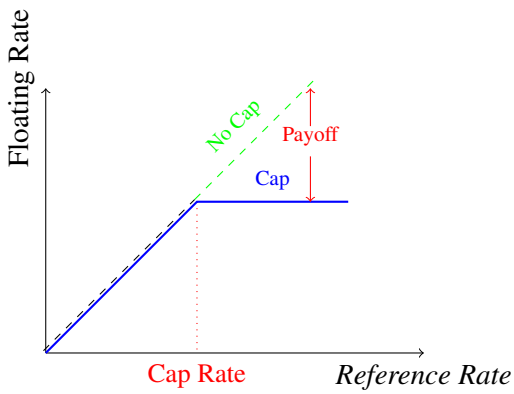

Figure 2.8: Cap Payoff

Remark 2.5.1.

- The total n (where n is a finite number) number of call options underlying the cap are called *caplets*.
- In fact, an interest rate cap is a portfolio of European put options on zero-coupon bonds. See page 654 in [18].

[6]For more detailed information see [18].

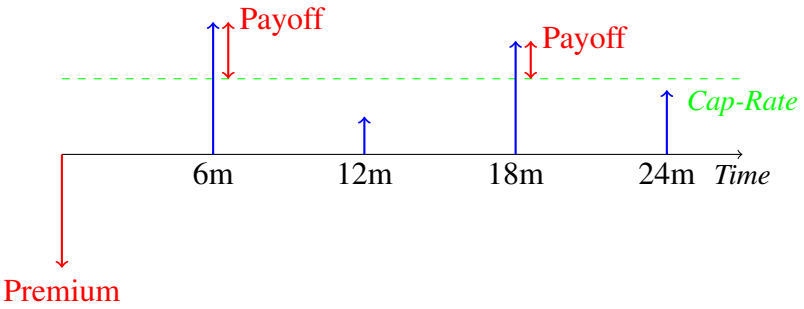

Figure 2.9: Payoff from a cap when the floating-rate exceeds the cap-rate

2.6 Interest Rate Floor

Interest rate floor has the same methodology as interest rate cap, but the payoff function of a floor has positive value, when the interest rate on the underlying floating rate note becomes less than a predetermined level. This level is called floor rate [18]. Figure 2.10 illustrates the floor's payoff and Figure 2.11 depicts the payoff to a floor with 6-month tenors.

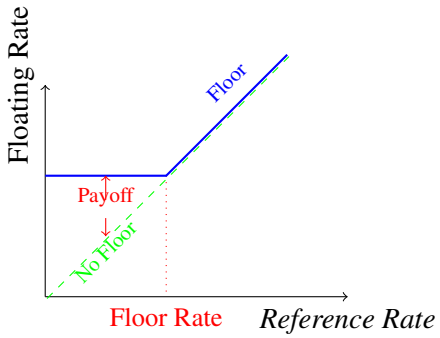

Figure 2.10: Floor Payoff

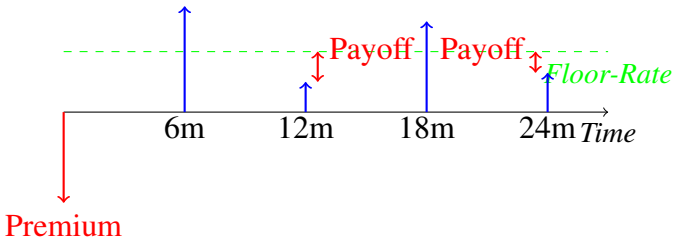

Figure 2.11: Payoff from a floor when the floating-rate does not reach the floor-rate

Remark 2.6.1.

- Every option which is involved in a floor is called *floorlet*.
- An interest rate floor can be seen as a portfolio of call option on zero-coupon bonds or as a portfolio of put option on interest rate. See page 654 in [18].

2.7 Interest Rate Collar

Interest rate collar or collar can be described as a financial instrument whose function is to keep the interest rate on the underlying floating-rate note between two levels, namely cap and floor. In other words, collar can be seen as a combination of a short position in floor and a long position in cap. Figure 2.12 illustrates how the payoff for a collar works.

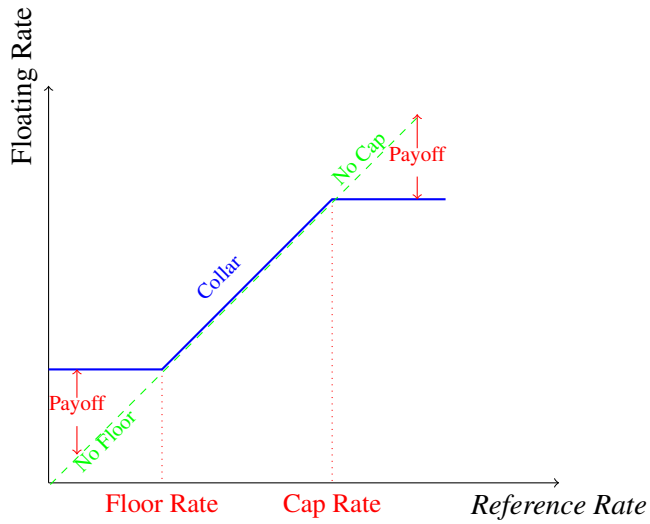

Figure 2.12: Collar Payoff

Remark 2.7.1.

- The most usual construction of floor is in such a way that the cost of entering in a floor is zero. This means, initial price of cap and floor set to be equal.
- The relationship between the prices of caps and floors is known by *put-call parity* and is given by (see page 654–655 in [18])

$$\text{Value of cap} = \text{Value of floor} + \text{value of swap}.$$

2.8 Derivative Securities

In the last section of this chapter, we start by introducing the mathematical definitions of the most commonly interest rate derivatives in the market. In the rest of this chapter, we considerably use the textbook in the subject, i.e., [22]. Now, let's go through the meaning of the derivative securities and we start by money market account.

2.8.1 Money Market Account

Suppose that a riskless deposit in a bank with initial principal (face value) $F(0) = 1$. Further, consider the amount of the initial deposit after t periods of time denotes by B_t i,e, B_t is called the money market account. Then, the amount of interest is paid for t-period of time can be calculated by $B_{t+1} - B_t$. After that, the compounded interest which is the interest rate[7] $r > 0$ whose value is proportional to the amount B_t, is given by

$$B_{t+1} - B_t = rB_t, \qquad t = 0, 1, 2, \ldots,$$

which implies

$$B_t = (1+r)^t, \qquad t = 0, 1, 2, \ldots.$$

If the interest rate r is annually compounded and it is paid n times per year, then the value of money market account after m periods of time will be

$$B_m = \left(1 + \frac{r}{n}\right)^m, \qquad m = 0, 1, 2, \ldots. \tag{2.1}$$

If we suppose that $t = m/n$, then the last equation will change to

$$B_t = \left(1 + \frac{r}{n}\right)^{nt}, \qquad m = 0, 1, 2, \ldots.$$

If we decrease the time intervals between the payments close to zero, or alternatively increase the number of payments n to infinity[8], then we can approximate $B(t)$ with the following limit

$$B(t) = \left[\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n\right]^t = e^{rt}, \quad t \geq 0. \tag{2.2}$$

which is nothing but, the value of money market account with continuous compounding interest rate after time t.

The Interest Rate Varies During The Time

Now, let's consider the more realistic case. Suppose the interest rate will vary during the time. That is,

$$r(t) = r_i \quad \text{if} \quad t_{i-1} \leq t < t_i, \quad ,t_0 = 0, \quad i = 1, 2, \ldots.$$

Then from (2.2) we have $B(t_1) = e^{r_1 t_1}$, $B(t_2)/B(t_1) = e^{r_1(t_2 - t_1)}$ and so on. Therefore, if $t_{n-1} \leq t < t_n$, we can calculate the value of money market account for time t by following formula

$$B(t) = \exp\left\{\sum_{k=1}^{n-1} r_k \delta_k + r_n(t - t_{n-1})\right\}, \qquad \delta_k \equiv t_k - t_{k-1}.$$

[7] In 2015 we have seen negative interest rates.

[8] See page 66 in [22].

Using our knowledge in calculus, we know that the integral of any (Riemann) integrable function is the limit of the sum, which implies

$$\int_0^t r(u)du = \lim_{n\to\infty}\left[\sum_{k=1}^{n-1} r_k\delta_k + r_n(t-t_{n-1})\right].$$

Let's end this discussion by introducing the following theorem.

Theorem 2.8.1. *Let $r(t)$ be the time-t instantaneous interest rate. If $r(t)$ is continuously compounded, then the value of money market account at time t will be*

$$B(t) = \exp\left\{\int_0^t r(u)du\right\}, \qquad t \geq 0, \tag{2.3}$$

provided that the integral exists.

2.8.2 Yield To Maturity

The rate of return on an investment per unit of time in the continuous time scale is defined by yield to maturity (yield). To describe this in mathematical language, let an investor at time t, buys a security for $S(t)$ amount of money and such a security at maturity time T pays $S(T)$ amount of money to the investor. Then, $R(t,T)$ can be described as a rate of return per unit of time and is given by

$$R(t,T) = \frac{S(T)-S(t)}{(T-t)S(t)}, \qquad t \leq T.$$

Which can be rewritten as

$$S(T) = S(t)[1+(T-t)R(t,T)].$$

Now, denote the rate of return per unit of time with n-times compounded interest rates per year by R_n. Then, using (2.1) the equation above can be rewritten as

$$S(T) = S(t)\left[1+\frac{(T-t)R_n(t,T)}{n}\right]^n, \qquad n = 1,2,\ldots.$$

Here, n denotes the number of interest rate compoundings per year. Define

$$Y(t,T) = \lim_{n\to\infty} R_n(t,T)$$

as the rate of return per unit of time in continuous time compounding frame. Then, we can use (2.2) and we obtain

$$S(T) = S(t)e^{(T-t)Y(t,T)}, \qquad t \leq T,$$

or equivalently,

$$Y(t,T) = \frac{1}{T-t} \ln \frac{S(T)}{S(t)}, \qquad t \leq T.$$

In the case that, the security is default-free discount bond with maturity T, i.e., $S(t) = v(t,T)$ and $S(T) = 1$, we will get

$$Y(t,T) = -\frac{\ln v(t,T)}{(T-t)}, \qquad t \leq T. \tag{2.4}$$

which is nothing but the definition of the yield of the discount bond. See page 68 in [22].

2.8.3 Spot Rates

Definition 2.8.1. Let the yield curve of the default-free discount bonds given by $Y(t,T)$, $t < T$. Then, the time-t **instantaneous interest rate (spot rate)** is defined by following limit

$$r(t) = \lim_{t \to T} Y(t,T), \tag{2.5}$$

provided that the limit exists.

Substituting (2.4) in (2.5) we obtain

$$r(t) = -\lim_{t \to T} \frac{\ln v(t,T)}{T-t} = -\frac{\partial}{\partial T} \ln v(t,T)\big|_{T=t}. \tag{2.6}$$

2.8.4 Forward Yields and Forward Rates

Definition 2.8.2. Let $f(t,T,\tau)$ be the time-t yield of the default-free discount bond over the future time interval $[T,\tau]$. Such a yield is called **forward yield**, i.e., $f(t,T,\tau)$ and is given by

$$\frac{v(t,\tau)}{v(t,T)} = e^{-(\tau - T) f(t,T,\tau)}, \qquad t < T < \tau.$$

The equivalent form of last equation can be rewritten as

$$f(t,T,\tau) = -\frac{\ln \dfrac{v(t,\tau)}{v(t,T)}}{\tau - T}, \qquad t < T < \tau.$$

Definition 2.8.3. The **instantaneous forward rate** is defined by

$$\begin{aligned} f(t,T) &= \lim_{\tau \to T} f(t,T,\tau) \\ &= -\lim_{h \to 0} \frac{\ln v(t,T+h) - \ln v(t,T)}{h} \\ &= -\frac{\partial}{\partial T} \ln v(t,T), \qquad t \leq T, \end{aligned} \tag{2.7}$$

provided that the limit exists.

Although we can not see such a forward rate in the market, we can obtain following equation from (2.7)

$$v(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}, \qquad t \leq T. \tag{2.8}$$

Remark 2.8.1.

- Equation (2.8) shows us that the forward rates are the key parameter to recover the discount bond prices.
- Comparing (2.6) and(2.7), we have $r(t) = f(t,t)$. Which implies the spot rates as well as the money market account are obtained from the forward rates.
- It can be shown that for the deterministic interest rate $r(T) = f(t,T), \quad t \leq T$ should hold. Moreover, when the interest rate is not stochastic but deterministic, from (2.8) we obtain (see page 70 in[22])

$$v(t,T) = \exp\left\{-\int_t^T r(s)ds\right\} = \frac{B(t)}{B(T)}, \qquad t \leq T.$$

- The forward delivery price $F_T(t)$ of an underlying asset $S(t)$ at time t with maturity time T which is not paying any dividend is given by

$$F_T(t) = \frac{S(t)}{v(t,T)}, \qquad t \leq T. \tag{2.9}$$

2.8.5 Interest Rate Derivatives

Interest rate derivative are financial derivatives in which their payoffs are due to the interest rates level. From 1980s and 1990s the trading volume of interest rate derivatives in exchange trading market and over-the-counter market has been increased sharply. Since then, the most commonly used over-the-counter interest rate option derivatives are interest rate caps or floors [9], bond options and swap options, i.e. swaptions. The problem with interest rate derivatives is the difficult procedure of evaluating them and this problem is mainly caused by following reasons [18]

- The interest rate behaves more complicated and it might have more sudden movements and even jumps in its process comparing with an exchange rate or stock price in the market[10].

[9] As we mentioned, collar is a combination of caps and floors and it has been very popular since economic crisis in 2007 and 2008.

[10] For example, in less than 6 months the Central Bank in Sweden decreased the risk-free interest rate for three times. Firstly, in the autumn 2014 the risk-free interest rate became 0%, then in February 2015 it was decreased to -0.1% and finally in March 2015 it was reduced even more to -0.25%.

- The procedure of evaluation should describe the behavior of whole zero-coupon yield curve.
- The different spot points on the yield curve have significantly different volatilities.
- To calculate the price we use the interest rate as a discounting factor and it is a key parameter in calculation of the payoff.

In more mathematical language, let $v(t,\tau)$ be the price of a bond (it does not have to be a discount bond) at time t with maturity at time τ. Then, the payoff of an obtainable contingent claim X, say European call option at maturity T can be described by

$$h(X) = \max\{v(T,\tau) - K, 0\}, \qquad t \leq T < \tau,$$

where K stands for strike price. Using (2.9) in the current case, gives us the price of the discount bond with maturity τ in following form

$$v_T(t,\tau) = \frac{v(t,\tau)}{v(t,T)}, \qquad t \leq T < \tau.$$

Chapter 3

Securities Market Model

In Chapter 3, definitions and concepts might be found in [22] and [24]. In this chapter, we will introduce two most important approaches to evaluate the price/value of a security, namely risk-neural evaluation and forward-neutral evaluation. To do so, we must be aware that the dynamics of risky assets (i.e., price processes) can be described by stochastic differential equations. To understand the nature of stochastic processes, we start presenting some of the most important and fundamental definitions and theorems which are involved in stochastic calculus and namely in financial mathematics. We define probability and stochastic terms and we will continue our explanation in discrete time approach and we will expand it to continuous time approach. After that, we will look at stochastic differential equations and their roles in evaluating of pricing formulas. Then, we will look at risk-neutral evaluation of option pricing formula and will look briefly at Black–Scholes–Merton formula as an example. After that, we consider the role of stochastic volatility and we look at some works in this area. Finally, we close this chapter by presenting forward-neutral evaluation which is the most commonly used evaluation when interest rates are stochastic.

3.1 Stochastic Processes

It is obvious that, the dynamics of risky assets is based on random movements. It is common to assume that the dynamics of these movements follow stochastic processes. Therefore, the price process and return process on an investment is assumed to be the solution of stochastic differential equations. It seems to be necessary to refresh our mind with some of the fundamental definitions related to stochastic calculus. Let's go through them.

A *sample space*, Ω, is a set of all possible outcomes ω of an experiment. An *event* is a special subset of Ω.

Definition 3.1.1. A family of events, $\mathscr{F}$, is called a σ-filed if

1. $\Omega \in \mathscr{F}$,
2. if $A \in \mathscr{F}$, then $\Omega \setminus A \in \mathscr{F}$,
3. if $A_n \in \mathscr{F}, n = 1, 2, \ldots$, then $\cup_{n=1}^{\infty} A_n \in \mathscr{F}$.

Fix a σ-field $\mathfrak{F}$. An event is an element of $\mathfrak{F}$.

Definition 3.1.2. A real **random variable** is a measurable function $\mathbf{X} : \Omega \to \mathbb{R}$, such that for any $x \in \mathbb{R}$, the following set

$$\{\omega \in \Omega : \boldsymbol{X}(\omega) \leq x\},$$

is an event.

Definition 3.1.3. A **multivariate random variable** is a function

$$\mathbf{X} = (X_1, X_2, ..., X_n) : \Omega \to \mathbb{R}^n$$

such that for any $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, the set (intersection)

$$\cap_{j=1}^{n} \{\omega \in \Omega : X_j(\omega) \leq x_j\}$$

is an event.

Definition 3.1.4. A **stochastic process** is defined as a family of random variables $\{X(t) : t \in \mathscr{T}\}$. Where, $\mathscr{T}$ denotes a set of time epochs and it is a fixed subset of the set $\mathbb{R}$ of real numbers.

Following the Definition 3.1.4 the stochastic process $X(t)$ can be completely determined by the multivariate random variables given below

$$\mathbf{X} = (X(t_1), X(t_2), \ldots, X(t_n)),$$

Here, n is a positive integer and $t_1, t_2, \ldots, t_n$ are n pairwise different time epochs.

Remark 3.1.1. Actually, $X(t)$ is a function of two variables

$$X(t, \omega) : \mathscr{T} \times \Omega \to \mathbb{R}$$

Definition 3.1.5. A **price process** is a multivariate stochastic process denoted by

$$\mathbf{S}(t) = (S_0(t), S_1(t), \ldots, S_n(t))^{\top}.$$

Definition 3.1.6. Let $\boldsymbol{\theta}(t) = (\theta_0(t), \theta_1(t), \ldots, \theta_n(t))^\top$ be a portfolio at time $t \geq 1$. Then the **portfolio process** or a trading strategy is defined by following multivariate stochastic process

$$\{\boldsymbol{\theta}(t) : 1 \leq t \leq T\}.$$

Now, we are prepared to deal with next steps. In the two next sections, we present the definitions and theorem in discrete and continuous time models which will eventually lead us to present our return and price processes and present risk-neutral and forward-neutral measurements.

3.2 Mathematical Explanation for Discrete Time Models

In this section, we start dealing with fundamental theorems and definitions in financial mathematics which will help us to take our next steps in finding prices of financial derivatives. It might be necessary to emphasize that, some of the materials we present in this section are (in this thesis) the same in continuous time models. This section is extremely important, because it contains all the necessary economical definitions and theorems which are interpreted in financial mathematics language. After reading this section, the reader is expected to be familiar with meaning of value process, gain process, arbitrage opportunity, risk-neutral probability measure, asset pricing theorem and some other theorems and definitions.

3.2.1 Conditional Expectation

Let's refresh our mind with the meaning of conditional expectation and be familiar to the common notations in the price evaluation of a derivative. We start with following definitions and theorems.

Definition 3.2.1. Random variable X is called **integrable random variable** if $\mathsf{E}[|X|] < \infty$.

Definition 3.2.2. Random process $X(t)$ is called **integrable random process** if for any t, $\mathsf{E}[|X(t)|] < \infty$.

Definition 3.2.3. A **filtration** is a sequence of information $\{\mathscr{F}_t; t = 0, 1, \ldots, T\}$ or $\{\mathscr{F}_t\}$ satisfying $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_T \subset \mathscr{F}$.

Definition 3.2.4. Let X be a random variable. Then, X is called **$\mathscr{F}_t$-measurable** if $\{x_1 < X \leq x_2\} \in \mathscr{F}_t$ for any $x_1 < x_2$.

Theorem 3.2.1. *Let X be an integrable random variable and $\mathscr{F}_t$ be a filtration. There exists random variable Z such that*

- *Z is $\mathscr{F}_t$-measurable.*
- *For any $\mathscr{F}_t$-measurable random variable Y, the following equality holds, $\mathsf{E}[YZ] = \mathsf{E}[YX]$.*

let us end this section by the following definition.

Definition 3.2.5. A random variable Z which satisfies the conditions of Theorem 3.2.1 is called **conditional expectation** of X given filtration $\mathscr{F}_t$ (or under filtration $\mathscr{F}_t$) and is denoted by $\mathsf{E}[X|\mathscr{F}_t]$ or for convenience in the shorter equivalent form by $\mathsf{E}_t[X]$.

3.2.2 Markov Chain and Markov Property

Definition 3.2.6. Let $\{X_n\}$ be a stochastic process with a finite state space $\mathscr{N}$. If $\{X_n\}$ has the following **Markov Property**:
for each n and every $i_0, i_1, \ldots, i_n, j \in \mathscr{N}$,

$$P\{X_{n+1} = j | X_0 = i_0, \ldots, X_n = i_n\} = P\{X_{n+1} = j | X_n = i_n\}.$$

then it is called a **Markov Chain**.

Therefore, the distribution of X_{n+1} depends only on the current state X_n, not on the whole history. In other words, it just depends on today's information (no memory).

Theorem 3.2.2. *For any Markov Chain*

$$\begin{aligned} &P\{X_{n+1} = i_{n+1}, \ldots, X_{n+m} = i_{n+m} | X_0 = i_0, \ldots, X_n = i_n\} \\ =&P\{X_{n+1} = i_{n+1}, \ldots, X_{n+m} = i_{n+m} | X_n = i_n\}, \quad m \geq 1. \end{aligned}$$

Therefore, once the current state X_n is known, prediction of future distributions cannot be improved by adding any knowledge of the past.

Theorem 3.2.3. *Let $\{X_n\}$ be a stochastic process. $\{X_n\}$ has **Markov Property** if and only if the past and the future states are conditionally independent given the present state, i.e.,*

$$\begin{aligned} &P\{X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_{n+1} = i_{n+1}, \ldots, X_{n+m} = i_{n+m}\} \\ =&P\{X_0 = i_0, \ldots, X_{n-1} = i_{n-1} | X_n = i_n\} \times P\{X_{n+1} = i_{n+1}, \ldots, X_{n+m} = i_{n+m} | X_n = i_n\}. \end{aligned}$$

3.2.3 Value Process and Gain Process

Assume a simple financial market model which consist of one risk-free security[1], i.e., $S_0(t)$; say bond for example, and $n \geq 1$ risky securities, i.e., $S_j(t),\ \ 1 \leq j \leq n$; say stocks as an example, where $t \in T$ is a positive integer and we put $\mathscr{T} = \{0, 1, \ldots, T\}$. Then using the Definition 3.1.5 for $\mathbf{S}(t)$, we can define our value process. That is,

Definition 3.2.7. Let $\mathbf{d}(t) = (d_1(t), d_2(t), \ldots, d_n(t))^\top$ be dividend vector on securities and adapted to the filtration $\{\mathscr{F}\}$. Then the **value process**, $V(t)$ is

$$V(t) = \begin{cases} \sum\limits_{j=0}^{n} \theta_j(1) S_j(0), & t = 0, \\ \sum\limits_{j=0}^{n} \theta_j(t) \left[S_j(t) + d_j(t) \right], & 1 \leq t \leq T. \end{cases}$$

where $\theta(t)$ is the portfolio process given in Definition 3.1.6 and $S(t)$ is the stochastic price process (An example of price process will be given in (3.12)).

Remark 3.2.1. In the formula above, the value of portfolio is given before any transaction cost. We can calculate the value of portfolio exactly after the transaction costs are taken in account by $V(t) = \sum\limits_{j=0}^{n} \theta_j(t+1) S_j(t)$. See page 101 in [22].

Following the recent remark, we can now define a self-financing portfolio.

Definition 3.2.8. A portfolio is called **self-financing** if

$$V(t) = \sum_{j=0}^{n} \theta_j(t+1) S_j(t), \qquad 1 \leq t \leq T.$$

Definition 3.2.9. Define $D_j(t) = \sum_{s=1}^{t} d_j(s)$ as a cumulative dividend paid to the jth security until time t. The **gain process** is

$$G_j(t) = S_j(t) + D_j(t), \tag{3.1}$$

and here comes the following important theorem. See page 103 in [22].

Theorem 3.2.4. *The **portfolio value** is*

$$V(t) = V(0) + \sum_{j=0}^{n} \sum_{u=0}^{t-1} \theta_j(u+1) \Delta G_j(u)$$

if and only if $\boldsymbol{\theta}(t)$ is self-financing , where $\Delta G_j(t) = G_j(t+1) - G_j(t)$.

[1] We have denoted and we will denote this risk-free security by money market account $B(t)$.

3.2.4 Contingent Claims, Replicating Portfolios, and Arbitrage Opportunities

After the fundamental definitions in previous parts, now we introduce another four important and relevant definitions in finance.

Definition 3.2.10. A random variable X which represent the payoff to a financial instrument at time T is called **contingent claim**.

Definition 3.2.11. Let $\boldsymbol{\theta}(t)$ be a self-financed trading strategy. $\boldsymbol{\theta}(t)$ is a **replicating portfolio** of a contingent claim X, if

$$X = V(0) + \sum_{j=0}^{n} \sum_{t=0}^{T-1} \theta_j(t+1)\Delta G_j(t)$$

Definition 3.2.12. Contingent claim X is called **attainable contingent claim** if there exists a replicating portfolio for contingent claim X.

Definition 3.2.13. The existence of a self-financing trading strategy $\boldsymbol{\theta}(t)$ such that

$$\begin{cases} V(0) = 0, \\ V(T) \geq 0, \\ \mathsf{P}\{V(T) > 0)\} > 0. \end{cases}$$

is called **arbitrage opportunity**.

We end this section with following theorem which is of the most fundamental theorem in finance and is an assumption in lots of security market models. For more details, see page 106 in [22].

Theorem 3.2.5. ***(No-Arbitrage Pricing Theorem)*** *Suppose that there is no arbitrage opportunity in the market. Then the fair price of an attainable contingent claim X with replicating trading strategy $\boldsymbol{\theta}(t)$ is $V(0)$.*

3.2.5 Martingale Probability Measure

We mentioned two important features of risk-neutral world in Section 2.2.2. Now, we can see what these features in mathematical language means. We start with martingale and we will continue with risk-neutral probability measure in the next part.

Definition 3.2.14. An integrable stochastic process $\{X(t); t = 0, 1, \ldots, T\}$ defined on the probability space $(\Omega, \mathscr{F}, \mathsf{P})$ with filtration $\{\mathscr{F}_t : t \in \mathscr{T}\}$ is called a **martingale** if

- $\mathsf{E}_t\left[|X(t)|\right] < \infty$,
- $\mathsf{E}_t\left[X(t+1)\right] = X(t), \qquad 0 \leq t \leq T-1.$

where the first condition implies that the expectation for the absolute value of stochastic process X_t must be finite for any time t and the second condition simply implies that, the today's expectation, i.e., E_t, with tomorrow's stochastic information, i.e., $X(t+1)$ is nothing but today's value of the stochastic process, i.e., $X(t)$.

3.2.6 Risk-Neutral Probability Measure

Before, we define the risk-neutral probability measure we need to define equivalent probability measures.

Definition 3.2.15. let $\mathscr{F}$ be a σ-field of subsets of a set Ω. Two probability measures P_1 and P_2 defined on $\mathscr{F}$ are called **equivalent** if

$$\forall\ A \in \mathscr{F} \quad \mathsf{P}_1\{A\} > 0 \Leftrightarrow \mathsf{P}_2\{A\} > 0.$$

Now, we can define risk-neutral probability measure with following definition.

Definition 3.2.16. Let $\{\mathscr{F}_t : t \in \mathscr{T}\}$ be a filtration on probability space $(\Omega, \mathscr{F}, \mathsf{P})$. A probability measure P^* is called a **risk-neutral probability measure** or a **martingale measure** if

- P^* is equivalent to P,
- All the gain processes, discounted with respect to the money market account $S_0(t)$, are martingale under probability measure P^*.

3.2.7 Asset Pricing Theorem

The relation between no-arbitrage price and risk-neutral probability measure can be interpreted in the asset pricing theorem.

Theorem 3.2.6. ***(Asset Pricing Theorem, Version 1)*** *The following statements are equivalent*

1. *There are no arbitrage opportunities.*

2. *There exists a risk-neutral probability measure. If this is the case, the price of an attainable contingent claim X can be calculated by*

$$V(0) = \mathsf{E}^* \left[\frac{X}{S_0(T)} \right] \tag{3.2}$$

where, E^* *stands for expectation operator under martingale probability measure (risk-neutral probability measure)* P^* *and* $S_0(t) = B(t)$ *for every replicating trading strategy.*

Remark 3.2.2. Theorem 3.2.6 implies the no-arbitrage pricing of contingent claim X includes following steps

- Find a risk-neutral probability measure P^* (sometimes is called Q),
- Calculate the expectation of (3.2) under P^*.

Theorem 3.2.7. ***(Asset Pricing Theorem, Version 2)*** *The following statements are equivalent*

- *A security market is complete*[2].
- *There exists a* ***unique*** *risk-neutral probability measure. If this is the case, the price of an attainable contingent claim X can be calculated by*

$$V(0) = \mathsf{E}^* \left[\frac{X}{S_0(T)} \right]$$

where, $S_0(t) = B(t)$ *for every replicating trading strategy.*

Remark 3.2.3. Equation (3.2) simply means a martingale probability measure is unique[3]. And if (3.2) holds then both 1 and 2 in Theorem 3.2.6 hold.

3.3 Mathematical Explanation for Continuous Time Models

In continuous time models, some definitions are the same as their definitions in discrete time models. So, we will simply omit to rewrite such definitions in this section. Now, we can see what Brownian motion and martingale mean in continuous time models. The meaning of Brownian motion and martingale are extremely important to state the dynamics of price processes. Let us look at them in more detail and after this section, we will be ready to present stochastic differential equation of price and return processes.

[2] A securities market is complete if every contingent claim is attainable and is said to be incomplete otherwise.

[3] If the securities market contains no arbitrage opportunites, then it is complete if and only if there exists a unique risk-neutral probability measure. See Theorem 6.4 at page 112 in [22]

3.3.1 Brownian Motion

Definition 3.3.1. The stochastic process $W(t)$, $0 \leq t \leq T$, defined on the probability space $(\Omega, \mathscr{F}, \mathsf{P})$ is called a **standard Brownian motion**[4] if

- $W(0) = 0$.
- $W(t)$ is continuous on time interval $[0,T]$ with probability $\mathsf{P} = 1$.
- $W(t)$ has independent increments.
- the increment $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$.

Theorem 3.3.1. *Let $W(t) - W(s)$ be a normal random variable. Then a Brownian Motion, $G(t)$, with drift coefficient $\mu \in \mathbb{R}$ and diffusion coefficients $\sigma > 0$ will be*

$$G(t) = \mu t + \sigma W(t),$$

Here, μ and σ^2 may be time-dependent.

3.3.2 Geometric Brownian Motion

Definition 3.3.2. Let $G(t)$ be a Brownian motion with drift and diffusion coefficients μ and σ. Further, define $S(0)$ as a positive real number. Then the process

$$S(t) = S(0)e^{G(t)} = S(0)e^{\mu t + \sigma W(t)}$$

is called a **Geometric Brownian Motion**.

For more details about Brownian motion, multiple dimensions Brownian motion and geometric Brownian motion see Chapter 3.1 and 3.2 in [13].

3.3.3 Diffusion Process

Define $X(t)$ as a continuous time stochastic process and let $\Delta X(t) = X(t + \Delta t) - X(t)$.

Definition 3.3.3. A **Diffusion Process** is a continuous time Markov process $X(t)$, such that

- $X(t)$ has continuous sample paths;
- The following limits exist:

$$\mu(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathsf{E}\left[\Delta X(t) | X(t) = x\right],$$
$$\sigma^2(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathsf{E}\left[(\Delta X(t))^2 | X(t) = x\right] \neq 0.$$

[4] Also called Wiener Process as well.

Definition 3.3.4. The function $\mu(x,t)$ is called **drift function** and **diffusion function** defines with $\sigma(x,t)$.

3.3.4 Martingales

Definition 3.3.5. An integrable **continuous time** stochastic process $\{X(t)\}$ defined on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ with filtration $\{\mathscr{F}_t : 0 \leq t \leq T\}$ is called a **martingale** if [22, 24]

- $\mathsf{E}[|X(t)|] < \infty \quad$ for each $\quad t \in [0,T]$
- $\mathsf{E}[X(s)|\mathscr{F}_t] = X(t), \quad 0 \leq t < s \leq T.$

In other words, today's expectation of our process for tomorrow, is the same as current value of our process.

3.3.5 Dividends, Value Process and Gain Process

Definition 3.3.6. Let $\mathbf{d}(t) = (d_1(t), d_2(t), \ldots, d_n(t))^\top$ be dividend vector on securities and adapted to the filtration $\{\mathscr{F}_t\}$. Then the **value process**, $V(t)$ is

$$V(t) = \sum_{j=0}^{n} \theta_j(t) S_j(t), \quad 0 \leq t \leq T.$$

where $\boldsymbol{\theta}(t)$ is the portfolio process given in Definition 3.1.6 and $\boldsymbol{S}(t)$ is the price process given in Definition 3.1.5.

Definition 3.3.7. The cumulative dividend $(D_j(t))$ paid by j-th security until time t is given by

$$D_j(t) = \int_0^t d_j(s) ds.$$

Definition 3.3.8. The **gain process** on security j, is simply the sum of price process $S_j(t)$ and dividend process $D_j(t)$, i.e.,

$$G_j(t) = S_j(t) + D_j(t),$$

or in differential form it can be written as

$$dGj(t) = dS_j(t) + d_j(t)dt$$

3.3.6 Contingent Claims and Replicating Portfolio

Definition 3.3.9. Portfolio $\{\boldsymbol{\theta}(t)\}$ is said to be **self-financing** if

$$V(t) = V(0) + \sum_{j=0}^{n} \int_0^t \theta_j(s) dG_j(s).$$

Definition 3.3.10. A self-financing trading strategy $\{\boldsymbol{\theta}(t); 0 \leq t \leq T\}$, is said to be the **replicating portfolio** of a contingent claim X such that

$$X = V(T) = V(0) + \sum_{j=0}^{n} \int_0^t \theta_j(s) dG_j(s)$$

Remark 3.3.1. Contingent claim X is said to be attainable if there exists a replicating portfolio for X.

3.4 Stochastic Differential Equations (SDE)

As we discussed before, in finance, the price of an attainable contingent claim is its discounted expected payoff. The price process in continuous time can usually be described in terms of stochastic differential equation. To begin with, we would look at the SDEs as a limit of difference equation following the ideas in [22]. Then, we consider SDE and we close this section by presenting Itô formula. So, let's start with the idea of stochastic difference equation.

3.4.1 Stochastic Difference Equation

To explain the idea, consider a colorful drop of ink (as a tiny particle) is dropped in a glass of clean water. A deterministic velocity function $\mu(x,t)$ for each time $t \geq 0$ at each point $x \in \mathbb{R}$ can be described with some really small and microscopic movements. Still, there exists some other movements. That is the chaotic movement of molecules which can be described by $W(t)\sigma(x,t)$. Where $W(t)$ represents the standard Brownian motion and $\sigma(x,t)$ is a deterministic function. If we decrease the time in a very short interval Δt, the change in the position of particle (drop) can be written as

$$\Delta X(t) = \mu\left(X(t),t\right)\Delta t + \sigma\left(X(t),t\right)\Delta W(t) \tag{3.3}$$

where, $\Delta W(t) \equiv W(t+\Delta t) - W(t)$ represent the increment of $\{W(t)\}$. We suppose that $X(t) = x$. If $\lim_{\Delta t \to 0}\{X(t)\}$ exists, then it is a strong Markov process with continuous sample paths.

That means our process $\{W(t)\}$ is the Brownian motion. Moreover, from (3.3) it can be seen that

$$\lim_{\Delta t \to 0} \frac{\mathsf{E}\left[\Delta X(t) | X(t) = x\right]}{\Delta t} = \mu(x,t),$$
$$\lim_{\Delta t \to 0} \frac{\mathsf{E}\left[(\Delta X(t))^2 | X(t) = x\right]}{\Delta t} = \sigma^2(x,t)$$

Consequently, the limiting process should be a diffusion process with drift and diffusion functions $\mu(x,t)$ and $\sigma(x,t)$. See [22] and [24] for more details.

3.4.2 Stochastic Differential Equation (SDE)

By formal change of increment to differentials form, i.e., letting $\Delta t \to 0$ in (3.3), we obtain

$$dX(t) = \mu\left(X(t),t\right)dt + \sigma\left(X(t),t\right)dW(t), \qquad 0 \le t \le T. \tag{3.4}$$

The differential (3.4) has the following integral form

$$X(t) - X(0) = \int_0^t \mu\left(X(u),u\right)du + \int_0^t \sigma\left(X(u),u\right)dW(u),$$

where the first integral is ordinary (path-by-path) or Riemann integral and the second integral is Itô integral.

Remark 3.4.1. The solution to (3.4) (if it exists) is called Itô process [24, 22]. Moreover, (3.4) can be rewritten in the following form

$$dX = \mu(X,t)dt + \sigma(X,t)dW, \qquad 0 \le t \le T. \tag{3.5}$$

3.4.3 Itô Formula

Let $\{X(t)\}$ be the solution to (3.2). Then, stochastic process $\{Y(t)\}$ can be obtained from $X(t)$ via a smooth function $f(x,t)$ in following form

$$Y(t) = f\left(X(t),t\right), \qquad 0 \le t \le T. \tag{3.6}$$

Now, assuming that the partial derivative of $f(x,t)$ is continuous, we can present Itô formula in following theorem. See page 238–239 in [22] for more details.

Theorem 3.4.1. ***(Itô Formula)*** *Let $X(t)$ be a stochastic process (Itô process) given by (3.5). Further, let $f(x,t)$, be a function continuously differentiable in t, i.e., time variable, and twice continuously differentiable in x, i.e., space variable. Then, $Y(t) = f(X(t),t)$ is again Itô process satisfying the following equation*

$$dY = \mu_Y(t)dt + \sigma_Y(t)dW$$

where

$$\begin{aligned}
\mu_Y(t) &= \frac{\partial f}{\partial t}(x,t) + \frac{\partial f(X,t)}{\partial x}\mu(X,t) + \frac{1}{2}\frac{\partial^2 f(X,t)}{\partial x^2}\sigma^2(X,t),\\
\sigma_Y(t) &= \frac{\partial f(X,t)}{\partial x}\sigma(X,t).
\end{aligned}$$

Definition 3.4.1. **(Itô Differential)** If the following Itô integral exists

$$I(t) = \int_0^t X(u)dW(u),$$

it can be alternatively expressed by Itô differential. That is

$$dI(t) = X(t)dW(t).$$

Theorem 3.4.2. ***(Itô Division Rule)*** *Let $X(t)$ and $Y(t) > 0$ be two stochastic processes in following forms*

$$\begin{aligned}
\frac{dX}{X} &= \mu_X(t)dt + \sigma_X(t)dW,\\
\frac{dY}{Y} &= \mu_Y(t)dt + \sigma_Y(t)dW.
\end{aligned}$$

Then, the process $Z(t) = X(t)/Y(t)$ satisfies following SDE

$$\frac{dZ}{Z} = \mu_Z(t)dt + \sigma_Z(t)dW,$$

where

$$\begin{aligned}
\mu_Z(t) &= [\mu_X(t) - \mu_Y(t)] - \sigma_Y(t)[\sigma_X(t) - \sigma_Y(t)],\\
\sigma_Z(t) &= \sigma_X(t) - \sigma_Y(t).
\end{aligned}$$

See the Itô product rule at page 250 in [22]. See the proofs of division and product rules at [24] or [22].

3.5 Return Process

We are really close to start pricing a contingent claim under both risk-neutral and forward-neutral measure. But, still we need to find a proper SDE for our return and price processes and we need some tools to have a proper probability measure. So, we will begin with stating the return and price processes and we close this section by presenting two important theorems in the price evaluation procedure of a derivative.

Definition 3.5.1. The following equation is **return process**

$$R(t)dt = \frac{dS(t) + d(t)dt}{S(t)}, \quad 0 \leq t \leq T. \tag{3.7}$$

Note: $d(t)$ stands for dividend process and $S(t)$ for price process.

Remark 3.5.1. The return process which was used by Black, Scholes [5] and Merton [26] to derive Black–Sholes–Merton model was originally proposed by Samuelson [30] in year 1965 and has the following form [8, 24]

$$R(t)dt = \mu(S(t),t)dt + \sigma(S(t),t)dW(t) \tag{3.8}$$

where $\mu(S,t)$ is called the mean rate of return and $\sigma(S,t)$ is called the volatility .

Proposition 3.5.1. *The price process of risky assets has following form [24]*

$$dS(t) = [\mu(S(t),t)S(t) - d(t)]\,dt + \sigma(S(t),t)S(t)dW(t), \quad 0 \leq t \leq T \tag{3.9}$$

Proof. Equating the right hand sides of (3.7) and (3.8) we obtain

$$\begin{aligned}
\frac{dS(t) + d(t)dt}{S(t)} &= \mu(S(t),t)dt + \sigma(S(t),t)dW(t) \\
\frac{dS(t)}{S(t)} &= \mu(S(t),t)dt + \sigma(S(t),t)dW(t) - \frac{d(t)dt}{S(t)} \\
\frac{dS(t)}{S(t)} &= \left[\mu\left(S(t),t\right) - \frac{d(t)}{S(t)}\right]dt + \sigma(S(t),t)dW(t) \\
\Rightarrow dS(t) &= [\mu(S(t),t)S(t) - d(t)]\,dt + \sigma(S(t),t)S(t)dW(t).
\end{aligned}$$

□

Remark 3.5.2. For simplicity, (3.9) usually is written in following form

$$dS = [\mu(S,t)S - d(t)]\,dt + \sigma(S,t)SdW.$$

In the case that the risky asset gives no dividend the term $d(t)$ will be zero and will get following equation

$$dS = \mu(S,t)Sdt + \sigma(S,t)SdW.$$

and if mean rate of return and volatility assume to be constant (like in Black–Scholes–Merton Model) then the price process will be written even in simpler following form

$$dS = \mu Sdt + \sigma SdW.$$

3.5.1 Feynman–Kac Formula

Definition 3.5.2. Let $\{X(t)\}$ be a solution of SDE (3.4). Then for a sufficiently smooth function $f(x,t)$, the operator L_t is (often) called the **generator** of $X(t)$ and is defined by

$$L_t f(x,t) = \mu(x,t)f_x(x,t) + \frac{1}{2}\sigma^2(x,t)f_{xx}(x,t).$$

Theorem 3.5.1. ***(Feynman–Kac formula)*** *Suppose $X(t)$ is the solution to the SDE 3.4) with boundary condition $X(0) = x$ and define L_t as the generator of $X(t)$. Now, let $f(x,t)$ be a solution to following PDE and boundary condition*

$$\begin{cases} f_t(x,t) + L_t f(x,t) = r(x,t)f(x,t) \\ f(x,t) = g(x) \end{cases}$$

then

$$f(X(t),t) = \mathsf{E}\left[g(X(t))\exp\left\{-\int_t^T r(X,u)du\right\}\middle| X(t) = x\right].$$

3.5.2 Girsanov Theorem

Theorem 3.5.2. ***(Girsanov theorem)*** *Define stochastic process $\beta(t)$ adapted to the the filtration $\{\mathscr{F}_t : t \in [0,T]\}$ which is generated by the standard Brownian motion $W(t)$ on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$. Further, let following expectation to be finite (i.e. process $\beta(t)$ satisfies Novikov condition),*

$$\int_0^T \beta^2(u)du < \infty, \quad \textit{almost surely}, \quad \mathsf{E}\left[\exp\left\{\frac{1}{2}\int_0^T \beta^2(u)du\right\}\right] < \infty, \tag{3.10}$$

further, put

$$Y(T) = \exp\left\{\int_0^T \beta(u)dW(u) - \frac{1}{2}\int_0^T \beta^2(u)du\right\},$$

then the following process, $\widetilde{W}(t)$, is also a standard Brownian motion on $(\Omega, \mathscr{F}_T, \widetilde{\mathsf{P}})$ [22, 24]

$$\widetilde{W}(t) = W(t) - \int_0^t \beta(u)d(u), \quad 0 \le t \le T.$$

3.6 Pricing Contingent Claims via Risk-Neutral Method

In this section, we will go through a procedure to find the price of an obtainable contingent claim under martingale probability measure. To begin with, we introduce following assumptions and notations which should be included.

Assumptions and Notations

- A probability space $(\Omega, \mathscr{F}, \mathsf{P})$ with physical (empirical) probability measure P;
- The filtration $\{\mathscr{F}_t : t \in [0,T]\}$ is generated by the standard Brownian motion $W(t)$;
- A non-negative $\mathscr{F}_t$-adapted stochastic process $r(t)$ (instantaneous spot interest rate);
- A positive $\mathscr{F}_t$-adapted stochastic process $\sigma(t)$ which satisfies $\int_0^T \mathsf{E}[\sigma^2(t)]dt < \infty$;
- An $\mathscr{F}_t$-adapted stochastic process $\delta(t)$ which satisfies $\int_0^T \mathsf{E}[|\delta^2(t)|]dt < \infty$, where $\delta(t)$ is continuous dividend rate;
- The money market account satisfies the following SDE and boundary condition

$$\begin{cases} dB(t) = r(t)B(t)dt \\ B(0) = 1 \end{cases} \tag{3.11}$$

- The stock price process satisfies following SDE and boundary condition

$$\begin{cases} dS(t) = [\mu(t)S(t) - \delta(t)]\,dt + \sigma(t)S(t)dW \\ S(0) = S_0 \end{cases} \tag{3.12}$$

Price Evaluation Denote the price of European contingent claim (value process) with $\pi(t) = V(t)$ whose payoff is denoted by $h(X)$. Let us refresh our memory that $\pi(T) = h(X(T))$. Replicate the claim by a self-financing dynamics strategy, meaning that allocate $\theta_B(t)$ units of the money in the money market account and $\theta_S(t)$ units in the underlying stock. So using the Definition 3.1.6, we have the following portfolio process or trading strategy as follow [22, 24].

$$\boldsymbol{\theta}(t) = (\theta_0(t), \theta_1(t))^\top = (\theta_B(t), \theta_S(t))^\top,$$

then our price process of European option will be

$$\pi(t) = \theta_B(t)B(t) + \theta_S(t)S(t),$$

using the definition for self-financing portfolio, i.e., Definition 3.3.9, we have

$$\pi(t) = \pi(0) + \int_0^t \theta_B dB(u) + \int_0^t \theta_S dG(u), \tag{3.13}$$

here, the gain process $G(t)$ is

$$G(t) = S(t) + \int_0^t \delta(u)du. \tag{3.14}$$

Now, using the Definition 3.4.1 we express (3.14) in its Itô differential form. That is

$$dG(t) = dS(t) + \delta(t)dt, \tag{3.15}$$

substituting $dS(t)$ from (3.12) in (3.15) we obtain

$$\begin{aligned} dG(t) &= [\mu(t)S(t) - \delta(t)]\,dt + \sigma(t)S(t)dW + \delta(t)dt \\ &= S(t)\,[\mu(t)dt + \sigma(t)dW]\,, \qquad 0 \le t \le T, \end{aligned} \tag{3.16}$$

under the physical probability measure P. Now, we introduce $\lambda(t)$ which is the **market price of risk**. See page 256 in[22].

$$\lambda(t) = \frac{\mu(t) - r(t)}{\sigma(t)}, \tag{3.17}$$

moreover, define the process dW^*

$$dW^* = \lambda(t)dt + dW, \tag{3.18}$$

substituting (3.17) in (3.18) and rearranging the equation we have

$$\begin{aligned} dW^* &= \frac{\mu(t) - r(t)}{\sigma(t)} + dW, \\ \sigma(t)dW &= r(t) - \mu(t) + \sigma(t)dW^*, \end{aligned} \tag{3.19}$$

substituting (3.19) in (3.16) we get

$$\begin{aligned} dG(t) &= S(t)\mu(t)dt + S(t)\,[r(t) - \mu(t) + \sigma(t)dW^*]\,, \\ dG(t) &= S(t)\,[r(t)dt + \sigma(t)dW^*]\,, \qquad 0 \le t \le T. \end{aligned}$$

substituting the last equation to (3.13), using (3.11), i.e., $dB(t) = r(t)B(t)dt$ and using Itô differential's definition, we will have

$$\begin{aligned} \pi(t) &= \pi(0) + \int_0^t \theta_B(u)r(u)B(u)du + \int_0^t \theta_S(u)S(u)r(u)du + \int_0^t \theta_S(u)S(u)\sigma(u)dW^*(u) \\ &= \pi(0) + \int_0^t r(u)\big[\theta_B(u)B(u) + \theta_S(u)S(u)\big]du + \int_0^t \theta_S(u)S(u)\sigma(u)dW^*(u) \\ &= \pi(0) + \int_0^t r(u)\pi(u)du + \int_0^t \theta_S(u)S(u)\sigma(u)dW^*(u), \\ \Rightarrow d\pi(t) &= r(t)\pi(t)dt + S(t)\theta_S(t)\sigma(t)dW^*(t). \end{aligned} \tag{3.20}$$

To solve the SDE (3.20) one needs to be sure that the process $W^*(t)$ is a standard Brownian motion. Let the following process satisfy condition (3.10)

$$\beta(t) = \frac{W(t) - \mu(t)}{\sigma(t)},$$

using Girsanov theorem (Theorem 3.5.2) there exists a probability measure Q which makes the process $W^*(t)$ a standard Brownian motion. For more details, see page 256 in [22].

The solution to SDE (3.20) is given by [24]

$$\pi(t) = \pi(0)\exp\left\{\int_0^t r(u)du\right\} + \int_0^t S(u)\theta_S(u)\sigma(u)dW^*(u).$$

Define $S^*(u) = S(u)/B(u)$ as the denominated stock price process. Now if we put $B(t)$ as a numériare and $\pi^*(t) = \pi(t)/B(t)$ we then have

$$\pi^*(t) = \pi^*(0) + \int_0^t S^*(u)\theta_S(u)\sigma(u)dW^*(u),$$

which means **the denominated claim price is a martingale under** Q. Finally considering $\pi(T) = h(S(T))$ we will obtain the price of a European option by

$$\pi(t) = B(t)\mathsf{E}^{\mathsf{Q}}\left[\frac{h(S(T))}{B(T)}\middle|\mathscr{F}_t\right], \qquad 0 \le t \le T. \tag{3.21}$$

3.6.1 An Example, Black–Scholes–Merton Model

In this section, we will review how to derive Black–Scholes–Merton PDE and the price of European call option using Black–Scholes–Merton Model. The objective here, is just reviewing this model and see how they applied the theories we have mentioned in this chapter in derivation of their model. So, we will omit lots of calculation to keep this section short. We will follow the derivations in [5].

Black–Scholes–Merton PDE

Black-Scholes and Merton used Samuelson's market model which consist of two financial instruments [8]. Riskless, i.e., bond which we denote by $B(t)$ and risky asset, i.e. stock price process $S(t)$. Then, we will have following ordinary differential equation for bond, where interest rate r assumed to be constant :

$$\begin{cases} dB(t) = rB(t)dt, \\ B(0) = 1, \end{cases}$$

which implies the return on bond is nothing but risk-free interest rate. That is $B(t) = e^{rt}$. Moreover, the price of risky asset $S(t)$ satisfies following stochastic differential equation

$$\begin{cases} dS(t) = \mu S(t)dt + \sigma S(t)dW, \\ S(0) = S_0, \end{cases}$$

where μ is constant mean return rate (drift coefficient), $\sigma > 0$ is a constant volatility (diffusion coefficient) and dW is a standard Brownian motion. Now, we can construct a portfolio which consists of our bond and stock and form our value process $V(t)$ such that our portfolio remain self-financed. After doing that and applying Itô formula and some algebra we will obtain Black-Scholes PDE[5]

$$V_t + rSV_S + \frac{1}{2}V_{SS}\sigma^2 S^2 = rV.$$

Remark 3.6.1. rV, represents that the expected return in our investment is nothing but risk-free interest rate. In other words, in risk-neutral world the expected return on risky asset must be equal to risk-free interest rate.

Black–Scholes–Merton Formula For European Call Option

We have already got Black–Scholes–Merton PDE. To have an attainable contingent claim X, we know that $V(T) = X$ should holds. Further, the payoff to European call option, is $\max\{S(T) - K, 0\}$. So, we have a system of differential equation with the following boundary condition

$$\begin{cases} V_t + rSV_S + \frac{1}{2}V_{SS}\sigma^2 S^2 = rV, \\ \\ X = V_T = \max(S_T - K, 0). \end{cases}$$

Applying Itô formula we obtain

$$dV = rV dt + \sigma S V_S dW.$$

after doing a little algebra, we get

$$V_t = e^{-r(T-t)}\mathsf{E}[V_T].$$

Thus, the price of European call option, i.e., π_c, under martingale probability measure Q is given by

$$\pi_c(t, S(t)) = V(t, S(t)) = e^{-r(T-t)}\mathsf{E}^{\mathsf{Q}}[max\{S_T - K, 0\}|\mathscr{F}_t]. \tag{3.22}$$

[5]The full derivation of Black-Scholes-Merton PDF is available in the appendix of my Bachelor thesis in following address: http://mdh.diva-portal.org/smash/record.jsf?pid=diva2:679217

Remark 3.6.2. The last equation, implies that the price of an attainable contingent claim, here is European Call option, under equivalent martingale probability measure is its discounted expected payoff.

Applying Itô to price process $S(T)$ with boundary condition $S(0) = S_0$ we will get the lognormally distributed value of stock price $S(T)$,

$$S_T = S_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W_T - W_t\right)\right\}, \tag{3.23}$$

substituting (3.23) in (3.22) and considering the fact that the price process follows Geometric Brownian Motion we denote, i.e., $W_T - W_t = \sqrt{T-t}z$ where,

$$z \sim N\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right],$$

and after a little algebra we will get the price of European Call Option by following formula[6]

$$\begin{aligned}
\pi_c(t, S(t)) &= S_t N(d_1) - Ke^{-r(T-t)} N(d_2), \\
d_1 &= \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}, \\
d_2 &= \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}} = d_1 - \sigma\sqrt{(T-t)}.
\end{aligned}$$

3.7 Introduction to Stochastic Volatility

As we discuss it in Chapter 1, the price can have some jumps in their sample path. An example of this was depicted in Figure 1.3, the price of USD versus CHF decreased by almost 27.5% in about half an hour. So, we present two models in this section to be familiar with stochastic volatility and jump diffusion models.

3.7.1 Merton's Mixed Jump-Diffusion Model

In 1975, Robert C. Merton argued that the dynamics of price process, cannot be always described by a continuous sample path. The classical example of this, is a jump diffusion and can be described by a sudden movement in prices, for example by an economics crisis or some important news. He discussed that the changes in the stochastic part of price process are the sum of two significantly different component, normal vibrations and abnormal vibrations. The

[6]The full derivation of Black-Scholes-Merton formula for European call option is available in the appendix of my Bachelor thesis in following address: http://mdh.diva-portal.org/smash/record.jsf?pid=diva2:679217.

abnormal vibrations can be included in the stochastic part of price process as a jump diffusion. Merton explained the Black-Scholes-Merton formula can be valid only if the stock price fluctuations will not violate the local Markov property, i.e., in the short time intervals the price varies only by small amount [27]. The price process for Mixed Jump-Diffusion model is given by

$$\frac{dS}{S} = (\alpha - \lambda j)dt + \sigma dW + dq \tag{3.24}$$

where α is the instantaneous expected return on stock, σ^2 is is the instantaneous variance of the return, dW is a standard Brownian motion, dq is a Poisson process generating the jumps, λ is average (mean) number of jumps per year and j is average jump size measured as a percentage of the asset price.

Remark 3.7.1. σ^2 is conditional on the arrival of new important information, i.e., the Poisson event does not occur. Moreover, the processes dW and dq are assumed to be independent [18].

SDE (3.24) has three parts, deterministic part, a part for normal vibration and a part for abnormal vibration. So, let's rewrite (3.24) in following form

$$\frac{dS}{S} = \begin{cases} (\alpha - \lambda j)dt + \sigma dW, & \text{if} \quad \text{Poisson event does not occur,} \\ (\alpha - \lambda j)dt + \sigma dW + (Y - 1), & \text{if} \quad \text{Poisson event occurs.} \end{cases}$$

where, Y is a IID random variables describing the arrival of new important information. I.e., if $S(t)$ is the stock price at time t, ignoring the continuous part $S(t+h) = S(t)Y$. We can see if no Poisson event occurs, then $\lambda = dq = 0$. Which gives us the Black-Scholes-Merton formula [5, 26]. With the boundary condition $S(0) = S_0$, assuming that the sample path for $S(t)$ has finite jumps, we get

$$\frac{S(t)}{S} = \exp\left\{\left(\alpha - \frac{\sigma^2}{2} - \lambda j\right)t + \sigma W(t)\right\} Y(n)$$

Here, $W(t)$ is a Gaussian random variable, with $\mu = 0$ and $\sigma = t$, n represents the numbers of Poisson distributed with parameter λt, and for IID Y_is, $1 \leq i \leq n$, $Y(n)$ is given by

$$Y(n) = \begin{cases} 1, & \text{if} \quad n = 0, \\ \prod_{i=1}^{n} Y_i, & \text{if} \quad n \geq 1. \end{cases}$$

An Option Pricing Formula

To get the option pricing formula, Merton defined a random variable X_n whose distribution is a product of n IID distributed random variable, such that each one of them is identically distributed to the random variable Y. So, we will have $X_0 \equiv 1$. Moreover, ε_n is defined as the

expectation operator over the distribution of X_n. Then, the current stock price at time T will be [27]

$$\pi(S(t),T) = \sum_{n=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^n}{n!} \left(\varepsilon_n\{\pi(SX_n e^{-\lambda jT}, T; K, \sigma^2, r)\}\right) \tag{3.25}$$

where $\pi(t, S(t); K, r, \sigma^2)$ is the Black-Scholes-Merton formula for pricing a European option. Formula (3.25) is not a closed-form solution, however it does work for approximations. Merton consider two significantly different cases which are considerably simplifying (3.25). In the two next parts we will look at these two cases separately.

Possibility of Immediate Ruin

To describe the possibility of immediate ruin Merton refers to the work by Samuelson which tells us with the positive probability of immediate ruin, the stock price goes to zero. In other words, if Poisson event occurs, then the stock price goes to zero and consequently, we have $Y \equiv 0$, $X_n = 0$, (for $n \neq 0$) , $j = -1$, and

$$\pi(S(t),T) = e^{-\lambda T}\pi(Se^{\lambda T}, T; K, \sigma^2, r) = \pi(S, T; K, r+\lambda)$$

Letting, $r' \equiv r + \lambda$, this formula is identical to Black-Scholes-Merton formula. However, the interest rate is higher.

Random Variable Y has a Log-normal Distribution

In this special case, where the logarithm size of the percentage jump is normal (see page 601 in [18]), i.e., Y has log-normal distribution, we will denote σ_Y^2 as variance of the logarithm of Y. Further, assume that $E[Y] = 1$. Therefore, the expected change in stock price will be zero, if the Poisson event has occurred. Moreover, X_n will in this case have a log-normal distribution as well and its variance will be σ_X^2 and $E_n[X_n] = 1$. Then, we will have $j = 0$ and (see page 17 in [27])

$$\pi_n(S(t),T) \equiv \pi(S, T; K, \sigma^2 + \left(\frac{n}{T}\right)\sigma_Y^2, r)$$

Denoting $\sigma'^2 = \sigma^2 + \left(\frac{n}{T}\right)\sigma_Y^2$, then our formula for π_n will be exactly the same as Black-Schole-Merton formula. Then, we will obtain (see page 17 in [27])

$$\pi(S(t),T) = \sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!}\pi_n(S(t),T)$$

where $\lambda' = \lambda(1+j)$.

3.7.2 Heston's Model

In 1993, Heston proposed his stochastic volatility model, which contains square-root diffusion. The price process $\{S(t)\}$ in Heston's model is given by [16]

$$dS(t) = \mu S(t)dt + \sqrt{\Upsilon(t)}S(t)dW_1(t)$$

where $W_1(t)$ is a Brownian motion, μ is mean-rate of return and $\sqrt{\Upsilon(t)}$ is the square-root diffusion . The square volatility process $\{\Upsilon(t)\}$ which follows a square-root diffusion, is given by [16]

$$dV(t) = \alpha(b - \sigma(t))dt + \sigma\sqrt{\Upsilon(t)}dW_2(t)$$

where $W_2(t)$ is a Brownian motion independent[7] of $W_1(t)$, α and b are constants.

Remark 3.7.2. Heston's model has an important and practical applications in simulating stock price. See [13] for detailed information and explanations.

3.8 Pricing Contingent Claims via Forward-Neutral Method

In Section 3.6, we took risk-free interest rate as a numériare and we went through the pricing of contingent claim under risk-neutral probability measure. Now, let's introduce the default-free discount bond as a numériare. Doing so, we can deal with stochastic interest rate and the models so called term-structure models. After introducing the forward-neutral method in this chapter, we will look at term-structure models in next chapter. To begin with, suppose the price process of a discount bond, i.e., $v(t,T)$ under risk-neutral probability measure Q at time t and maturity time T satisfies the following SDE

$$\frac{dv(t,T)}{v(t,T)} = r(t)dt + \sigma_v(t)dW^*, \qquad 0 \le t \le T, \tag{3.26}$$

where

- $\sigma_v(t)$ is the volatility of discount bond,
- $r(t)$ denotes the risk-free interest-rate.

Assume that stock pays no dividends and the price process of stock $S(t)$ under risk-neutral probability measure Q satisfies the following SDE

$$\frac{dS}{S} = r(t)dt + \sigma(t)dW^*, \qquad 0 \le t \le T, \tag{3.27}$$

[7] $\{W_1(t), W_2(t)\}$ is a two-dimensional Brownian motion. To read more about multi-dimensional Brownian motions, see Chapter 3.1 and 3.2 in [13].

where, $\sigma(t)$ is the volatility of stock price. Because, the stock pays no dividends, the **forward price** can be expressed by

$$S^T(t) \equiv \frac{S(t)}{v(t,T)}, \qquad 0 \le t \le T,$$

applying Itô division rule (i.e., Theorem 3.4.2), we obtain

$$\frac{dS^T}{S^T} = -\sigma_v(t)[\sigma(t) - \sigma_v(t)]dt + [\sigma(t) - \sigma_v(t)]dW^*.$$

Define the process $\{W^T(t)\}$ such that the following equality holds

$$[\sigma(t) - \sigma_v(t)]dW^T = -\sigma_v(t)[\sigma(t) - \sigma_v(t)]dt + [\sigma(t) - \sigma_v(t)]dW^*.$$

Then, using Girsanov's theorem, we know that, there exists such a probability measure Q such that the process $\{W^T(t)\}$ is indeed a standard Brownian motion. Observe that

$$\frac{dS^T}{S^T} = -\sigma^T(t)dW^T, \qquad \sigma^T(t) \equiv \sigma(t) - \sigma_v(t). \tag{3.28}$$

Q^T is called **forward-neutral** probability measure, because the forward price process $\{S^T(t)\}$ is indeed martingale under our new probability measure Q^T. See [21] for the formal definition of forward-neutral probability measure or see page 263 in [22] for brief details.

Now, consider the following scenario

- $h(X)$: payoff function of an European contingent claim written on the stock,
- T: maturity time of the contingent claim,
- $\pi_C(t)$: time-t price of the contingent claim,
- $\theta(t)$: units of underlying stock,
- $b(t)$: units of discount bond
- $\pi_C(T) = h(S(T))$,
- $\pi_C^T(t) = \pi_C(t)/v(t,T)$, i.e., time-$t$ forward price
- X: $\mathscr{F}_T$-measurable random variable which stands for the payoff of a contingent claim.

Now, consider that our contingent claim is replicated a self-financing portfolio (dynamics) strategy, such that it will consist of $b(t)$ units of discount bond and $\theta(t)$ units of the underlying stock, i.e,

$$\begin{aligned}\pi_C(t) &\equiv b(t)v(t,T) + \theta(t)S(t) \\ &= \pi_C(0) + \int_0^t b(u)dv(u,T) + \int_0^t \theta(u)dS(u),\end{aligned}$$

applying Itô formula yields

$$\pi_C^T(t) = \pi_C^T(0) + \int_0^t \theta(u)\sigma^T(u)S^T(u)dW^T(u), \qquad 0 \le t \le T. \tag{3.29}$$

Thus, we can conclude that, under some technical conditions, the forward price process $\{\pi_C^T(t)\}$ is martingale under Q^T. Since $\pi_C(T) = h(S(T))$, the price of our European contingent claim can be written as [22]

$$\pi_C(t) = v(t,T)\mathsf{E}^{\mathsf{Q}^T}\left[h(S(T))\middle|\mathscr{F}_t\right], \qquad 0 \le t \le T. \tag{3.30}$$

Remark 3.8.1.

- Despite the fact that the price process when underlying security pays dividend will not be a martingale under Q^T, the forward-neutral method can be applied [22].
- In contrary with risk-neutral method, the forward-neutral has practical usage in stochastic interest rates. From (3.21) and (3.30) we have

$$\pi_C(t) = B(t)\mathsf{E}^{\mathsf{Q}}\left[\frac{X}{B(T)}\middle|\mathscr{F}_t\right] = v(t,T)\mathsf{E}^{\mathsf{Q}^T}\left[X\middle|\mathscr{F}_t\right], \tag{3.31}$$

  where $X = h(S(T))$ is the $\mathscr{F}_T$-measurable random variable and stands for the payoff of an attainable contingent claim [22].
- Another advantage of forward-neutral method over risk-neutral method is that we need two dimensional distribution of $(X, B(T))$ in risk-neutral evaluation, while the marginal distribution of X is enough to price the contingent claim under forward-neutral method. This property of forward-neutral evaluation has a considerable role in Gaussian interest-rate frameworks. See page 264 in [22] for more details.

Chapter 4

Interest Rate Models (Term-Structure Models)

In this chapter, we review some different models and procedures to price default-free discount bonds and interest-rate derivatives. To begin with, denote the time-t price of discount bond with maturity at time T by $v(t,T)$. Thus, $v(t,T)$ is called the **term structure** of interest rate. Further, in this chapter we mainly follow the ideas in Chapter 15 of [22], Chapter 30 of [18] and the explanations in [24].

4.1 Spot-Rate Models (Equilibrium Models)

Define $r(t)$ as the time-t instantaneous spot rate under physical probability measure P. In one factor equilibrium model, the process for spot rate (short rate) r contains just one source of uncertainty. It is common to describe risk-neutral process for the spot rate (short rate) by following Itô process

$$dr = a(m-r)dt + \sigma r^{\gamma} dW, \quad t \geq 0, \tag{4.1}$$

where a, m, σ and γ are constants. It is important to remind us that, unless $\gamma = 0$ or $1/2 \leq \gamma \leq 1$, the path-by-path solution may not exist. The drift function in (4.1) describes a restoring force directed to the level m. Moreover, the magnitude of m is proportional to the distance. See page 271 in [22].

Remark 4.1.1.

- Here, drift and diffusion functions are assumed to be functions of r and independent of time.
- There exists an important different between stock prices and interest rates and that is, in average; the high interest rates have negative trend while the low interest rates have positive trend in long-run. This phenomenon is known as **mean reversion** and this model is called mean-reverting model.

4.1.1 Rendleman–Bartter Model

Suppose that $a = -\mu, m = 0$ and $\gamma = 1$ in (4.1). Then we will obtain the risk-neutral process for r [28, 18]

$$dr = \mu r dt + \sigma r dW,$$

where μ and σ are assumed to be constant, i.e., r follows geometric Brownian motion .

Remark 4.1.2. This model is known as geometric Brownian motion model. This model always provides a positive spot rate, since it is log-normally distributed. However, in the down side, the mean of the money-market account diverges.

4.1.2 Vasicek Model

Let $a, m > 0$ and $\gamma = 0$ in (4.1), then we get the Vasicek model which satisfies the following SDE [32]

$$dr = a(m - r)dt + \sigma dW, \quad t \geq 0.$$

Vasicek model is an important mean-reverting model. In other words, the spot rate (short rate) is pulled to m level at rate a. The Vasicek SDE is linear and has the following closed form solution

$$r(t) = m + (r(0) - m)e^{-at} + \sigma \int_0^t e^{-a(t-s)} dW(s).$$

Remark 4.1.3. The process $\{r(t)\}$ is known as an Ornstein-Uhlenbeck process. In this process $r(t)$ is normally distributed. The advantage[1] of Vasicek's model is that the spot rate with a positive probability can be negative [22].

4.1.3 Cox–Ingersoll–Ross (CIR) Model

Let $a, m > 0$ and $\gamma = 0.5$ in (4.1). The CIR model [6] satisfies the following SDE

$$dr = a(m - r)dt + \sigma \sqrt{r} dW, \quad t \geq 0.$$

The CIR model has considerably been used by many authors for pricing interest-rate derivatives[2]. The CIR model on contrast by Vasicek model does not allow the spot rate become negative. The mathematical explanation of this important property of CIR model comes from Feller argument [6, 22]. This property claims that, the spot rate $r(t)$ is

[1] Before the market experienced negative interest rates, it was counted as a downside of this model.

[2] Before introducing the negative interest rates in the market.

- if $\sigma^2 \leq 2\alpha \Rightarrow r(t) > 0$
- if $\sigma^2 > 2\alpha \Rightarrow r(t) \geq 0$

Additionally, the probability distribution of $r(t)$ is known and the corresponding density function of it can be expressed by (see page 272 in [22])

$$f_t(r) = ce^{-c(u+r)} \left(\frac{r}{u}\right)^{q/2} I_q(2c\sqrt{ur}), \qquad r \geq 0,$$

where

$$c = \frac{2a}{\sigma^2(1-e^{-at})}, \quad u = r(0)e^{-at}, \quad q = \frac{2am}{\sigma^2} - 1,$$

and the following expression for $I_q(x)$, known as the modified Bassel function of the first kind and order of q.

$$I_q(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+q}}{k!\Gamma(k+q+1)}$$

Moreover, the $\Gamma(x)$ is used to denote the gamma function.

4.1.4 Longstaff–Schwartz Stochastic Volatility Model

In 1992, Longstaff and Schwartz [23] proposed an equilibrium term structure model with the following stochastic volatility model for the spot rate $r(t)$ and the square volatility $\Upsilon(t)$

$$\begin{cases} dr = \mu_r(r,\Upsilon,t)dt + \Upsilon dW_1 \\ \\ d\Upsilon = \mu_\Upsilon(r,\Upsilon,t)dt + \sigma_\Upsilon(r,\Upsilon,t)dW_2, \end{cases}$$

where μ_r represents the drift function of r, μ_Υ represents the drift function of Υ and σ_Υ is the diffusion function of Υ. It can be shown [22] that with $dW_1 dW_2 = 0$ the spot rates follow Vasicek model and squared volatility will follow a geometric Brownian motion. It is good to know that, in this model the spot rate $r(t)$ can become negative with positive probability[3].

4.1.5 A Problem with Equilibrium models

The equilibrium models do not really fit today's term structure of interest-rate. Although, choosing proper parameters in these models can provide an approximation in such a way that these models fit to many term structures in practice, but they are not exact ones. To see this disadvantage consider that a 1% error in the price of the underlying bond may cause a 25% error in an option price. See page 689–690 in [18]. To avoid these errors and to have a better fit, we introduce no-arbitrage models in next section.

[3]Which now can be seen as an advantage.

4.2 Spot Rate Models (No-Arbitrage Models)

The main difference of no-arbitrage models comparing with equilibrium models is that in no-arbitrage models today's term structure of interest rate is an input while in equilibrium models, today's term structure of interest rate is an output. In other words, no-arbitrage models designed in such a manner that they can be exactly consistent with today's term structure of interest-rate.

4.2.1 Ho–Lee Model

The first no-arbitrage model of the term structure was proposed by Ho and Lee [17] in 1986. The original model was developed in the form of lattice approximation, namely binomial model of bond price. The model contains of two parameters. Firstly, the spot rate (short rate) standard deviation and secondly the market price of risk of the short rate. It has been shown that, the limit of this model under risk-neutral valuation converges to the continuous-time model with following SDE

$$dr = \phi(t)dt + \sigma dW$$

where σ is a constant and denotes the instantaneous standard deviation of the short rate and $\phi(t)$ is a function of time, such that it can ensure us the model fits the initial term structure. The analytical solution to $\phi(t)$ yields

$$\phi(t) = \frac{\partial F(0,t)}{\partial t} + \sigma^2 t$$

where $F(0,t)$ is the instantaneous forward rate with maturity at time t.

Remark 4.2.1. In fact, the function $\phi(t)$ measures the average direction which r moves at time t. However, this movement is independent of r. The problem of Ho–Lee model arises in the pricing of interest rate derivatives. That is the irrelevancy of the model's parameters that measures the market price of risk. See page 690 in [18].

In the Ho–Lee model, it is possible to analytically evaluate the zero-coupon bonds and European options on zero-coupon bonds. The discount bond price of zero-coupon bond is given by

$$v(t,T) = H(t,T)e^{-r(t)(T-t)}, \qquad T \geq t,$$

where

$$\ln H(t,T) = \ln \frac{v(0,T)}{v(0,t)} + (T-t)F(0,t) - \frac{\sigma^2}{2} t(T-t)^2.$$

As we can see in formulas above, the future can be calculated from today's term structure.

4.2.2 Hull–White (One-Factor) Model

In 1990, Hull and White [19] developed Vasicek model in such a way that their model provide an exact fit to the initial term structure. In Hull–White one-factor model[4], the spot rate $\{r(t)\}$ under the risk-neutral probability measure Q satisfies the following SDE

$$dr = (\phi(t) - ar)dt + \sigma dW^*, \qquad 0 \leq t \leq T,$$

Here a and σ are positive constants, $\phi(t)$ is a deterministic function of time and $W(t)$ is a standard Brownian motion under risk-neutral probability measure Q. This model like the Ho–Lee model can be characterized with mean reversion at rate a, at the same time this model has a time-dependent reversion level like Vasicek model. In other words, at time t the spot rate reverts to $\phi(t)/a$ at rate a. Moreover, it is easy to see that the Ho–Lee is special case of Hull–White one factor model with $a = 0$.

Remark 4.2.2. The Hull–White model can be extend in the form of trinomial tree and this lattice approach has practical usage when there does not exist an analytical solution to price American options or other derivatives. See Chapter 30.6–30.8 of [18] for more details.

4.2.3 Black–Derman–Toy Model

In 1990, a binomial-tree model for log-normal spot-rate (short-rate) process was proposed by Black, Derman and Toy [3]. It is possible to show that the corresponding stochastic process to this model has following form

$$d\ln r = (\phi(t) - a(t)\ln r)\,dt + \sigma(t)dW,$$

where

$$a(t) = -\frac{\sigma'(t)}{\sigma(t)},$$

and $\sigma'(t)$ is the derivative of σ with respect to t. In this model, on contrary with Ho–Lee and Hull–White models, the interest rate cannot be negative, since the Wiener process dW can lead the term $\ln r$ to become negative, but r will always be positive. Further, this model has two more disadvantages. First, there are no analytic properties in this model. Second and more serious problem is the interrelation between volatility parameter σ and the reversion rate parameter $a(t)$. That is, the reversion rate is only positive in the case that the short rate is a decreasing function of time.

This model can be more useful if σ assumed to be constant, then $\sigma'(t) = 0$ and consequently $a = 0$. This means no mean reversion and the model will be simplified to following form which can be considered as log-normal version of Ho–Lee model.

$$d\ln r = \phi(t)dt + \sigma dW$$

[4]It is also called Hull–White model (1990) or the extended vasicek model.

4.2.4 Black–Karasinski Model

The extension to Black–Derman–Toy model was done by Black and Karasinski [4] in 1991. The advantage of this model is that in Black–Karasinsky model reversion rate and volatility are determined independently of each other. The most general log-normal SDE of this model has following form

$$d\ln r = (\phi(t) - a(t)\ln r)\,dt + \sigma(t)dW$$

In fact there is no relationship between $a(t)$ and $\sigma(t)$. Moreover, in practical usage the reversion rate and volatility are often assumed to be constant.

4.2.5 Hull–White (Two-Factor) Model

In 1994, Hull and White [20] extended their Hull–White one-factor model to so called Hull–White two-factor model. They chose the parameter $\phi(t)$ to be a deterministic function of time, so that the model become consistent with initial term-structure. The model under a given risk-probability measure Q has the following form

$$\begin{cases} dr = (\phi(t) + u - ar)\,dt + \sigma_r dW_1^*, \\ \\ du = -budt + \sigma_u dW_2^*, \end{cases}$$

where a, b, σ_r and σ_r are positive constant. W_1 and W_2 are standard Brownian motion with correlation coefficient ρ, meaning that $dW_1 dW_2 = \rho dt$. Moreover, the mean reverting level $\phi(t) + u(t)$ of the spot rate under risk-neutral probability measure Q is assumed to vary stochastically during the time.

4.3 Pricing of Discount Bonds

As we mentioned in the beginning of this chapter, $v(t,T)$ denotes the price of default-free discount bond which is a contingent claim written on the spot rate $r(t)$. Further, suppose the spot rate process $\{r(t)\}$ satisfies the following SDE under physical probability measure P

$$dr = \mu(r,t)dt + \sigma(r,t)dW, \qquad t \geq 0. \tag{4.2}$$

Remark 4.3.1. Since the spot rate is not tradable in market, in this framework, the discount bond is not replicated by using underlying variable, i.e., the spot rate. That is the main difference between security pricing, namely Black–Scholes–Merton model and pricing a default-free discount bond.

4.3.1 Market Price of Risk and the Risk-Neutral Method

Let $F(t) = v(t,T)$ be the time-t price of a default-free discount bond with maturity T. Further, let $f(r,t)$ be a smooth function such that $F(t) = f(r(t),t),\ \ 0 \le t \le T$. Then applying Itô formula (Theorem 3.4.1) we obtain

$$\frac{dF}{F} = \mu_F(r(t),t)dt + \sigma_F(r(t),t)dW, \qquad 0 \le t \le T,$$

where,

$$\begin{cases} \mu_F(r,t) = \dfrac{1}{f(r,t)}\left(f_r(r,t)\mu(r,t) + f_t(r,t) + \frac{1}{2}f_{rr}(r,t)\sigma^2(r,t)\right), \\ \\ \sigma_F(r,t) = \dfrac{f_r(r,t)\sigma(r,t)}{f(r,t)}. \end{cases}$$

The no arbitrage opportunities argument conducts us to the argument that, the excess rate of return per unit of risk must be equal for two different bonds. That is the market price of risk is independent of maturity and is defined by

$$\lambda(t) = \frac{\mu_F(r(t),t) - r(t)}{\sigma_F(r(t),t)}. \tag{4.3}$$

Suppose the market price of risk, i.e., $\lambda(t)$, is known in the market. Now, using Girsanov theorem (Theorem 3.5.2) we claim that there exists an equivalent probability measure Q to the physical probability measure P such that the following process is a standard Brownian motion

$$dW^* = dW + \lambda(t)dt, \qquad 0 \le t \le T. \tag{4.4}$$

Define the denominated price $F^*(t) = F(t)/B(t)$ of the discount bond with maturity T, where the money-market account is $B(t) = \exp\left\{\int_0^t r(s)ds\right\}$. Then by Itô's division rule (Theorem 3.4.2), we obtain

$$\frac{dF^*}{F^*} = (\mu_F(t) - r(t))dt + \sigma_F(t)dW, \qquad 0 \le t \le T.$$

Suppose the market price of risk is known, then from (4.3) and (4.4) we obtain

$$\frac{dF^*}{F^*} = \sigma_F(t)dW^*, \qquad 0 \le t \le T.$$

meaning that, the price process $\{F(t)\}$ is a martingale under Q and hence, Q is risk-neutral probability measure. Now, on the base of risk-neutral valuation the denominated price is

$$F^*(t) = \mathsf{E}^{\mathsf{Q}}\left[\frac{1}{B(T)}\middle|\mathscr{F}_t\right], \qquad 0 \le t \le T.$$

The discount bond will pay one unit of money for sure at maturity time T. So, the price formula for a default-free discount bond when the market price of risk is known will be

$$v(t,T) = \mathsf{E}^{\mathsf{Q}}\left[e^{-\int_t^T r(s)ds}\middle|\mathscr{F}_t\right], \qquad 0 \le t \le T. \tag{4.5}$$

Now, is the time to obtain the SDE for the spot-rate process under the risk-neutral probability measure Q to calculate the bond price. Using (4.2) and (4.4) we obtain

$$dr = [\mu(r,t) - \sigma(r,t)\lambda(t)]dt + \sigma(r,t)dW^*, \qquad 0 \le t \le T. \tag{4.6}$$

Comparing the spot-rate process under physical probability measure P and equivalent risk-neutral probability measure Q, we see that only the drift term in these SDEs differs. To value the discount bond, we need to set the drift $\mu(r,t)$ in a way that

$$m(t) \equiv \mu(r,t) - \sigma(r,t)\lambda(t)$$

In the case, "risk-adjusted" drift, i.e., $m(t)$ is only a function of $r(t)$ and time t, then the discounted bond price can be calculated by

$$v(t,T) = \mathsf{E}^{\mathsf{Q}}\left[e^{-\int_t^T r(s)ds}\middle|r(t)\right], \qquad 0 \le t \le T.$$

It is possible to show that, in the Markovian case, i.e., $m(t) = m(r(t),t)$, the bond price function satisfies the following PDE (see page 276 in [22])

$$f_t(r,t) + m(r,t)f_r(r,t) + \frac{1}{2}\sigma^2(r,t)f_{rr}(r,t) = rf(r,t) \tag{4.7}$$

with boundary condition $f(r,T) = 1$.

4.3.2 Affine Models

There exists a class of spot-rate models such that the PDE (4.7) can be used to solve them easily. To begin with, let us look at the following theorem.

Theorem 4.3.1. ***(Affine Model)*** *Under risk-neutral probability measure Q, suppose the risk-adjusted drift $m(r,t)$ and the diffusion coefficient $\sigma(r,t)$ of the spot rate $r(t)$ have following forms*

$$\begin{cases} m(r,t) = \alpha_1(t) + \alpha_2(t)r, \\ \sigma^2(r,t) = \beta_1(t) + \beta_2(t)r, \end{cases}$$

where, $\alpha_i(t)$ and $\beta_i(t)$, $i = 1,2$, are deterministic functions of time t. Then, the default-free discount-bond price $v(t,T)$ is given by

$$v(t,T) = e^{a_T(t)+b_T(t)r}, \qquad 0 \le t \le T, \tag{4.8}$$

where, $a_T(t)$ and $b_T(t)$ satisfy the following simultaneous ODE and boundary conditions

$$\begin{cases} b_T'(t) = -\alpha_2(t)b_T(t) - \dfrac{\beta_2(t)b_T^2(t)}{2} + 1, \\ \\ a_T'(t) = -\alpha_1(t)b_T(t) - \dfrac{\beta_1(t)b_T^2(t)}{2}, \\ \\ a_T(T) = b_T(T) = 0. \end{cases}$$

Proof. Substituting the values of $m(r,t)$ and $\sigma^2(r,t)$ in (4.7), then we will have

$$f_t(r,t) + (\alpha_1(t) + \alpha_2(t)r)f_r(r,t) + \frac{\beta_1(t) + \beta_2(t)r}{2} f_{rr}(r,t) = rf(r,t) \tag{4.9}$$

Now, suppose for some smooth deterministic functions $a_T(t)$ and $b_T(t)$ the discounted bond price is given by

$$v(t,T) = f(r(t),t) = f(r,t) = \exp\{a_T(t) + b_T(t)r\}, \qquad 0 \le t \le T.$$

Further, to guarantee that $f(r,T) = 1$ holds, we must have $a_T(T) = b_T(T) = 0$. Calculating following partial derivative we have

$$\begin{aligned} f_t &= \frac{\partial f(r,t)}{\partial t} = (a_T'(t) + b_T'(t)r)f, \\ f_r &= \frac{\partial f(r,t)}{\partial r} = b_T(t)f, \\ f_{rr} &= \frac{\partial^2 f(r,t)}{\partial r^2} = b_T^2(t)f. \end{aligned}$$

substituting the partial derivatives in (4.9) and doing little bit algebra we will obtain

$$\left(b_T'(t) + \alpha_2(t)b_T(t) + \frac{\beta_2(t)b_T^2(t)}{2} - 1\right)r + \left(a_T'(t) + \alpha_1(t)b_T(t) + \frac{\beta_1(t)b_T^2(t)}{2}\right) = 0$$

In the last equation, denote the expressions in the first parentheses by x and second parentheses by y. Then, since $r \neq 0$, we get one acceptable solution, i.e., $x = y = 0$. □

Remark 4.3.2. Note that the Vasicek, CIR and Hull–White models are categorized in the affine models class.

Definition 4.3.1. The rate of return per unit of time in continuous compounding is called **yield-to-maturity** or just yield. In particular case, if the security is the default-free discount bond with maturity T, i.e., $S(t) = v(t,T)$ with boundary condition $S(T) = 1$, then the yield to maturity can be obtained from

$$Y(t,T) = -\frac{\ln v(t,T)}{T-t}, \qquad t \le T. \tag{4.10}$$

From (4.8) and (4.10), the yield of discount bond is given by

$$Y(t,T) = -\frac{b_T(t)}{T-t}r(t) - \frac{a_T(t)}{T-t}.$$

Remark 4.3.3. Since $a_T(t)$ and $b_T(t)$ are deterministic functions in time t, the randomness of price in (4.8) is only because of the spot-rate $r(t)$.

Special Case $\sigma(r,t) = \sigma(t)$

If the diffusion coefficient in affine models be just a deterministic function of time, i.e., $\sigma(r,t) = \sigma(t)$, then we will have $\beta_1(t) = \sigma^2(t)$ and $\beta_2(t) = 0$. Moreover, we get following ODE

$$\begin{cases} b_T'(t) = -\alpha_2(t)b_T(t) + 1, & 0 \le t \le T, \\ b_T(t) = 0. \end{cases}$$

and the solution to the ODE above is simply

$$b_T(t) = -\int_t^T e^{\int_t^s \alpha_2(u)du} ds, \qquad 0 \le t \le T. \tag{4.11}$$

after calculating $b_T(t)$ we can calculate $a_T(t)$ as follow

$$a_T(t) = \int_t^T \alpha_1(u)b_T(u)du + \frac{1}{2}\int_t^T \sigma^2(u)b_T^2(u)du.$$

4.3.3 Pricing a Discount Bond Via Vasicek Model

To find the default-free discount bond price using Vasicek model, we assume that the market price of risk is a constant, i.e., $\lambda(t) = \lambda$. Then, from (4.6), we obtain a new SDE under risk-neutral probability measure Q as follow

$$dr = a(\bar{r} - r)dt + \sigma dW^*, \tag{4.12}$$

where $\bar{r}$ is a positive constant and represents the risk-adjusted, mean-reverting level.

$$\bar{r} = m - \frac{\sigma}{a}\lambda.$$

It is obvious that the SDE (4.12) is an affine model with the risk-adjusted drift $m(r,t) = a\bar{r} - ar$ and diffusion coefficient $\sigma(r,t) = \sigma$. Since we have a constant diffusion coefficient, we can calculate $b_T(t)$ and $a_T(t)$ as follow

$$\begin{aligned} b_T(t) &= -\int_t^T e^{\int_t^s \alpha_2(u)du} ds = -\frac{1-e^{-a(T-t)}}{a}, \\ a_T(t) &= \int_t^T \alpha_1(u)b_T(u)du + \frac{1}{2}\int_t^T \sigma^2(u)b_T^2(u)du \\ &= -(b_T(t) + T - t)\left(\bar{r} - \frac{\sigma^2}{2a^2}\right) - \frac{\sigma^2}{4a}b_T^2(t). \end{aligned}$$

So, the default-free discount bond price using Vasicek model becomes

$$v(t,T) = H_1(T-t)e^{-H_2(T-t)r(t)}, \qquad 0 \le t \le T,$$

where, functions H_1, H_2 have following expressions

$$H_2(t) = \frac{1-e^{-at}}{a},$$

and

$$H_1(t) = \exp\left\{ \frac{(H_2(t)-t)(a^2\bar{r}-\sigma^2/2)}{a^2} - \frac{\sigma^2 H_2^2(t)}{4a} \right\}.$$

4.3.4 Pricing a Discount Bond Via CIR Model

Let the market price of risk be

$$\lambda(t) = \frac{a(m-\bar{r})}{\sigma\sqrt{r(t)}}, \qquad 0 \le t \le T,$$

then, using the SDE (4.6), the new SDE under risk-neutral probability measure Q is given by

$$dr = a(\bar{r}-r)dt + \sigma\sqrt{r(t)}dW^*, \qquad 0 \le t \le T, \tag{4.13}$$

here, $\bar{r}$ represents the risk-adjusted mean-reverting level. Again, the SDE (4.13) is an affine model with the risk-adjusted mean $m(r,t) = a\bar{r} - ar$ and diffusion coefficient $\sigma^2(r,t) = \sigma^2 r$. The simultaneous ODE for this model gives

$$\begin{cases} b_T'(t) = ab_T(t) - \dfrac{\sigma^2 b_T^2(t)}{2} + 1, \\ \\ a_T'(t) = -a\bar{r}b_T(t). \end{cases}$$

Let $\gamma = \sqrt{a^2+2\sigma^2}$. Then after after solving the above ODE, we get

$$b_T(t) = -\frac{2(e^{\gamma(T-t)}-1)}{(a+\gamma)(e^{\gamma(T-t)-1})+2\gamma},$$
$$a_T(t) = \frac{2a\bar{r}}{\sigma^2}\ln\frac{2\gamma e^{(a+\gamma)(T-t)/2}}{(a+\gamma)(e^{\gamma(T-t)-1})+2\gamma}.$$

Finally, the discount-bond price in the CIR model is given by (see example 15.2 in [22])

$$v(t,T) = H_1(T-t)e^{-H_2(T-t)r(t)}, \qquad 0 \le t \le T,$$

where

$$H_1(t) = \left(\frac{2\gamma e^{(a+\gamma)t/2}}{(a+\gamma)(e^{\gamma t}-1)+2\gamma}\right)^{2a\bar{r}/\sigma^2},$$
$$H_2(t) = \frac{2(e^{\gamma t}-1)}{(a+\gamma)(e^{\gamma t}-1)+2\gamma}.$$

4.3.5 Pricing a Discount Bond Via Hull–White (One-Factor) Model

To begin with, we know that in Hull–White model the spot-rate process $\{r(t)\}$ is already given under risk-neutral probability measure Q. As mentioned before, the spot-rate SDE has following form

$$dr = (\phi(t) - ar)dt + \sigma dW^*, \qquad 0 \le t \le T,$$

Hull–White model is an affine model. So, the default-free discount bond price is given by (4.8). The parameters of the risk-adjusted drift $m(r,t)$ and the diffusion coefficient $\sigma(r,t)$ are given by

$$\begin{cases} m(r,t) = \alpha_1(t) + \alpha_2(t)r = \phi(t) - ar, \\ \sigma^2(r,t) = \beta_1(t) + \beta_2(t)r = \sigma^2(t). \end{cases}$$

Here, the diffusion coefficient is only a function of time t. As we discussed in the Section 4.3.1 (Special Case $\sigma(r,t) = \sigma(t)$), since $\beta_1(t) = \sigma^2(T)$ and $\beta_2(t) = 0$, we can calculate a_T and b_T in the following procedure

$$\begin{aligned}
b_T(t) &= -\int_t^T e^{\int_t^s \alpha_2(u)du} ds = -\int_t^T e^{-a(s-t)} ds, \\
\text{put} \quad u &= -a(s-t) \Rightarrow \frac{du}{ds} = -a \qquad \text{then we have} \\
b_T(t) &= \frac{1}{a}\int_t^T e^u du = \frac{1}{a}\int_0^{-a(T-t)} e^u du = \frac{1}{a}e^{-a(T-t)} - \frac{1}{a}e^{-a(0)} = \frac{1}{a}(e^{-a(T-t)} - 1), \\
a_T(t) &= \int_t^T \alpha_1(u) b_T(u) du + \frac{1}{2}\int_t^T \sigma^2(u) b_T^2(u) du \\
&= \int_t^T \phi(u)\frac{1}{a}(e^{-a(T-t)} - 1) du + \frac{1}{2}\int_t^T \sigma^2(u)[\frac{1}{a}(e^{-a(T-t)} - 1)]^2 du.
\end{aligned}$$

Now, we can substitute $b_T(t)$ and $a_T(t)$ into (4.8), we get

$$\begin{aligned}
v(t,T) &= H_1(t,T)e^{-H_2(t,T)r(t)}, \\
H_2(t,T) &= \frac{1 - e^{-a(T-t)}}{a}, \\
H_1(t,T) &= \exp\left\{\frac{1}{2}\int_t^T \sigma^2 H_2^2(u,T) du - \int_t^T \phi(u) H_2(u,T) du\right\}.
\end{aligned}$$

Remark 4.3.4. Using the definition of forward rate, i.e.,

$$f(t,T) = -\frac{\partial}{\partial T}\ln v(t,T), \quad t \le T,$$

it is possible to show that (see Exercise 15.5 in [22])

$$\phi(t) = af(0,t) + \frac{\partial}{\partial t}f(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

Chapter 5

Pricing Interest Rate Derivatives

From the beginning of this chapter up to Section 5.4.3, we follow the ideas and concepts stated in Chapter 15 in [22], Chapter Appendix in [13] and Chapter 28 in [18]. In Section 5.4.3, we follow the concepts in [29] and in Section 5.5 we mainly use [7] and [31] as our references. To price the interest derivatives like bond options, caps and swaps, we should consider the fact that the interest rates are stochastic and therefore forward-neutral method plays a vital role in the pricing of interest rate derivatives. This means that, the volatility structure in future can be significantly different from what we can see in the today's market. As we have seen in the previous chapters our models were initially constructed just on short rate r and later we introduced the stochastic interest rate or term structure of interest rates. Now, we can introduce some of the most commonly used interest rate derivatives .

5.1 Terminology

We have already defined the money market account and the return on default-free bonds in Section 2.8. Now, consider the most commonly used compounded interest rate, r. That is the fraction of a year over which interest is compounded and we will denote it by δ. Once again, assume that an investor has one unit of money, then the interest over δn years will be $(1+\delta r)^n - 1$. Here, we should emphasize which day-count convention is applied, because the precise length of nominally equal 3-month or 6-month interval might vary. In general, for unequal fractions of $\delta_1, \delta_2, \ldots$ we can calculate the growth of one unit of money by

$$\prod_{i=1}^{n}(1+\delta_i r)$$

Now, consider the case that a coupon bond gives multiple payments c_i at different dates T_i, $i = 1, \ldots, n$ and a principal payment of one at maturity time T_n. Then, the value of our portfolio (money-market account, i.e., $B_c(t)$) consisting of such a coupon bond at any time prior to

maturity with $l(t)$ as the next coupon date will be

$$B_c(t) = v(t,T_n) + \sum_{i=l(t)}^{n} c_i v(t,T_i), \qquad T_{l(t)-1} < t \leq T_{l(t)}. \tag{5.1}$$

As we discussed it in Section 2.8, the return on such a portfolio with a constant and continuously compounded interest rate will be e^{rT}. Further, when we have a stochastic interest rate $r(t)$ (short-rate) the value of our money-market account was given by (2.2). Finally, in Section 4.3, we discussed explicitly that the price of such a default-free discount bond is given by (4.5).

We remember from Section 2.8 that the continuously compounded yield for a zero-coupon bond with maturity T was given by (2.4) and (4.10), i.e.

$$Y(t,T) = -\frac{\ln v(t,T)}{(T-t)}, \qquad t \leq T, \quad v(t,T) = e^{-Y(t,T)(T-t)}. \tag{5.2}$$

In Chapter 4, we went through different models and approaches to find a proper value for the default-free discount bond $v(t,T)$ or equivalently the continuously compounded yield. Put all the discussion above together, (5.1) can be rewritten as

$$B_c(t) = e^{-Y_c(t)(T_n-t)} + \sum_{i=l(t)}^{n} c_i e^{-Y_c(t)(T_n-t)}, \qquad T_{l(t)-1} < t \leq T_{l(t)}.$$

In the market, yields are mostly quoted on a semi-annual basis. Let $Y_\delta(t,T)$ be the yield with compounding interval δ, then when $n = (T-t)/\delta$ is an integer, we have

$$v(t,T) = \frac{1}{(1+\delta Y_\delta(r,T))^n}, \qquad t \in (0,T).$$

Now, we are taking one more step to complete our puzzle. As we discussed in Section 2.8, the forward rate is a type of interest rate which is set today for lending or borrowing at some predetermined date in the future. An investor would go to a contract to borrow one unit of money at time T_1 and pay the loan back at time T_2. The forward rate at $F(t,T_1,T_2)$ applies to this contract. In other words, the investor get one unit of money at time T_1 and gives back $1+F(t,T_1,T_2)(T_2-T_1)$ at time T_2. We showed in (2.7), the forward rates are determined by bond prices. Say, an investor funds a purchase of a zero-coupon bond with maturity T_1 at time t,by issuing k-number of bonds maturing at T_2, $t < T_1 < T_2$. Then, we obtain

$$kv(t,T_2) = v(t,T_1),$$

This means that getting one unit of money at time T_1 and paying back k units of money at time T_2. This procedure has some transaction costs which prevents any arbitrage opportunity. So we will have

$$k = 1 + F(t,T_1,T_2)(T_2-T_1)$$

substituting $k = v(t,T_1)/v(t,T_2)$ yields to

$$F(t,T_1,T_2) = \frac{1}{T_2 - T_1}\left(\frac{v(t,T_1) - v(t,T_2)}{v(t,T_2)}\right). \tag{5.3}$$

Before, the financial crisis in 2007, the most widely used benchmark for interest rates was the London Interbank Offered rates (LIBOR). The LIBOR rate is calculated every day and it is an average of rates which is offering by a number of selected banks in London. Different LIBOR rate are set separately for different maturity times and different currencies. A forward LIBOR rate can be seen as a special case of (5.3) whose $\delta = T_2 - T_1$ is fixed for its life time. Therefore, the δ-year forward LIBOR rate at time t whose maturity is T can be shown by

$$L(t,T) = F(t,T,T+\delta) = \frac{1}{\delta}\left(\frac{v(t,T) - v(t,T+\delta)}{v(t,T+\delta)}\right). \tag{5.4}$$

Remark 5.1.1. In the evaluation of (5.4) as a special case of (5.3), we have ignored the credit risk. We assumed that all payments would occur at their predetermined dates, i.e., the borrower never goes to default. But, as we have seen, even some banks faced bankruptcy. And this is one of the reason that collateral agreement has become more popular.

We can use (5.3) to derive a continuously compounded forward rate, i.e., $f(t,T_1,T_2)$

$$\begin{aligned} \exp\{f(t,T_1,T_2)(T_2 - T_1)\} - 1 &= F(t,T_1,T_2)(T_2 - T_1) \\ f(t,T_1,T_2) &= \frac{\ln v(t,T_1) - \ln v(t,T_2)}{T_2 - T_1} \end{aligned}$$

Now, let $f(t,T)$ be a continuously compounded forward rate at time t with maturity T. Then, $\lim_{h\to 0} f(t,T+h)$ gives us (2.7), further putting $v(T,T) = 1$ yields (2.8) and finally comparing with (2.4) or equivalently (5.2) gives

$$Y(t,T) = \frac{1}{T-t}\int_t^T f(t,u)du;$$

which means the yields are averages over the forward rates.

Remark 5.1.2. Since yields are averages over the forward rates, the forward rates have more proper potential to build term-structure models.

5.2 Bond Options

Define

- $v(t,\tau)$: the time-t price of the default-free discount bond maturing at time τ,
- $h(X)$ the payoff function of a contingent claim written on $v(t,\tau)$ with maturity $T < \tau$,

then, the time-t price of this claim using forward-neutral method is given by (see page 270 in[22])

$$\pi(t) = v(t,T)\mathsf{E}^{\mathsf{Q}^T}[h(v,T,\tau))|\mathscr{F}_t], \quad t \leq T < \tau. \tag{5.5}$$

Therefore, one needs the univariate distribution of $v(T,\tau)$ under our probability measure, which is forward-neutral probability measure Q^T. Now, assume that the market is arbitrage-free and the default-free discount bond under risk-neutral probability measure Q satisfies following SDE

$$\frac{dv(t,\tau)}{v(t,\tau)} = r(t)dt + \sigma(t,\tau)dW^*, \quad 0 \leq t \leq \tau, \tag{5.6}$$

where $r(t)$ is default-free spot rate and $\{W^*(t)\}$ is a standard Brownian motion under probability measure Q.

Now, the T-forward price of the discount bond is denoted by

$$v^T(t,\tau) = \frac{v(t,\tau)}{v(t,T)}, \quad t \leq T < \tau.$$

Replacing the volatility $\sigma(t,\tau)$ by $\sigma(t,T)$, $v(t,T)$ satisfies SDE (5.6). Hence, Itô division (Theorem 3.4.2) for some mean rate of return $\mu^T(t)$ gives

$$\frac{dv^T(t,\tau)}{v^T(t,\tau)} = \mu^T(t)dt + [\sigma(t,\tau) - \sigma(t,T)]dW^*, \quad 0 \leq t \leq T.$$

Change of measure must be done in a way such that our forward-neutral probability measure Q^T makes our T-forward price process, $\{v^T(t,\tau)\}$, into a martingale. Thus, we obtain

$$\mu^T(t)dt + [\sigma(t,\tau) - \sigma(t,T)]dW^* = [\sigma(t,\tau) - \sigma(t,T)]dW^T,$$

where, $\{W^T(t)\}$ is a standard Brownian motion under T-forward probability measure Q^T. Now, can introduce the T-forward discount-bond price SDE by

$$\frac{dv^T(t,\tau)}{v^T(t,\tau)} = [\sigma(t,\tau) - \sigma(t,T)]dW^T, \quad 0 \leq t \leq T, \tag{5.7}$$

and the process $\{v^T(t,\tau)\}$ is a martingale under forward-probability measure Q^T.

5.2.1 Affine Model

We start this part by introducing a proposition which will help us to find the price of our interest derivative.

Proposition 5.2.1. *Let the price process $\{S(t)\}$ satisfies following SDE (see Proposition 14.3 in [22])*

$$dS(t) = \sigma(t)S(t)dW, \qquad t \geq 0$$

where $\{W(t)\}$ is standard Brownian motion. Now, suppose $\sigma(t)$ to be a deterministic function of time t which satisfies the condition $\int_0^T \sigma^2(t)dt < \infty$. Then defining $\psi^2 = \int_0^t \sigma^2(u)du$, we have

$$\begin{aligned} S(t) &= S(0)\exp\left\{-\frac{\psi^2}{2} + \int_0^t \sigma(u)dW(u)\right\}, \\ \mathsf{E}\left[\max(S(t)-K,0)\right] &= S(0)\Phi(d) - K\Phi(d-\psi), \\ d &= \frac{\ln(S(0)/K)}{\psi} + \frac{\psi}{2}, \qquad \psi > 0. \end{aligned}$$

Now, let's consider an example (see Example 15.3 in [22]) of the affine model (Theorem 4.3.1). Using (4.8) we obtain

$$\begin{aligned} v^T(t,\tau) &= \frac{v(t,\tau)}{v(t,T)} = \exp\left\{a_\tau(t) - a_T(t) + [b_\tau(t) - b_T(t)]r(t)\right\}, \\ \frac{dv^T(t,\tau)}{v^T(t,\tau)} &= [b_\tau(t) - b_T(t)][\sigma(r(t),t)][-b_T(t)\sigma(r(t),t)dt + dW^*] \end{aligned}$$

where $\sigma^2(r(t),t) = \beta_1(t) + \beta_2(t)r(t)$. Now, we change our probability measure to T-forward-probability measure in a way that

$$dW^T = -B_T(t)\sigma(r(t),t)dt + dW^* \tag{5.8}$$

which leads us to

$$\frac{dv^T(t,\tau)}{v^T(t,\tau)} = [b_\tau(t) - b_T(t)][\sigma(r(t),t)]dW^T, \quad 0 \leq t \leq T. \tag{5.9}$$

If we assume that diffusion coefficient $\sigma(t)$ is a deterministic function of time, then $\beta_1(t) = \sigma^2(t)$ and $\beta_2(t) = 0$. Now, using (4.11) we get

$$b_\tau(t) - b_T(t) = -\int_T^\tau \exp\left\{\int_t^s \alpha_2(u)du\right\}ds.$$

Putting the equation above in (5.9) gives

$$\frac{dv^T(t,\tau)}{v^T(t,\tau)} = \left(-\sigma(t)\int_T^\tau \exp\left\{\int_t^s \alpha_2(u)du\right\}ds\right)dW^T, \quad 0 \leq t \leq T.$$

Which is a deterministic function of time. For convenient, denote the expression in parentheses with $\theta(t)$. Thus, in this case, the forward price $v_T(t,\tau)$ has a log-normal distribution under Q^T. Now, we set

$$\sigma_F^2 = \int_0^T \left(-\sigma(t)\int_T^\tau \exp\left\{\int_t^s \alpha_2(u)du\right\}ds\right)^2 dt = \int_0^T \theta^2(t)dt,$$

then, using Proposition 5.2.1 we obtain

$$\mathsf{E}^{\mathsf{Q}^T}\left[\max\left\{V^T(T,\tau)-K,0\right\}|\mathscr{F}_t\right]=v^T(t,\tau)\Phi(d)-K\Phi(d-\sigma_F),$$

where

$$d=\frac{\ln[v^T(t,\tau)/K]}{\sigma_F}+\frac{\sigma_F}{2}.$$

Because, $v^T(T,\tau)=v(T,\tau)$ and $v^T(t,\tau)=v(t,\tau)/v(t,T)$, the price of a call option premium is

$$\pi_C(t)=v^T(t,\tau)\Phi(d)-K\Phi(d-\sigma_F),\quad t\leq T<\tau.$$

Remark 5.2.1. Since, our affine model has a deterministic volatility, the prices of interest-rate derivatives are irrelevant to the risk-adjusted, mean reverting models. And that is why we have not seen the term $\alpha_1(t)$ in our pricing formula.

5.3 Forward LIBOR and Black's Formula

Definition 5.3.1. The interest rate defined by

$$L_i(t)=\frac{v(t,T_i)-v(t,T_{i+1})}{\delta_i v(t,T_{i+1})},\quad 0\leq t\leq T_i, \tag{5.10}$$

where, $0\leq T_0<\cdots<T_i<T_{i+1}<\ldots$ (are set or maturities or tenor dates) and $\delta_i\equiv T_{i+1}-T_i>0$ (the length of intervals between tenor dates) is called the T_i**-forward LIBOR rate** at time t.

Observe that,

$$1+\delta_i L_i(t)=\exp\left\{\int_{T_i}^{T_{i+1}}f(t,s)ds\right\}=\frac{v(t,T_i)}{v(t,T_{i+1})}. \tag{5.11}$$

Therefore, $L_i(T_i)=L(T_i,T_{i+1})$ is the **LIBOR Rate** at time T_i.

We know that any security price $\pi(t)$ which is not paying dividends, the forward price defined by $\pi^T(t)=\pi(t)/v(t,T)$ under forward probability measure Q^T is martingale. Therefore,

$$\frac{\pi(t)}{v(t,T)}=\mathsf{E}^{\mathsf{Q}^T}\left[\frac{\pi(\tau)}{v(\tau,T)}\middle|\mathscr{F}_t\right],\quad t\leq\tau\leq T. \tag{5.12}$$

Now, we substitute $T=T_{i+1}$ and $\pi(t)=v(t,T_i)-v(t,T_{i+1})$ in (5.12). We obtain

$$\frac{v(t,T_i)-v(t,T_{i+1})}{v(t,T_{i+1})}=\mathsf{E}^{\mathsf{Q}^{T_{i+1}}}\left[\frac{v(\tau,T_i)-v(\tau,T_{i+1})}{v(\tau,T_{i+1})}\middle|\mathscr{F}_t\right].$$

and from (5.10) we obtain

$$L_i(t)=\mathsf{E}^{\mathsf{Q}^{T_{i+1}}}\left[L_i(\tau)\middle|\mathscr{F}_t\right],\quad t\leq\tau\leq T,$$

thus, T_i-forward LIBOR, i.e., $L_i(t)$ under the forward-neutral probability measure $\mathsf{Q}^{T_{i+1}}$ is a martingale.

5.3.1 Caps and Floors

Recall that a cap is a portfolio of caplets.

Definition 5.3.2. Let the LIBOR rate $L_i(T_i) = L(T_i, T_{i+1})$ covers the interval $[T_i, T_{i+1}]$. Then an interest-rate derivative whose payoff is defined by

$$\delta_i \left(\max\{L_i(T_i) - K, 0\}\right), \quad \delta_i = T_{i+1} - T_i$$

is called a **caplet**. Here K denotes and is called **cap rate**.

As shown in Section 5.3, the T_i-forward LIBOR $L_i(t)$ under forward-neutral probability measure $\mathsf{Q}^{T_{i+1}}$ is a martingale. Thus, we can assume that the process $\{L_i(t)\}$ for some volatility $\sigma_i(t)$ follows the SDE in the form of

$$\frac{dL_i}{L_i} = \sigma_i(t) dW^{T_{i+1}}, \quad 0 \le t \le T_i, \tag{5.13}$$

where the process $\{W^{T_{i+1}}(t)\}$ is a standard Brownian motion under forward-neutral probability measure $\mathsf{Q}^{T_{i+1}}$. Now, we assume that $\sigma_i(t)$ is a deterministic function of time-t. Then, the LIBOR rate L_2 is log-normally distributed under forward-neutral probability measure $\mathsf{Q}^{T_{i+1}}$. Now, using Definition 5.3.2 the price of caplet at time t can be described by

$$\mathrm{Cpl}_i = B(t)\mathsf{E}^{\mathsf{Q}}\left[\frac{\delta_i \left(\max\{L_i(T_i) - K, 0\}\right)}{B(T_{i+1})}\middle| \mathscr{F}_t\right], \quad 0 \le t \le T_i,$$

applying the forward-neutral method and usuing (3.31), we obtain

$$\mathrm{Cpl}_i(t) = \delta_i v(t, T_{i+1})\mathsf{E}^{\mathsf{Q}^{T_{i+1}}}\left[\delta_i \left(\max\{L_i(T_i) - K, 0\}\right) | \mathscr{F}_t\right], \quad 0 \le t \le T_i.$$

define $\varsigma_i^2 = \int_t^{T_i} \sigma_i^2(s) ds$ (so called accumulated variance), then applying Proposition 5.2.1 we can obtain the formula to determine the **caplet price** by

$$\mathrm{Cpl}_i(t) = \delta_i v(t, T_{i+1}) \left[L_i(t)\Phi(d_i) - K\Phi(d_i - \varsigma_i)\right], \tag{5.14}$$

where

$$d_i = \frac{\ln(L_i(t)/K)}{\varsigma_i} + \frac{\varsigma_i}{2}, \qquad \varsigma_i > 0.$$

Hence, if $t < T_0$ then the **cap price** at time t is given by

$$\mathrm{Cap}(t) = \sum_{i=0}^{n-1} \delta_i v(t, T_{i+1}) \left[L_i(t)\Phi(d_i) - K\Phi(d_i - \varsigma_i)\right].$$

In the same manner we can find the floorlet and floor prices.

Remark 5.3.1. If we put $\delta_i = 1$ then (5.14) will be the same as Black's formula. In other words, letting σ_i to be deterministic volatility then $L_i(t)$ is log-normally distributed and the cap price can be calculated using the Black's formula. Black's formula has a considerable usage when we use implied volatilities.

5.3.2 Swaptions

As we mentioned in Section 2.4, in a standard swaption, i.e., interest rate swap option, one of the two parties A and B promise to pay interest at the floating rate and the other party at fixed rate. This exchange payments fixed on a notional principal.

Consider an interest rate swap that starts from $T_0 > t$ is issued at the current time t. Moreover, assume (for simplification) the notional principal is one unit of money and the exchange of cash flows are going to happen at dates $T_i = T_0 + \delta i, \quad i = 1, \ldots, n$. Denote the swap rate by $S(t)$. Consider a scenario that party A will pay a fix rate of $\delta S = S'$ to party B at dates T_i. Recall that $v(t,T)$ is the time-t price of default-free discount bond with maturity time T, then the present value of total payments at the fix side becomes

$$V_{\text{FIX}} = S'v(t,T_1) + \cdots + S'v(t,T_n) = \delta S \sum_{i=1}^{n} v(t,T_i). \tag{5.15}$$

On contrary, party B is going to pay a floating interest rate to party A at time T_{i+1}. This floating interest rate is called δ-LIBOR and let's denote it by $\delta_i L_i(T_i) = L_i'$. Since we are in time t_0 and we do not have access to the real value of LIBOR rate at time T_{i+1}, therefore $L(T_i, T_{i+1})$ is a random variable. So, we have to choose forward-neutral valuation to find the value of floating side. In general, let's say the floating rate at time T_{i+1} is denoted by $\delta_i L_i(T_i)$. So, the aggregate value of payment by floating side will be (see Example 15.4 in [22])

$$V_{\text{FL}} = B(t)\mathsf{E}^{\mathsf{Q}}\left[\sum_{i=0}^{n-1} \frac{\delta_i L_i(T_i)}{B(T_{i+1})}\middle|\mathscr{F}_t\right] = B(t)\sum_{i=0}^{n-1} \delta_i \mathsf{E}^{\mathsf{Q}}\left[\frac{L_i(T_i)}{B(T_{i+1})}\middle|\mathscr{F}_t\right],$$

And from (3.31), we have

$$B(t)\mathsf{E}^{\mathsf{Q}}\left[\frac{L_i(T_i)}{B(T_{i+1})}\middle|\mathscr{F}_t\right] = v(t,T_{i+1})\mathsf{E}^{\mathsf{Q}^{T_{i+1}}}\left[L_i(T_i)\middle|\mathscr{F}_t\right],$$

since the process $\{L_i(t)\}$ is a martingale under forward-neutral probability measure under $\mathsf{Q}^{T_{i+1}}$, the two equations above yields to

$$V_{\text{FL}} = \sum_{i=0}^{n-1} v(t,T_{i+1})\delta_i L_i(t),$$

observing that

$$v(t,T_{i+1})\delta_i L_i(t) = v(t,T_i) - v(t,T_{i+1}),$$

we will get

$$V_{\text{FL}} = v(t,T_0) - v(t,T_n).$$

Thus, the **swap rate** at time t is given by

$$S(t) = \frac{v(t,T_0) - v(t,T_n)}{\delta \sum_{i=1}^{n} v(t,T_i)}, \qquad 0 \le t \le T_0. \tag{5.16}$$

Remark 5.3.2. Since swap rates fluctuate randomly, thus they can be used as an underlying asset for an option. Such kind of options in the market are called **swaptions**.

5.4 Black's Formula and Negative Interest Rates

In this section, we look at the original Black's model, Black's model revised and the usage of Black's formulas in evaluating the price of Caps, Floors and Swaptions. Then, we go through the problem with negative interest rates and the lognormal properties of Black's model and Black's volatility.

5.4.1 Black's Model (Formula)

In 1976, Fisher Black proposed an extension to the Black-Scholes-Merton Model which is known as Black's Model (Formula). He tried to evaluate the price of European future options in his work. In this evaluation process, Black assumed that in the risk-neutral world the future prices are lognormal . As a result, Black obtained future (forward) option prices for European call, i.e., π_C and European put, i.e., π_P in following forms [2]

$$\pi_C(t) = e^{-r(T-t)}\left[f_0 N(d_1) - KN(d_2)\right],$$
$$\pi_P(t) = e^{-r(T-t)}\left[KN(-d_2) - f_0 N(-d_1)\right],$$

where

$$d_1 = \frac{\ln(f_0/K) + \sigma^2(T-t)/2}{\sigma\sqrt{T-t}},$$
$$d_2 = \frac{\ln(f_0/K) - \sigma^2(T-t)/2}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

Here, T the is maturity time, K is the strike price and σ is a constant and represents the volatility of future prices.

Remark 5.4.1.

- In this version, interest rate assumed to be constant. Recall that, $f_0 = f(0,0) = r(0)$.
- Under some technical conditions, it is possible to show that the volatility of future price and the volatility of underlying asset are the same (see page 370–371 in [18]).

5.4.2 Black's Formula and Evaluation of LIBOR, Caps and Swaptions

The Black's model has been used (was the market praxis when the rate were non-negative) for the evaluation of Caps, Floors and Swaptions. Black's model is used on individual Caplets and Floorlets as well [29]. But as we discussed before, in the evaluation of such financial instruments the interest rate is stochastic. So, let's start with looking at Black's revised model with stochastic interest rate.

Black's Model with Stochastic Interest Rate

Let K be the strike price, T the maturity time and S the asset price. Then using forward-neutral evaluation, we will have the following relation to evaluate the price of a European call option, π_C

$$\pi_C(t) = v(t,T)\mathsf{E}^{\mathsf{Q}^T}\left[\max\{S_T - K, 0\}\,|\mathscr{F}_t\right],$$

further, define $f(0,0) = S(0)$ and $f_T = f(T,T) = S_T$. Then we get

$$\pi_C(t) = v(t,T)\mathsf{E}^{\mathsf{Q}^T}\left[\max\{f_T - K, 0\}\,|\mathscr{F}_t\right].$$

Now, assume that in our forward-neutral world the forward price f_T is lognormally distributed. Thus, the standard deviation of $\ln(f_T)$ will be equal to $\sigma_f\sqrt{T-t}$. In other words, we are dealing with a stochastic forward price process which has a constant volatility σ_f. Before going any further, we introduce the following corollary and we will use its result in next step.

Corollary 5.4.1. *Assume f_T is lognormally distributed. Further, let the standard deviation of $\ln(f_T)$ be σ_f. Then we will have*

$$\mathsf{E}\left[\max\{f_T - K, 0\}\right] = \mathsf{E}[f_T]N(d_1) - KN(d_2).$$

where

$$d_1 = \frac{\ln(\mathsf{E}[f_T]/K) + \sigma_f^2/2}{\sigma_f},$$
$$d_2 = \frac{\ln(\mathsf{E}[f_T]/K) - \sigma_f^2/2}{\sigma_f}.$$

For the proof and more details, see the Appendix of Chapter 14 in [18].

Considering the fact that, the expectations in the Corollary 5.4.1 can be rewritten in our forward-neutral probability measure Q^T, and $\mathsf{E}^{\mathsf{Q}^T}[f_T] = \mathsf{E}^{\mathsf{Q}^T}[S_T] = f_0$, we obtain

$$\begin{aligned}
\pi_C(t) &= v(t,T)\left[f_0N(d_1) - KN(d_2)\right],\\
d_1 &= \frac{\ln(f_0/K) + \sigma_f^2(T-t)/2}{\sigma_f\sqrt{T-t}},\\
d_2 &= \frac{\ln(f_0/K) - \sigma_f^2(T-t)/2}{\sigma_f\sqrt{T-t}}.
\end{aligned} \tag{5.17}$$

And in the same manner the price of European put option will be

$$\pi_P(t) = v(t,T)\left[KN(-d_2) - f_0N(-d_1)\right].$$

We have seen an example of Black's formula to evaluate the price of cap by (5.14). It is possible to see the price evaluation of floor, swaption and LIBOR using Black's formula as well[1].

5.4.3 Black's Volatility

Black's model with stochastic interest rate has been considerably used in the market. In many literature, the volatility of forward rates, σ_f, is denoted by, σ_B, i.e., Black's volatility and considered to be constant. As an example, the price of the European bond options (namely, forward bond price), caps, floors and swaptions can be calculated by Black's formula[2].

Another commonly usage of Black's volatility, is to estimate the volatility of normally distributed forward price process, i.e. σ_N [29]. Before introducing the negative interest rates, one could claim that the price of a European option using either the normal or lognormal forward price process must be equal. Through such a claim it is possible to estimate the σ_N when we know the σ_B. This property shows its vital importance when we consider the fact that if the forward bond price are lognormal, then the swap rates and forward rates are normal. On the other hand, if forward rates are lognormal, thus the swap rates and forward bond prices are normal. To see this better, let's look at the formula to estimate the normal and lognormal volatilities.

Calculating σ_N using known σ_B

To begin with, let us refresh our mind with the dynamics of forward rates. For today's forward rate (namely cap/swap rate), we have following dynamics

1. **Lognormaly distributed forward rate, i.e., Black's Model**

$$\begin{cases} dF = \sigma_B F dW \\ F(0) = f_0 \end{cases}$$

   where σ_B is the implied Black's (lognormal) volatility and the solution to the above SDE for a European call option is given by (5.17).

2. **Normaly distributed forward rate**

$$\begin{cases} dF = \sigma_N dW \\ F(0) = f_0 \end{cases}$$

   where σ_N is the "normal" or the annualized "basis point" or "absolute" volatility and the

[1] See [29, 18]
[2] See [29, 18] for detailed explanations and examples.

solution to the above SDE for a European call option is given by [29]

$$\pi_{C_N}(t) = v(t,T)\left[(f_0 - K)N(d) + \frac{\sigma_N^2}{\sqrt{2\pi}}e^{-d^2/2}\right],$$
$$d = \frac{f_0 - K}{\sigma_N\sqrt{T-t}}. \tag{5.18}$$

Now, from the discussion above we can find a relation between normal and lognormal Black's volatility. In fact, it can be shown that for a Swaption with fix rate (strike price) K, the normal volatility σ_N and the lognormal volatility σ_B give the same price of a European option, i.e., $\pi_{C_N}(t) = \pi_{C_B}(t)$. The correspondieng relation for normal and Black's volatility can be described by (see page 570–571 in [29])

$$\sigma_N = \frac{\sigma_B\,(f_0 - K)}{\ln(f_0/K)\left[1 + \frac{1}{24}\left(1 - \frac{1}{120}\left[\ln(f_0/K)\right]^2\right)\sigma_B^2\tau + \frac{1}{5760}\sigma_B^4\tau^2\right]}, \quad \frac{f_0}{K} > 0, \quad f_0 \neq K. \tag{5.19}$$

Here τ stands for the exercise date in years. As we can see in the formula above, we have two important conditions. The first condition is to avoid dividing by zero, $f_0 \neq K$ must be fulfilled, because $\ln(1) = 0$. To avoid this technical complication, the alternative formula of (5.19) is given by [29]

$$\sigma_N = \frac{\sigma_B\sqrt{f_0 K}\left(1 + \frac{1}{24}\left[\ln(f_0/K)\right]^2\right)}{1 + \frac{1}{24}\sigma_B^2\tau + \frac{1}{5760}\sigma_B^4\tau^2}, \quad \text{for} \quad \left|\frac{f_0 - K}{K}\right| < 0.001.$$

Although the alternative formula would work with the given condition, but it does not cover at the money case, i.e. $f_0 = K$. Moreover, to get the Black's volatility knowing the normal volatility, one can use some numerical methods (Newton-Raphson method for example).

Second is to have $(f_0/K) \in \mathbb{R}^+$. Since we have seen negative interest rate, this condition cannot always hold. So, we might look after some models in which their price processes are normally distributed and they can give us negative prices or rates.

5.5 Bachelier's "Theory of Speculation"

After introducing the negative rates, Black's model might not be effective in the price evaluation of interest rate derivatives. Recall, the $f_0 = f(0,0) = r(0)$ and if $r(0) < 0 \Rightarrow f_0 < 0$. Further, we know that the logarithmic function $\ln(x)$ is defined for $x \in \mathbb{R}^+$. Thus, it might be a good idea to search for some models which can give the negative prices.

To begin with, Samuelson's representation of Brownian motion to define the price and return processes have been used in the derivation of price formulas. The most famous model in this field might be Black-Scholes-Merton model. Further, the most important property of Samuelson's representation of Brownian motion (Geometric Brownian motion) is its strength to secure positive prices, i.e., the price process is lognormally distributed.

After introducing negative interest rates, a model which can give the negative prices might have lots of practical usages in the price evaluation of financial derivatives. Surprisingly, there was a model which could give the negative prices. In 1900, a French PHD student whose name was Louis Bachelier proposed "Theory of Speculation" (Théorie de la Spéculation) in his thesis. Bachelier introduced a mathematical finance model. But, his work was used and seen mostly in probability theories and stochastic analysis for several decades till Samuelson proposed his price process using Bachelior arguments. After seeing the negative interest rates, we can say that the best feature of Bachelier's representation of Brownian motion is that the asset prices are normally distributed and they can be negative at any time. The Bachelier model can be seen as a version of Black-Scholes-Merton model which is defined for normal random variables instead of lognormal random variables. Now, let's see how the Bachelier model looks like.

To begin with, Bachelier presented the foundations for the modern theory of Brownian motion. He argued, assumed and obtained followings [7][3]

1. In short interval of time, the small fluctuations in stock price must be independent of current price.

2. The price process is assumed to be memory less and it is independent of past behavior (discrete sample path). That is what we call it Markov property in the modern mathematical language.

3. Applying Central Limit Theorem to the memory less property of price process, he showed that the increments of price process are independent and normally distributed (independent Gaussian random variables). That is Brownian motion as diffusion limit of random walk.

4. Using memory less property of price process, Bachelier obtained an equation (known as Chapman-Kolmogorov equation) to derive the connection with the heat equation.

[3] In [7] the Bachelier's original thesis in French and a translated English version are available.

5.5.1 Bachelier's Formula

Bachelier used an equilibrium model to derive his pricing formula. On the contrary, we know that Black-Scholes-Merton formula is derived using no-arbitrage argument. The Bachelier security price process SDE (matching our notation in this work) can be seen as

$$dS(\tau) = S(t)\sigma dW(\tau), \qquad 0 \leq t \leq \tau \leq T,$$

where the process $\{W(t)\}$ is a standard Brownian motion and $\sigma > 0$ is a constant and stands for the volatility. Solving Bachelier SDE we will obtain

$$S(T) = S(t)\left[1 + \sigma\left(W(T) - W(t)\right)\right].$$

Now, to find a fair price of European call option with strike price K and knowing that Bachelier set the interest rate $r(t) = 0$, we can rewrite (3.21) in following form

$$\begin{aligned}\pi_C(t) &= B(t)\mathsf{E}^{\mathsf{Q}}\left[\frac{h(S(T))}{B(T)}\middle|\mathscr{F}_t\right] = e^{-r(T-t)}\mathsf{E}^{\mathsf{Q}}\left[\max\left(S(T)-K,0\right)|\mathscr{F}_t\right] \\ &= \mathsf{E}^{\mathsf{Q}}\left[\max\left(S(T)-K,0\right)|\mathscr{F}_t\right], \qquad 0 \leq t \leq T.\end{aligned}$$

The result above is called Bachelier's "fundamental principle". See [31] for more details. Substituting $S(T)$ in the last equation we have

$$\pi_C(t) = \mathsf{E}^{\mathsf{Q}}\left[\max\left\{S(t)\left[1+\sigma\left(W(T)-W(t)\right)\right]-K,0\right\}|\mathscr{F}_t\right].$$

Bachelier's notation for volatility was given by normalized quantity of $H = S(t)\sigma/\sqrt{2\pi}$, which he called by the "coefficient of instability" ("nervousness") of the security price $S(t)$. Further, we know that Bachelier assumed that security price, i.e., $S(T)$ is normally distributed with mean $\mu = S(t)$ and standard deviation $s_d = S(t)\sigma\sqrt{(T-t)}$. That means

$$S(T) \sim N\left[(S(t), S^2(t)\sigma^2(T-t))\right].$$

Now, let us transform our normal random variable $S(T)$ to the standard normal random variable x, through the following relation

$$x = \frac{S(T)-\mu}{s_d} = \frac{S(T)-S(t)}{S(t)\sigma\sqrt{(T-t)}},$$

where the probability density function of the standard normal distribution is given by $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. We refresh our mind by representing the relations between the cumulative distribution function, i.e., $\Phi(x)$ and the probability density function, i.e. $\phi(x)$ by following formulas [33]

$$\Phi(x) = N(x) = \int_{-\infty}^{x}\phi(u)du, \qquad \Phi(-x) = N(-x) = 1-\Phi(x) = \int_{x}^{\infty}\phi(u)du.$$

Further, we introduce the point where $S(T) = K$

$$x_0 = \frac{K - S(t)}{S(t)\sigma\sqrt{(T-t)}}.$$

Now, since the Bachelier's price process follows a Brownian motion, the stochastic part (in modern language is called Wiener process) of Bachelier's price process can be rewritten as

$$W(T) - W(t) = \sqrt{(T-t)}x$$

Thus, the time-t Bachelier's security price of European call option will be

$$\begin{aligned}
\pi_C(t) &= \int_{-\infty}^{\infty} \max\left\{S(t)\left[1+\sigma\left(\sqrt{(T-t)}x\right)\right] - K, 0\right\}\phi(x)dx \\
&= \int_{x_0}^{\infty} \left(S(t)\left[1+\sigma\left(\sqrt{(T-t)}x\right)\right] - K\right)\phi(x)dx \\
&= \int_{x_0}^{\infty} \left(S(t) + S(t)\sigma\left[\sqrt{(T-t)}x\right] - K\right)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= [S(t)-K]\int_{x_0}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx + S(t)\sigma\sqrt{(T-t)}\int_{x_0}^{\infty}\frac{1}{\sqrt{2\pi}}xe^{-x^2/2}dx
\end{aligned}$$

Where the first integral is nothing but the cumulative distribution function of a standard normal random variable, i.e., $\Phi(x)$. The second integral can be calculated by changing the variables $x^2/2 = u \Rightarrow du/dx = x$. Then we will get

$$\begin{aligned}
\pi_C(t) &= [S(t)-K]\Phi(-x_0) + \frac{S(t)\sigma\sqrt{(T-t)}}{\sqrt{2\pi}}\int_{x_0}^{\infty} e^{-u}du \\
&= [S(t)-K]\Phi(-x_0) + \frac{S(t)\sigma\sqrt{(T-t)}}{\sqrt{2\pi}}\left[-e^{-u}\right]_{x_0}^{\infty} \\
&= [S(t)-K]\Phi(-x_0) + S(t)\sigma\sqrt{(T-t)}\left[-\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right]_{x_0}^{\infty} \\
&= [S(t)-K]\Phi(-x_0) + S(t)\sigma\sqrt{(T-t)}\left(-\lim_{x\to\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2} + \frac{1}{\sqrt{2\pi}}e^{-x_0^2/2}\right) \\
&= [S(t)-K]\Phi(-x_0)S(t)\sigma\sqrt{(T-t)}\phi(-x_0).
\end{aligned}$$

Remark 5.5.1. Since the density function of a standard normal random variable is an even function, thus $\phi(x) = \phi(-x)$. We know that, the standard random variable has mean zero and variance one and its density function has a bell shape and is symmetric about line $x = 0$.

With the same approach we can find the price of European put option. To summarize, the Bachelier's formulas for European call and put option prices are

$$\begin{aligned}
\pi_C(t) &= [S(t)-K]N(d)+S(t)\sigma\sqrt{(T-t)}\phi(d), \qquad (5.20)\\
\pi_P(t) &= [K-S(t)]N(-d)-S(t)\sigma\sqrt{(T-t)}\phi(-d),\\
d &= \frac{S(t)-K}{S(t)\sigma\sqrt{(T-t)}}.
\end{aligned}$$

Here, the prices can become negative and this is the main difference of Bachelier's formula (the price process is normally distributed) with Black-Scholes-Merton formula where prices cannot be negative (price process is lognormally distributed).

Remark 5.5.2. It is good to remind that Bachelier expressed the prices in integral form and did not calculate the integrals. This is probably due to the market's structure at his time where the option prices (at least in Paris) set to be fixed and the strike prices K were fluctuated. Therefore, the proper price formula for Bachelier should have been the inverse version of the above formulas. See [31] for a deeper discussion.

5.5.2 Bachelier's Implied Volatility

Bachelier considered a case which he called "simple option" and we know it as "at the money option", i.e., $S(t) = K$. That means the term d in Bachelior's formula will be vanished and (5.20) reduced to following simple relation for European call option

$$\pi_C(t) = S(t)\sigma\sqrt{\frac{(T-t)}{2\pi}}.$$

In this manner Bachelier was able to determine the "coefficient of nervousness" of the security which we known as the implied volatility in modern mathematical finance.

As a result, volatility in the Bachelier's model for at the money option can be calculated by

$$\sigma = \frac{\pi_C(t)}{S(t)}\sqrt{\frac{2\pi}{(T-t)}}.$$

Chapter 6

HJM Framework and LIBOR Market Model (LMM)

To describe the evolution of the full term structure of interest rates, we need more advanced models and we follow the derivations and theories in [13]. Further to price interest rate derivatives, we need to construct our forward curve and the reason is that yields and bond prices are just reflections of forward rates up to maturity. To aim this goal, we will go through some traditional approaches in this chapter and we will continue with the current results of constructing forward curve after economic crisis in 2007 and 2008 in the next chapter.

6.1 Heath–Jarrow–Morton (HJM) Framework

In 1990 and 1992 David Heath, Bob Jarrow and Andy Morton (HJM) introduced [14] and [15], i.e., a new framework arguing the no-arbitrage conditions that must be fulfilled by a model of yield curve. For some ultimate maturity τ which is usually 20 or 30 years from now, the HJM model explains the dynamics of forward rate curve $\{f(t,T,\tau), \quad 0 \leq t \leq T \leq \tau\}$. As we mentioned in Section 2.8.4, the forward rate $f(t,T)$ defines the instantaneous continuously compounded rate at time t for risk-free lending and borrowing at time $T \geq t$. The graphical illustration of evaluation of forward curve is given in Figure 6.1. As we can see, at time 0 the forward curve $f(0,.)$ can be defined for maturity times in $[0,\tau]$. In this case, the short rate will be $r(0) = f(0,0)$. Further, for $t > 0$, the forward curve $f(t,.)$ can be defined for maturity times $[t,\tau]$. Thus, the short rate will be $r(t) = f(t,t)$.

The relation between bond price and forward rate were given in (2.7) and (2.8). In the HJM framework [13] the evolution of forward curve satisfies the following SDE

$$df(t,T) = \mu(t,T)dt + \boldsymbol{\sigma}(f,t,T)^{\top} d\boldsymbol{W}(t). \tag{6.1}$$

where

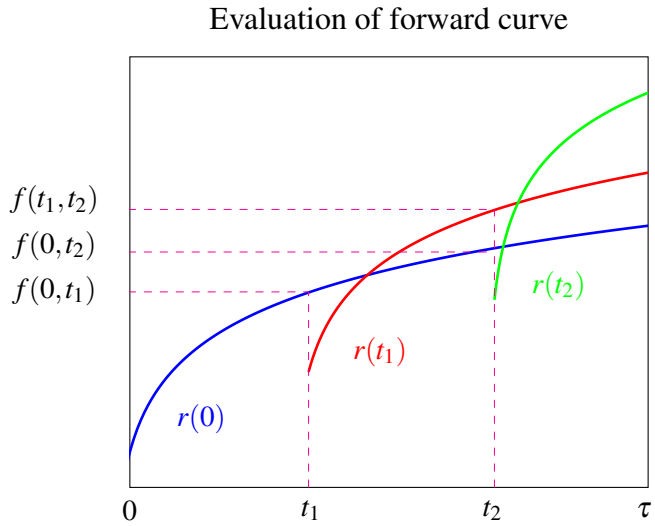

Figure 6.1: Evaluation of forward curve

- W is a standard-d dimensional Brownian motion
- d represents the number of factors
- μ and σ are scalars and $\mathscr{R}^d$-valued and can be either stochastic, or can depend on current and past level of forward rate.

6.1.1 Valuation Under Risk-Neutral Measure

Before talking about the forward measure, let's look at the evolution of HJM model under risk-neutral measure. To make evolution of the forward curve Markovian, we assume that μ and σ are deterministic functions of t, $T \geq t$ and the current forward curve $\{f(t,u), t \leq u \leq \tau\}$. In this manner, we will make sure that W is a standard Brownian motion under risk-neutral measure. As we know, if there is no arbitrage opportunities and in absence of dividend the risk-neutral measure tells us that the dynamics of asset price must be martingale when we it is divided by (money market account) numériare. This was given in (2.3). As Heath, Jarrow and Morton discussed in [15] the forward rates are not asset prices and our account is informal. To solve this technical difficulty and make the discounted bond price, i.e., $v(t,T)/B(t)$ a positive martingale, the HJM model provides us with following dynamics

$$\frac{dv(t,T)}{v(t,T)} = r(t)dt + \boldsymbol{\sigma}_v(f,t,T)^\top d\boldsymbol{W}(t), \qquad 0 \leq t \leq T \leq \tau. \tag{6.2}$$

where, $\boldsymbol{\sigma}_v$ are the bond volatilities and can be functions of current bond prices. Equation (2.7), provides necessary conditions for the evolution of the forward rate in (6.2). Applying the Itô formula we obtain

$$d\ln v(t,T) = \left[r(t) - \frac{1}{2}\boldsymbol{\sigma}_v(f,t,T)^\top \boldsymbol{\sigma}_v(f,t,T)\right] dt + \boldsymbol{\sigma}_v(f,t,T)^\top d\boldsymbol{W}(t).$$

Differentiating with respect to T and then interchange the order of differentiation with respect to t and T, gives (see page 152 in [13])

$$\begin{aligned} df(t,T) &= -\frac{\partial}{\partial T} d\ln v(t,T) \\ &= -\frac{\partial}{\partial T}\left[r(t) - \frac{1}{2}\boldsymbol{\sigma}_v(f,t,T)^\top \boldsymbol{\sigma}_v(f,t,T)\right] dt - \frac{\partial}{\partial T}\boldsymbol{\sigma}_v(f,t,T)^\top d\boldsymbol{W}(t). \end{aligned}$$

comparing the last equation and (6.1) implies that

$$\begin{aligned} \boldsymbol{\sigma}(f,t,T) &= -\frac{\partial}{\partial T}\boldsymbol{\sigma}_v(f,t,T), \\ \mu(t,T) &= -\frac{\partial}{\partial T}\left[r(t) - \frac{1}{2}\boldsymbol{\sigma}_v(f,t,T)^\top \boldsymbol{\sigma}_v(f,t,T)\right] \\ &= \left(\frac{\partial}{\partial T}\boldsymbol{\sigma}_v(f,t,T)\right)^\top \boldsymbol{\sigma}_v(f,t,T). \end{aligned}$$

The next step is to eliminate $\boldsymbol{\sigma}_v(f,t,T)$. Let $\boldsymbol{C} = (C_1,\ldots,C_d)^\top$ be a vector of constant values, then

$$\boldsymbol{\sigma}_v(f,t,T) = -\int_t^T \boldsymbol{\sigma}(f,t,u)du + \boldsymbol{C},$$

remember that as bond matures, i.e., $\lim_{t\to T} v(t,T) = 1$ and $\boldsymbol{\sigma}_v(f,T,T) = 0$. These results make our constant zero, i.e. $\boldsymbol{C} = 0$. Thus, the value for μ is reduced to

$$\mu(t,T) = \boldsymbol{\sigma}(f,t,T)^\top \int_t^T \boldsymbol{\sigma}(f,t,u)du. \tag{6.3}$$

In other words, (6.3) represents the risk-neutral drift in the absence of any arbitrage opportunity. Substituting (6.3) in (6.1) yields

$$df(t,T) = \left(\boldsymbol{\sigma}(f,t,T)^\top \int_t^T \boldsymbol{\sigma}(f,t,u)du\right) dt + \boldsymbol{\sigma}(f,t,T)^\top d\boldsymbol{W}(t), \tag{6.4}$$

which represent the arbitrage-free dynamics of the forward curve using risk-neutral evaluation. Now, we can rewrite (6.4) in following form

$$df(t,T) = \sum_{i=1}^d \left(\sigma_i(f,t,T)\int_t^T \sigma_i(f,t,u)du\right) dt + \sum_{i=1}^d \sigma_i(f,t,T)dW_i(t).$$

As we can see, the drift in (6.4) is the sum of the contributions of every single factors and at the same time each factor has a term contribution to the drift. Further, once the σ is specified then the drift can be determined.

Remark 6.1.1. The advantage of the HJM model comparing with the short rate models is that, if the initial forward curve $f(0,T)$ is chosen consistently with the bond prices given by (2.7), then the HJM model is automatically calibrated to an initial set of bond prices $v(t,T)$. While, the drift and diffusion coefficients in most short rate models are evaluated separately.

Special Case; constant σ and $d = 1$

As an example which is taken from page 153 in [13] , let's assume that $\sigma(f,t,T) \equiv \sigma$, then each increment $dW(t)$ will move the entire forward curve, i.e., $\{f(t,u),\ t \leq u \leq \tau\}$ by equal amount (parallel shifts caused by diffusion coefficient, i.e., $\sigma dW(t)$). This phenomena can be seen as an arbitrage opportunity. To see it better, we calculate the drift coefficient

$$\mu(t,T) = \sigma \int_t^T \sigma du = \sigma^2(T-t).$$

As we can see the drift coefficient varies by really small amount, which makes the parallel shifts almost impossible. Thus, the model can still be seen as an arbitrage free model. So, we can find a solution to (6.4)

$$\begin{aligned} f(t,T) &= f(0,T) + \int_0^t \sigma^2(T-u)du + \sigma W(t) \\ &= f(0,T) + \frac{1}{2}\sigma^2[T^2 - (T-t)^2] + \sigma W(t). \end{aligned}$$

and $r(t) = f(t,t)$ gives

$$dr(t) = df(t,T)\Big|_{T=t} + \frac{\partial}{\partial T} f(t,T)\Big|_{T=t} dt.$$

and since σ is constant in our case, we get

$$dr(t) = \left(\frac{\partial}{\partial T} f(0,T)\Big|_{T=t} + \sigma^2 t \right) dt + \sigma dW(t). \tag{6.5}$$

□

Remark 6.1.2.

- As we can see (6.5) is identical to the result of the Ho–Lee model in Section 4.2.1.
- A problem with risk-neutral measure is that, none of the forward rates becomes martingale.

6.1.2 Valuation Under Forward Measure

As we mentioned in the introduction of this chapter, volatility structure in future can be significantly different from what we can see in the today's market and therefore we should use the forward measure in our valuation. Further, in contrary with evaluation of the HJM model under risk-neutral measure, the evaluation of the HJM model under forward measure will give the forward rates martingale. The good news is that to change our measurement, we have to just fix the maturity which we will denote by T_F. Doing so, bond $v(t,T_F)$ becomes a numériare asset. Then, the forward probability measure, i.e., $\mathbb{Q}^{T_F}$ is defined relative to the risk-neutral

probability measure Q^B (where B stands for money market account $B(t)$ and given by (2.3)) in the following form

$$\left(\frac{d\mathsf{P}^{T_F}}{d\mathsf{P}^B}\right)_t = \frac{v(t,T_F)B(0)}{B(t)v(0,T_F)},$$

to refresh our mind, we know that the bond dynamics was given by (6.2). Thus, the equation above can be rewritten as

$$\left(\frac{d\mathsf{P}^{T_F}}{d\mathsf{P}^B}\right)_t = \exp\left\{-\frac{1}{2}\int_0^t \boldsymbol{\sigma}_v(f,u,T_F)^\top \boldsymbol{\sigma}_v(f,u,T_F)du + \int_0^t \boldsymbol{\sigma}_v(f,u,T_F)^\top d\boldsymbol{W}(u)\right\}.$$

To make our process a standard Brownian motion under forward measure, Q^{T_F}, we use Girsanov theorem (Theorem 3.5.2) and define the process W^{T_F} by

$$d\boldsymbol{W}^{T_F}(t) = -\boldsymbol{\sigma}_v(f,u,T_F)^\top dt + d\boldsymbol{W}(t).$$

Recall that, $\boldsymbol{\sigma}_v(f,t,T) = -\int_t^T \boldsymbol{\sigma}(f,t,u)du$, the forward rate dynamics in (6.4) transforms to

$$\begin{aligned} df(t,T) &= -\boldsymbol{\sigma}(f,t,T)^\top \boldsymbol{\sigma}_v(f,t,T)dt + \boldsymbol{\sigma}(f,t,T)^\top \left[\boldsymbol{\sigma}_v(f,t,T_F)^\top dt + d\boldsymbol{W}^{T_F}(t)\right] \\ &= -\boldsymbol{\sigma}(f,t,T)^\top \left[\boldsymbol{\sigma}_v(f,t,T) - \boldsymbol{\sigma}_v(f,t,T_F)\right] dt + \boldsymbol{\sigma}(f,t,T_F)^\top d\boldsymbol{W}^{T_F}(t) \\ &= -\boldsymbol{\sigma}(f,t,T)^\top \left(\int_T^{T_F} \boldsymbol{\sigma}(f,t,u)du\right) dt + \boldsymbol{\sigma}(t,T)^\top d\boldsymbol{W}^{T_F}(t), \quad t \le T \le T_F. \end{aligned} \tag{6.6}$$

Remark 6.1.3. Once again, remember that $f(t,T_F)$ is a martingale under Q^{F_T}.

6.1.3 The Discrete Drift (Risk-Neutral Measure)

To get a feasible simulation of the HJM forward rate dynamics, discrete approximation is essential. To do so, we need to discretisize both arguments of $f(t,T)$. To simplify our notations, we assume that the two set of dates are the same. Then, we will have our time grid in the form $0 = t_0 < t_1 < \cdots < t_M$. As usual, we denote our discretized variables with hat. Doing so, we will introduce $\hat{f}(t_i,t_j)$ as the forward rate and $\hat{v}(t_i,t_j)$ as the corresponding bond price with maturity t_j for time t_i, where $i \le j$. Then, we obtain a disretized version of (2.8) in the following form (see Chapter 3.6.2 in [13] for the derivations)

$$\hat{v}(t_i,t_j) = \exp\left\{-\sum_{l=i}^{j-1} \hat{f}(t_i,t_l)[t_{l+1}-t_l]\right\}. \tag{6.7}$$

One way to avoid any unnecessary discretization errors, is to set $\hat{v}(0,t_j) = v(0,t_j)$ for all maturities t_j on our discrete grid. Using (2.7), we see this hold if

$$\sum_{l=0}^{j-1} \hat{f}(0,t_l)[t_{l+1}-t_l] = \int_0^{t_j} f(0,u)du;$$

which equivalently means

$$\hat{f}(0,t_l) = \frac{1}{t_{l+1}-t_l}\int_{t_l}^{t_{l+1}} f(0,u)du, \qquad \text{for all} \qquad l = 0,1,\ldots,M-1. \tag{6.8}$$

The last equation suggests that over the interval $[t_l, t_{l+1}]$ one needs to initialize each $\hat{f}(0,t_l)$ to the average level of the forward curve $f(0,T)$. In other words, initializing the value of $f(0,t_l)$ at the right point or left point of this interval might cause undesirable errors. The graphical explanation of (6.8) is depicted on Figure 6.2. As soon as we specify our initial

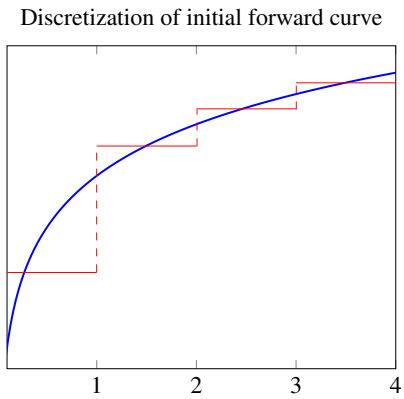

Figure 6.2: Each discretized forward rate is the average of the underlying forward curve over the discretized time interval.

forward curve, we can simulate our **single-factor model** [13] forward rates for $i = 1,2,\ldots,M$, as follows

$$\begin{aligned}\hat{f}(t_i,t_j) &= \hat{f}(t_{i-1},t_j) + \hat{\mu}(t_{i-1},t_j)[t_i - t_{i-1}] \\ &\quad + \hat{\sigma}(\hat{f},t_{i-1},t_j)\sqrt{t_{i-1}-t_j}Z_i, \qquad j = i,\ldots,M,\end{aligned} \tag{6.9}$$

- $Z_1,\ldots,Z_M$ are normal $N(0,1)$ independent and identically distributed (IID) random variables,
- $\hat{\mu}$ and $\hat{\sigma}$ are discrete components of the continuous time coefficients in (6.4).

Remark 6.1.4.

- In reality, calibration procedure might be used to specify $\hat{\sigma}$ in a way that it will be consistent with the market prices of high volume and actively traded derivative securities,
- $\hat{\sigma}$ is not usually specified explicitly in our simulations,
- We choose the discrete drift $\hat{\mu}$ in (6.9) in such a way that its continuous time limit converges to (6.3). This will guarantee the martingale, i.e., arbitrage-free, property for the discounted bond prices. See page 156–157 in [13].

One-Dimensional Drift Coefficient

$$\hat{\mu}(t_{i-1},t_j)[t_{j+1}-t_j] = \frac{1}{2}\left[\left(\sum_{l=i}^{j}\hat{\sigma}(\hat{f},t_{i-1},t_l)[t_{l+1}-t_l]\right)^2 - \left(\sum_{l=i}^{j-1}\hat{\sigma}(\hat{f},t_{i-1},t_l)[t_{l+1}-t_l]\right)^2\right]. \tag{6.10}$$

Corollary 6.1.1. *The limit of expression in (6.10) is equivalent to (6.3) with $d = 1$.*

Proof. Define, $t_i = ih,\ \ h > 0$. Set a time t and fix a maturity time T. Let $i, j \to \infty,\ \ h \to 0$ such that $jh = T,\ \ ih = t$. Then divide both hand side of (6.10) by $t_{j+1} - t_j = h$. Finally, for small value of h, we get following (see page 158 in [13])

$$\lim_{h\to 0}\frac{1}{2h}\left[\left(\int_t^T \sigma(f,t,u)du\right)^2 - \left(\int_t^{T-h}\sigma(f,t,u)du\right)^2\right],$$

let's denote the expressions in the first parentheses by $f(x+h)$ and second parentheses by $f(x)$. Then we have

$$\lim_{h\to 0}\frac{1}{2h}\left([f(x+h)]^2 - [f(x)]^2\right) = \lim_{h\to 0}\frac{1}{2h}\Big([f(x+h)-f(x)]\,[f(x+h)+f(x)]\Big)$$
$$= \frac{1}{2}\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)2f(x) = f'(x)f(x) = \sigma(f,t,T)\int_t^T \sigma(f,t,u)du,$$

in the last step, we used the fundamental theorem of calculus, i.e., $\frac{d}{dT}\int_t^T g(u)du = g(T)$. □

Multi-Dimensional Drift Coefficient

Let the number of factors (dimensions) be the number d. Then the combined drift is

$$\hat{\mu}(t_{i-1},t_j) = \sum_{k=1}^{d}\hat{\mu}_k(t_{i-1},t_j), \qquad k = 1,2,\ldots,d.$$

and

$$\hat{\mu}_k(t_{i-1},t_j)[t_{j+1}-t_j] = \frac{1}{2}\left[\left(\sum_{l=i}^{j}\hat{\sigma}_k(\hat{f},t_{i-1},t_l)[t_{l+1}-t_l]\right)^2 - \left(\sum_{l=i}^{j-1}\hat{\sigma}_k(\hat{f},t_{i-1},t_l)[t_{l+1}-t_l]\right)^2\right].$$

where $\hat{\sigma}_k$ represents the kth entry of vector $\hat{\boldsymbol{\sigma}}$. Finally, the **multi-factor model** forward rates has following form

$$\hat{f}(t_i,t_j) = \hat{f}(t_{i-1},t_j) + \hat{\mu}(t_{i-1},t_j)[t_i - t_{i-1}] + \sum_{k=1}^{d}\hat{\sigma}(\hat{f},t_{i-1},t_j)\sqrt{t_i - t_{i-1}}Z_{ik}, \quad j = i,\ldots,M, \tag{6.11}$$

and $\boldsymbol{Z}_i = (Z_{i1}, \ldots, Z_{id}), \quad i = 1, \ldots, M$ are IID standard normal random vectors. See page 158–159 in [13].

6.1.4 The Discrete Drift (Forward Measure)

Under forward measure evaluation, the HJM model for maturity time T_F is a martingale if $v(t,T)/v(t,T_F)$ is a martingale. That is, the bond matures at T_F is the numériare asset with forward measurement. In discrete working scale, let $t_M = T_F$, then for all i and for each j, $\hat{v}(t_i,t_j)/\hat{v}(t_i,T_M)$ should be martingale, i.e., (see 160 in[13])

$$\frac{\hat{v}(t_i,t_j)}{\hat{v}(t_i,T_M)} = \exp\left\{\sum_{l=i}^{M-1} \hat{f}(t_i,t_l)[t_{l+1}-t_l]\right\}.$$

which will introduce the discrete drift coefficient under forward measure as

$$\hat{\mu}(t_{i-1},t_j)[t_{j+1}-t_j] = \frac{1}{2}\left[\left(\sum_{l=j+1}^{M-1} \hat{\sigma}(\hat{f},t_{i-1},t_l)[t_{l+1}-t_l]\right)^2 - \left(\sum_{l=j}^{M-1} \hat{\sigma}(\hat{f},t_{i-1},t_l)[t_{l+1}-t_l]\right)^2\right]. \quad (6.12)$$

6.1.5 Implementation

A proper start for implementation of HJM model is by introducing the notation. To begin with, $\hat{f}(t_i,t_j)$ suggests that we are dealing with a, $M \times M$ matrix, but in fact we need to keep track of a vector of current rates. Table 6.1 illustrates our notations.

Variable	Stands for
t_i	current time, $i = 1, \ldots, M$
t_j	the maturity of current forward rate (relative maturity)
t_M	longest bond's maturity
f_j	forward rate $\hat{f}(t_i,t_{i+j-1})$ at time t_i
m_j	drift coefficient $\hat{\mu}(t_i,t_j)$ at time t_i
$s_j(k)$	diffusion coefficient $\hat{\sigma}_k(\hat{f}_j,t_i,t_j), \quad k = 1, \ldots, d$ at time t_i
k	$k = 1, \ldots, d$ refers the factor index in a d-factor model
$h_i = t_i - t_{i-1}$	vector$(h_1, \ldots, h_M)$ represents the time intervals between t_is at all the steps
D	the simulated value of discount factor
C	cumulative discounted cashflows from an interest rate derivative
P	payoff, i.e., cashflows at t_i (depending on instrument)

Table 6.1: Table Of Variables in HJM Simulation

Remark 6.1.5.

- Time and maturity grid is a set of dates $0 = t_0 < t_1 < \cdots < t_M$,
- The last forward rate will be subjected to the time interval $[t_{M-1}, t_M]$
- Initial vector of forward rates contains the M components $\hat{f}(0,0), \ldots, \hat{f}(0, t_{M-1})$, which is identical to (6.8) and will be denoted by $(f_1, \ldots, f_M)$. Further, we will use one instead of zero as the smallest index value,
- $\hat{r}(t_i) = \hat{f}(t_i, t_i)$, thus f_1 always refers to the current level of short rate.

So, the simulation step from t_{i-1} to t_i corresponding to (6.11) is given by

$$f_j \leftarrow f_{j+1} + m_j(t_i - t_{i-1}) + \sum_{k=1}^{d} s_j(k)\sqrt{t_i - t_{i-1}} Z_{ik}, \quad j = i, \ldots, M-i.$$

Therefore, in advancing from t_{i-1} to t_i we want

$$m_j = \hat{\mu}(t_{i-1}, t_{i+j-1}), \quad s_j(k) = \hat{\sigma}_k(\hat{f}, t_{i-1}, t_{i+j-1}) \tag{6.13}$$

But, to be able to simulate f_js, we need to simulate the values of $s_j(k)$ from each t_{i-1} to t_i in advance. Because, $\hat{\sigma}$ can depend on the current vector of forward rates. To make our procedure easy, we make two algorithms. The first algorithm calculates the discrete drift parameter at a fixed time step. The second algorithm is looping over time steps and it updates the forward curve at each steps.

An Algorithm to Calculate the Discrete Drift

Define

$$\hat{\mu}(t_{i-1}, t_j) = \frac{1}{2h_j}[B_{next} - B_{prev}], \quad A_{next}(k) = \sum_{l=i}^{j} \hat{\sigma}_k(\hat{f}_i, t_{i-1}, t_l) h_{l+1},$$

where

$$B_{next} = \sum_{k=1}^{d} \left(\sum_{l=i}^{j} \hat{\sigma}_k(\hat{f}_i, t_{i-1}, t_l) h_{l+1} \right)^2, \quad B_{prev} = \sum_{k=1}^{d} \left(\sum_{l=i}^{j-1} \hat{\sigma}_k(\hat{f}_i, t_{i-1}, t_l) h_{l+1} \right)^2.$$

The algorithm to calculate the discrete drift parameters $m_j = \hat{\mu}(t_{i-1}, t_{i+j-1})$ for each transition t_{i-1} to t_i is illustrated in Figure 6.3.

Figure 6.3 clearly reminds us that in our procedure to evaluate the forward curve from t_{i-1} to t_i, we should firstly evaluate the $s_j(k)$ and m_j by using the forward rates at step $i-1$. Secondly we can update the rates to obtain their corresponding values at step i.

```
inputs: j = 1,...,M − i, k = 1,...,d and h_1,...,h_M (h_l = t_l − t_{l−1})
calculate s_j(k) = σ̂_k(f̂, t_{i−1}, t_{i+j−1})
A_prev(k) = 0
for j = 0 to M − i do
    B_next = 0
    for k = 0 to d do
        A_next(k) = A_prev(k) + s_j(k) × h_{i+j}
        B_next = B_next + A_next(k) × A_next(k)
        A_prev(k) = A_next(k)
    end for
    m_j = (B_next − B_prev)/(2h_{i+j})
   B_prev = B_next
end for
return m_1,...,m_{M−i}.
```

Figure 6.3: An algorithm to calculate the discrete drift parameters

Remark 6.1.6.

- In practice, a single-factor model, i.e., $\hat{\sigma}(t_i,t_j) = \tilde{\sigma}(i,j)\hat{f}(t_i,t_j)$, ($\tilde{\sigma}$ is empirical volatility), can be and usually is used. It might be necessary to stress that this can be used under condition that the increments h_i are bounded away from zero,
- The evaluation of $\tilde{\sigma}(i-1,j)\hat{f}(t_{i-1},t_j)$ must be done firstly and then we can update forward rates $\hat{f}(t_{i-1},t_j)$ to $\hat{f}(t_i,t_j)$.

An Algorithm to Update Forward Curves

To simulate the evaluation of forward curve over time $t_0,t_1,\ldots.t_{M-1}$ and to calculate cumulative discounted cashflows from a desirable interest rate derivative we can use the algorithm which is illustrated in Figure 6.4. Now, we are able to value interest rate derivatives. Algorithm in Figure 6.4 enables us to evaluate the forward curve, calculate the payoff and discount the payoff of a financial security or derivative. Moreover, the discount factor used in Figure 6.4 is given by

$$\hat{D}(t_j) = \exp\left\{-\sum_{i=0}^{j-1}\hat{r}(t_i)[t_{i+1}-t_i]\right\},$$

where

$$\hat{r}(t_0) = \hat{f}(t_0,t_0), \quad \hat{r}(t_1) = \hat{f}(t_1,t_1), \quad \ldots \quad , \quad \hat{r}(t_M) = \hat{f}(t_M,t_M).$$

```
inputs: (f_1,...,f_M) ,i.e., initial curve and (h_1,...,h_M), i.e., intervals
D = 1, P = 0, C = 0
for i = 0 to M − i do
    D = D × exp(−f_1 × h_i)
    s_j(k) = σ̂_k(f̂, t_{i−1}, t_{i+j−1}), j = 1,...,M − i, k = 1,...,d
    call m_1,...,m_{M−i} from Figure 6.3
    generate Z_1,...,Z_d ~ N(0,1)
   for j = 1 to M − i do
      Sum = 0
      for k = 1 to d do
          Sum = Sum + s_j(k) × Z_k
          f_j = f_{j+1} + m_j × h_i + Sum × √h_i
       end for
    end for
    P = cashflow at t_i (depending on instrument)
    C = C + D × P
end for
return C.
```

Figure 6.4: An algorithm to simulate the evaluation of forward curve

6.2 LIBOR Market Model (LMM)

We have discussed the evaluation of an interest derivatives using the forward LIBOR in Section 5.3. Using forward LIBOR has three advantages. Firstly, LIBOR market models are based on observable market data and rates. Secondly, the LMM model is capable of admitting deterministic volatilities where as HJM framework does not have such a capability. Finally, the LMM model can be used to price caps and we know that collar is a mixture of cap and floor. Now, let's see how we can evaluate the forward LIBOR curve in the LMM framework. To do so, we will try to use the same approach and algorithm as we used in the HJM framework. We start with introducing our notations. Define

- $0 \leq T_0 < \cdots < T_M < T_{M+1}$: finite set of maturity or tenor dates
- $\delta_i = T_{i+1} - T_i, \quad i = 0, \ldots, M$: length of interval between tenor dates

then using our notations, we can rewrite (5.10) in the following form

$$L_n(t) = \frac{v_n(t) - v_{n+1}(t)}{\delta_n v_{n+1}(t)}, \quad 0 \leq t \leq T_n, \quad n = 0, 1, \ldots, M.$$

As it is depicted in Figure 6.5, each $L_n(t)$ corresponds to the forward rate for the time interval $[T_n, T_{n+1}]$, where $t \leq T_n$. Since, the LIBOR rates do not specify the discount factor in time interval T_{i+1} to time t, we define a new function of time η. Let $\eta : [0, T_{M+1}) \to \{1, \ldots, M+1\}$.

Take $\eta(t)$ as a unique integer satisfying $T_{\eta(t)-1} \leq t < T_{\eta(t)}$. Therefore, $\eta(t)$ modifies the index of the next tenor date at time t. There are two main major ways to evaluate of the forward-LIBOR rates. One is the spot measure and the other is the forward-measure which discussed it before. See Chapter 3.7.1 in [13]. Let's go through spot rate briefly.

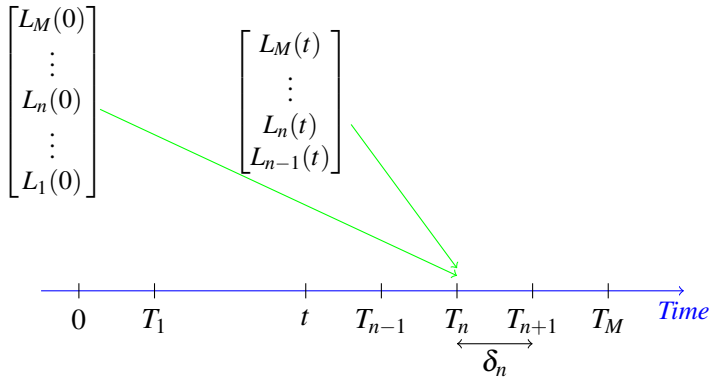

Figure 6.5: Evaluation of vector of forward rates

6.2.1 Spot Measure

Suppose the evaluation of the forward-LIBOR rate satisfies following SDE

$$\frac{dL_n(t)}{L_n(t)} = \mu_n(t)dt + \boldsymbol{\sigma}_n(t)^\top d\boldsymbol{W}(t), \quad 0 \leq t \leq T_n, \quad n = 1, \ldots, M,$$

where

- $\{W(t)\}$ is a d-dimensional standard Brownian motion process,
- μ_n, $\boldsymbol{\sigma}_n$ are drift and diffusion coefficient and can be dependent on time t and the vector of LIBOR rates, i.e., $(L_1(t), \ldots, L_M(t))$.

We do not say risk-neutral, but spot measure since, in this approach, the bond price is martingale when it is deflated (rather discounted) by the numériare asset. In other words, the SDE becomes martingale when we do not discount bond price by the continuously compounded rate, but we divide it by the numériare asset (see page 169 in [13]). After doing some algebra we obtain

$$\frac{dL_n(t)}{L_n(t)} = \sum_{j=\eta(t)}^{n} \frac{\delta_j L_j(t) \boldsymbol{\sigma}_n(t)^\top \boldsymbol{\sigma}_j(t)}{1 + \delta_j L_j(t)} dt + \boldsymbol{\sigma}_n(t)^\top d\boldsymbol{W}(t), \quad 0 \leq t \leq T_n, \quad n = 1, \ldots, M.$$

in fact we have evaluated the value of the drift term μ_n.

6.2.2 Implementation

Simulation in the HJM model requires discretization in both time and maturity while, in the LMM model we need only to discretize time argument. Let the simulation take place over the time grid $0 = t_0, \cdots < t_m < t_{m+1}$. Moreover, the tenor dates be $T_1, \ldots, T_{M+1}$. Then we introduce following simulation procedures.

Euler Scheme under Spot Measure

$$\hat{L}_n(t_{i+1}) = \hat{L}_n(t_i) + \mu_n(\hat{L}(t_i), t_i)\hat{L}_n(t_i)[t_{i+1} - t_i] + \hat{L}_n(t_i)\sqrt{t_{i+1} - t_i}\boldsymbol{\sigma}_n(t_i)^\top \boldsymbol{Z}_{i+1}, \tag{6.14}$$

where

$$\mu_n(\hat{L}(t_i), t_i) = \sum_{j=\eta(t_i)}^{n} \frac{\delta_j \hat{L}_j(t)\boldsymbol{\sigma}_n(t_i)^\top \boldsymbol{\sigma}_j(t_i)}{1 + \delta_j \hat{L}_j(t_i)}, \tag{6.15}$$

and

$$\hat{L}_n = \frac{v_n(0) - v_{n+1}(0)}{\delta_n v_{n+1}(0)}, \quad n = 1, \ldots, M.$$

This scheme may produce negative rates of $\hat{L}$.

Applying Logarithm of LIBOR Rate to Euler Scheme

$$\hat{L}_n(t_{i+1}) = \hat{L}_n(t_i) \exp\left\{ \left[\mu_n(\hat{L}(t_i), t_i) - \frac{1}{2}\|\boldsymbol{\sigma}_n(t_i)\|^2 \right] [t_{i+1} - t_i] + \sqrt{t_{i+1} - t_i}\boldsymbol{\sigma}_n(t_i)^\top \boldsymbol{Z}_{i+1} \right\} \tag{6.16}$$

where μ_n is given by (6.15). Choosing deterministic $\boldsymbol{\sigma}_n$ causes L_n to be close to lognormal and makes this approach becomes more attractive. This approach guarantee that all $\hat{L}_n > 0$.

Forward Measure

The value of $\hat{L}_n(t_{i+1})$ is given by (6.14) and

$$\mu_n(\hat{L}(t_i), t_i) = -\sum_{j=n+1}^{M} \frac{\delta_j \hat{L}_j(t_i)\sigma_n(t_i)^\top \sigma_j(t_i)}{1 + \delta_j \hat{L}_j(t_i)}.$$

Remark 6.2.1.

- In the algorithms above η is right continuous function in time,
- In the implementation of the LMM, no restriction is need to be made in choosing σ_n, however deterministic σ_n is usually used,
- Andreasen and Andersen [1] introduced σ_n as a product of a function of $L_n(t)$ and product of a deterministic function of time. This model has been very attractive in the market,
- The descretized process $\{L(t)\}$ is still martingale. See page 168–172 in [13].

6.2.3 Volatility Structure and Calibration

In our description for the LIBOR market model, we have not specified how one should specify σ_n. In practice, the market prices of actively traded derivatives (in particular caps and swaps) can be used to specify the coefficient of σ_n. In such a manner our model is said to be calibrated to market data. The components of this matrix contain the volatility of forward rates and correlation between them. Let's see some scenarios[1].

Inverting Black's Formula

Assume the market price of a caplet for some time interval $[T_n, T_{n+1}]$ is given. Suppose that all the other parameters of Black's formula are known, except $\boldsymbol{\sigma}_n$. Now, we can calculate an implied volatility (ϑ_n) by inverting Black's formula. Let $\boldsymbol{\sigma}_n$ be a deterministic $\mathscr{R}^d$-valued function which is satisfying

$$\vartheta_n^2 = \frac{1}{T_n}\int_0^{T_n} \|\boldsymbol{\sigma}_n(t)\|^2 dt$$

Applying these constraints on all σ_js, we make sure that the model is calibrated to all caplet prices. See Chapter 3.7.4 in [13].

One-Factor Model, i.e., $d = 1$

We know that, the LMM model specify the interest rates over tenor basis. Hence, we focus only on the functions $\boldsymbol{\sigma}_n(t)$ which are constants between tenor dates. Now, suppose that the model is one dimensional and thus each $\boldsymbol{\sigma}_n$ is a scalar. Then we will have [13]

$$\boldsymbol{\sigma}_n = \begin{pmatrix} \sigma_1(T_0) & & & \\ \sigma_2(T_0) & \sigma_2(T_1) & & \\ \vdots & \vdots & \ddots & \\ \sigma_M(T_0) & \sigma_M(T_1) & \dots & \sigma_M(T_{M-1}) \end{pmatrix}$$

[1] Similar approach applies in the HJM model, however it is slightly more complicated.

A caplet price, would impose a constraint along each row of matrix above, i.e.,

$$\int_0^{T_n} \sigma_n^2(t)dt = \sigma_n^2(T_0)\delta_0 + \sigma_n^2(T_1)\delta_1 + \cdots + \sigma_n^2(T_{n-1})\delta_{n-1}, \quad n = 1, \ldots, M.$$

Stationary Volatility Structure and $d = 1$

If $\boldsymbol{\sigma}_n(t)$ depends on n and t just through the difference between $T_n - t$, then our volatility structure is stationary. We can form our matrix for a single factor, stationary and piecewise constant volatility structure in the following form

$$\boldsymbol{\sigma}_n = \begin{pmatrix} \sigma(1) & & & \\ \sigma(2) & \sigma(1) & & \\ \vdots & \vdots & \ddots & \\ \sigma(M) & \sigma(M-1) & \ldots & \sigma(1) \end{pmatrix}$$

Multi-Factor Model, i.e., $d \geq 2$

In this approach, the $\boldsymbol{\sigma}_n(T_i)$s are vectors. So, we replace each element of matrix by its norm, i.e., $\|\boldsymbol{\sigma}_n(T_i)\|$. So, we have

$$\boldsymbol{\sigma}_n = \begin{pmatrix} \|\boldsymbol{\sigma}_1(T_0)\| & & & \\ \|\boldsymbol{\sigma}_2(T_0)\| & \|\boldsymbol{\sigma}_2(T_1)\| & & \\ \vdots & \vdots & \ddots & \\ \|\boldsymbol{\sigma}_M(T_0)\| & \|\boldsymbol{\sigma}_M(T_1)\| & \ldots & \|\boldsymbol{\sigma}_M(T_{M-1})\| \end{pmatrix}$$

and with constant piecewise values, we obtain

$$\int_0^{T_n} \|\boldsymbol{\sigma}_n(t)\|^2 dt = \|\boldsymbol{\sigma}_n(T_0)\|^2\delta_0 + \|\boldsymbol{\sigma}_n(T_1)\|^2\delta_1 + \cdots + \|\boldsymbol{\sigma}_n(T_{n-1})\|^2\delta_{n-1}, \quad n = 1, \ldots, M.$$

which causes the caplet implied volatilities to constraint the sums of squares in each row of our matrix. An important property of muti-factor model is its capability to analyze the correlations between forward rates of different maturities. As an example, the correlation between the increment $\ln L_j(t)$ and $\ln L_k(t)$ in (6.16) over a short time interval can be approximated by (see page 183 in [13])

$$\rho_{L_j,L_k} = \frac{\boldsymbol{\sigma}_k(t)^\top \boldsymbol{\sigma}_j(t)}{\|\boldsymbol{\sigma}_k(t)\|\|\boldsymbol{\sigma}_j(t)\|}.$$

These correlations have practical usages in the evaluation of swap prices. They can be used to match the market price of a swap or to match the empirical correlations.

Chapter 7

A New Framework Under Collateral Agreement (CSA)

In this chapter we are following the ideas in [9], [12] and [29]. To begin with, the financial crisis in 2007 and 2008 made considerable changes in the market. The dramatic increases in the basis swap spreads caused a divergence from implied rates and actual traded interest rates in the markets. As the result, lots of financial institutions started to use collateral agreements. By end of 2009, IDSA (International Swaps and Derivatives Association) reported that around 70% of the trade volumes for the OTC derivatives were collateralized. Consequently, after increasing the number of collateral agreements in the market, lots of financial institutions started discounting at O/N (Over Night) rates regulated by the CSA (Credit Support Annex). In other words, collateralization introduces a new market standards.

7.1 Unsecured versus Collateralized (Secured) Trade

Let's describe the unsecured trade in an old style implementation with an example. Consider a 3-month floating rate versus a 6-month floating rate plain vanilla swap. This scenario, is depicted in Figure 7.1. The green arrows, i.e, spread (waves) used to be considered zero (or close to zero), but from 2007 the spread have been taken in consideration as it represents the difference in risk levels and its value can be quite significant. See page in 481 [29].

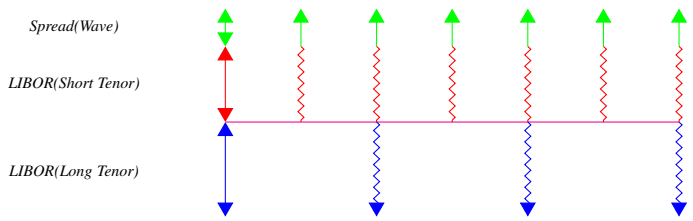

Figure 7.1: A 3-month floating against a 6-month floating rate

Now, let's see how an unsecured trade looks like. The main feature of unsecured trade is its external funding which commonly financed by a loan. Once again assume counterparties Red and Blue are going to make a trade. Consider following case. The Red firm purchase a derivative from the Blue firm where the value of the derivative is adjusted by the option payment. The Red firm is taking a loan to finance its trade and promise to pay a LIBOR rate as the interest rate of the loan it has taken. But, using the LIBOR as the discounting factor makes the present value of the loan equal to zero and thus the LIBOR is unproperly discounted and it is unsecured offer rate in the interbank market. This scenario is illustrated in Figure 7.2.

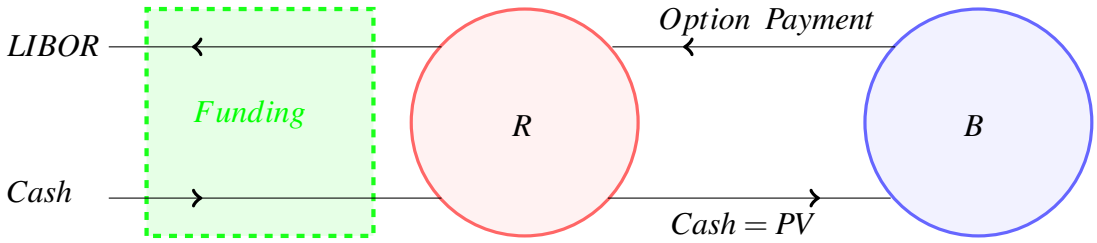

Figure 7.2: Unsecured trade with external funding.

Snapshot Lehman Brothers Bankruptcy

On 15th of September, 2008, Lehman Brothers recorded the biggest bankruptcy in the US history through failing to fulfill the bankruptcy protection. The main reasons for this disaster were, liquidity problems (lenders to Lehman refused to roll over their funding), high leverage (with ratio 31:1) and risky investments (large positions in mortgage derivatives). Lehman took part in OTC derivatives market with around 8,000 different counterparties and in most cases, these counterparties demanded collateral. Sorting out these contracts were not an easy task.
The Lehman bankruptcy created a systemic risk and a systemic risk can cause the other counterparties go to the default. The reason is due to the huge loss through the large number of OTC transactions between counterparties. To avoid this systemic risk, governments bailed out lost of financial institutions before they failed in 2008.
Although during 2007 and 2008 the number of trades decreased dramatically, the CDSs (Credit Default Swaps) were actively traded with high cost of protection. The volume of CDS for a company can be bigger than its debt. In Lehman case, this bank had about **$400 billions of CDS contracts** and $155 billion of Lehman Brothers dept was outstanding. A later investigation showed that the payout to the buyers of CDS's protection was 91.375% of principle.
As the result of financial crisis in 2007 and 2008 monitoring both liquidity and capital adequacy are applied in supervisions of banks [18].

Figure 7.3: Business Snapshot (Lehman Brothers Bankruptcy)

Due to new risks in the markets (see an example in Figure 7.3), major banks started demanding collateral to reduce the existing risks in the market and secure their trades. In other words, the

banks started evaluating the price of the derivatives considering the collateral agreement. In this framework, there is no need to an external funding or loan and the PV (present value) is equal to collateral. As we mentioned before, collar is a mixture of cap and floor. So, it depends to the direction of changes in PV, the Red firm pays/gets collateral to/from the Blue firm. Moreover, these transactions take place once a day. The holder of collateral pays interest rate at collar rate (indicated in the collateral agreement) to other firm and this interest rate is evaluated (discounted) at collateral rate. The situation is depicted in Figure 7.4. As we can see

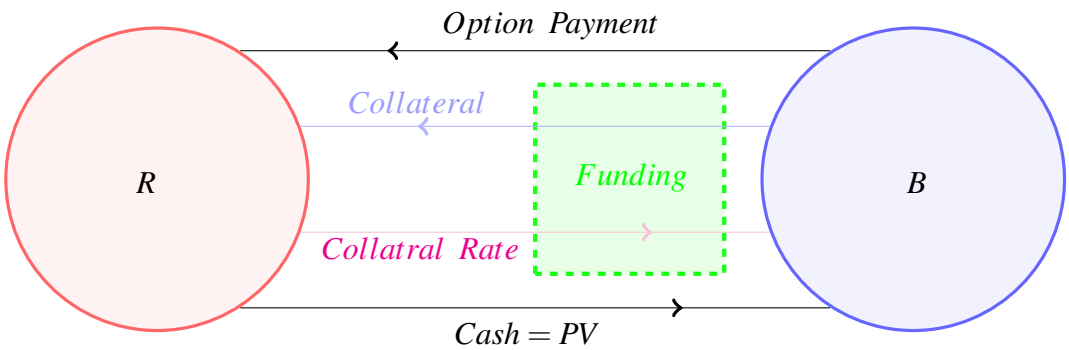

Figure 7.4: Secured trade with external funding.

in Figure 7.4, the funding in a secured trade is provided by the collateral agreement. As we said, the transactions between counterparties are taking place once a day and thus a LIBOR discounting is of course inappropriate. The most common collateral rate is O/N rate and to bootstrap the discount curve we are suggested to use OIS (Overnight Indexed Swap). A typical example of a collateral agreement (CSA) is depicted in Figure 7.5.

Base Currency	USD
Eligible Currency	USD, EUR, GBP
Independent Amount	5 Million
Haircuts	[Schedule]
Threshold	50 Million
Minimum Transfer Amount	500,000
Rounding	Nearest 100,000 USD
Valuation Agent	Red Firm
Valuation Date	Daily, New York Business Day
Notification Time	2:00 PM, New York Business Day
Interest Rate	OIS, EONIA, SONIA
Day Count	Act/360

Figure 7.5: Data in a collateral agreement.

7.2 Pricing a Derivative under Collateral Agreement

In this part, we will try to price a derivative under collateral agreement. To begin with, in today's market lots of counter parties go to trade with different currencies, thus we must take the exchange rates in our consideration. Discounting under CSA is given by the collateral rate specified in the contracts. This can be cash collateral in different currencies or some other rate. Many times the collateral rate is the O/N rate (OIS-rate). But in Sweden, this can also be the repo rate set by the central bank, Riksbanken. Before going any further, let's introduce the following assumptions

- In our procedure, we will deal with full collateralization (zero threshold) by cash.
- The collateral is supposed to be adjusted continuously with zero minimum transfer amount (MTA).

Let's denote the domestic currency with d and foreign currency with f. Now, we consider following.

7.2.1 Risk-Neutral Measure

In this part, we evaluate the price of a derivative under risk-neutral probability measure Q. To begin with, let us introduce some notations

- d stands for domestic currency,
- f stands for foreign currency,
- X is an obtainable contingent claim,
- $h(X(T))$ is the payoff function to contingent claim X. For simplicity we will omit writing X, i.e., $h(T)$,
- c stands for collateral rate which is specified in the collateral agreement CSA,
- r interest rate,
- R instantaneous return (cost if it becomes negative),
- λ is the exchange rate between currencies,
- π denotes price/present value.

Then, the price of an obtainable contingent claim X whose payoff is $h(X)$ is given by (3.21). Hence, for a single currency (say domestic) we have following price

$$\begin{aligned}\pi^{(d)}(t) &= B(t)\mathsf{E}^{\mathsf{Q}}\left[\frac{h^{(d)}(X(T))}{B(T)}\middle|\mathscr{F}_t\right] \\ &= \mathsf{E}^{\mathsf{Q}}\left[\exp\left\{-\int_t^T r^{(d)}(u)du\right\}\pi^{(d)}(T)\middle|\mathscr{F}_t\right], \qquad 0 \le t \le T.\end{aligned}$$

but, if we have two currencies we must take the instantaneous return (cost if it becomes negative) R in the account. Then the time-t instantaneous return (R) in terms of foreign currency which gives the collateral rate, is given by (see page 4 in [9] for more details)

$$R^{(f)}(t) = r^{(f)}(t) - c^{(f)}(t)$$

and the domestic counterparty subjects to post the $\pi^{(d)}(t)/\lambda^{(d,f)}(t)$ collateral amount at time t. Considering the money market account of domestic currency as the numériare, the collaterlized price of a derivative is given by

$$\begin{aligned}\pi^{(d)}(t) &= \mathsf{E}^{\mathsf{Q}_d}\left[\exp\left\{-\int_t^T r^{(d)}(u)du\right\}\pi^{(d)}(T)\middle|\mathscr{F}_t\right] \\ &\quad + \lambda^{(d,f)}(t)\mathsf{E}^{\mathsf{Q}_f}\left[\int_t^T \exp\left\{-\int_t^s r^{(f)}(u)du\right\}R^{(f)}(s)\frac{\pi^{(d)}(s)}{\lambda^{(d,f)}(s)}ds\middle|\mathscr{F}_t\right], \qquad 0 \le t \le T.\end{aligned}$$

now, we change the measure from Q_f to Q_d and we obtain

$$\begin{aligned}\pi^{(d)}(t) &= \mathsf{E}^{\mathsf{Q}_d}\left[\exp\left\{-\int_t^T r^{(d)}(u)du\right\}\pi^{(d)}(T)\right. \\ &\quad \left. + \int_t^T \exp\left\{-\int_t^s r^{(d)}(u)du\right\}R^{(f)}(s)\pi^{(d)}(s)ds\middle|\mathscr{F}_t\right], \qquad 0 \le t \le T.\end{aligned}$$

where the expression inside the expectation brackets (satisfying the integralbility condition which) is a martingale under risk-neutral probability measure Q_d. Thus the price process can be described by following SDE[1]

$$d\pi^{(d)}(t) = \left(r^{(d)}(t) - R^{(f)}(t)\right)\pi^{(d)}(t)dt + dM(t), \qquad 0 \le t \le T.$$

with some Q_d-martingale M.

7.2.2 Forward-Neutral Measure

Now, let's consider the forward-neutral probability measure Q^T. To do so, we state the theorem which was proposed by Fujji, Shimada and Takahashi in [9].

[1]To see full algebraic calculation, see page 486–492 in [29].

Theorem 7.2.1. *(Theorem 1 in [9]) Let the time-T derivative's payoff in terms of domestic currency and when the foreign currency is used as the collateral for contract, be $\pi^{(d)}(T)$. Further, define*

- $R^{(d,f)}(t) = R^{(d)}(t) - R^{(f)}(t)$,
- $R^{(d)}(t) = r^{(d)}(t) - c^{(d)}(t)$,
- $R^{(f)}(t) = r^{(f)}(t) - c^{(f)}(t)$,
- $v^{(d)}(t,T) = \mathsf{E}^{\mathsf{Q}^T_{(d)}}\left[\exp\left\{-\int_t^T c^{(d)}(s)ds\right\}\Big|\mathscr{F}_t\right]$ *stands for the default-free collateralized zero-coupon bond of domestic currency and is used as the numériare.*

Then the time-t price/value of such a derivative under collateralized forward-neutral probability measure $\mathsf{Q}^{T_{(d)}}$ is given by

$$\begin{aligned}\pi^{(d)}(t) &= \mathsf{E}^{\mathsf{Q}_{(d)}}\left[\exp\left\{-\int_t^T r^{(d)}(u)du + \int_t^T R^{(f)}(u)du\right\}\pi^{(d)}(T)\Big|\mathscr{F}_t\right] \\ &= v^{(d)}(t,T)\mathsf{E}^{\mathsf{Q}^T_{(d)}}\left[\exp\left\{\int_t^T R^{(d,f)}(u)du\right\}\pi^{(d)}(T)\Big|\mathscr{F}_t\right], \qquad 0 \le t \le T.\end{aligned}$$

as the result of theorem above, we have following corollary.

Corollary 7.2.1. *(Corollary in [9]) If the risk-free interest rate and collateral rate are considered to be in the same currency, then we will have*

$$\begin{aligned}\pi(t) &= \mathsf{E}^{\mathsf{Q}}\left[\exp\left\{-\int_t^T r(u)du + \int_t^T R(u)du\right\}\pi(T)\Big|\mathscr{F}_t\right] \\ &= \mathsf{E}^{\mathsf{Q}}\left[\exp\left\{-\int_t^T r(u)du + \int_t^T [r(u)-c(u)]du\right\}\pi(T)\Big|\mathscr{F}_t\right] \\ &= \mathsf{E}^{\mathsf{Q}}\left[\exp\left\{-\int_t^T c(u)du\right\}\pi(T)\Big|\mathscr{F}_t\right] \\ &= v(t,T)\mathsf{E}^{\mathsf{Q}^T}\left[\pi(T)|\mathscr{F}_t\right], \qquad 0 \le t \le T.\end{aligned}$$

7.2.3 Multiple Currencies

In the market, there exists possibility of signing a collateral agreement in several different eligible currencies. In other words, the collateral payer has a right to change the valid collateral currency whenever she/he favors. In such a scenario, the collateral payer would chose the optimal strategy. Now, let's define the following

- C is a set of all eligible foreign currencies f in collateral agreement,

- $R^{(d,f)}(t) = R^{(d)}(t) - R^{(f)}(t)$ the funding spread between currencies,
- $R^{(d,f)}(t) = -R^{(f,d)}(t) \quad t > 0$,
- $R^{(d,f)}(t) = R^{(d,k)}(t) + R^{(k,f)}(t) \quad t > 0$.

Then, the time-t price/value formula is given by (see page 492–493 in [29])

$$\begin{aligned}\pi^{(d)}(t) &= \mathsf{E}^{\mathsf{Q}_{(d)}}\left[\exp\left\{-\int_t^T \max_{f\in C}\left(R^{(d,f)}(s)\right)ds\right\}\pi^{(d)}(T)\middle|\mathscr{F}_t\right] \\ &= v^{(d)}(t,T)\exp\left\{-\int_t^T \max_{f\in C}\left(R^{(d,f)}(s)\right)ds\right\}\mathsf{E}^{\mathsf{Q}^T_{(d)}}\left[\pi^{(d)}(T)\middle|\mathscr{F}_t\right], \qquad 0 \le t \le T.\end{aligned}$$

Remark 7.2.1. It is usual to set the USD as the eligible collateral and accept the domestic currency. In this case, we will have following discount factor

$$\mathsf{E}^{\mathsf{Q}_{(d)}}\left[\exp\left\{-\int_t^T \max\left(R^{(d,USD)}(s),0\right)ds\right\}\middle|\mathscr{F}_t\right]$$

The question may arise is , what is an optimal strategy? The optimal strategy is to choose the currency which gives the highest interest rate. Such a currency is called CTD (Cheapest To Deliver). If a multiple currencies collateral contract consists of 4 different currencies (say USD, EUR,GBP and JPY), then we need to bootstrap 19 different forward-rate curves. The case is depicted in Figure 7.6.

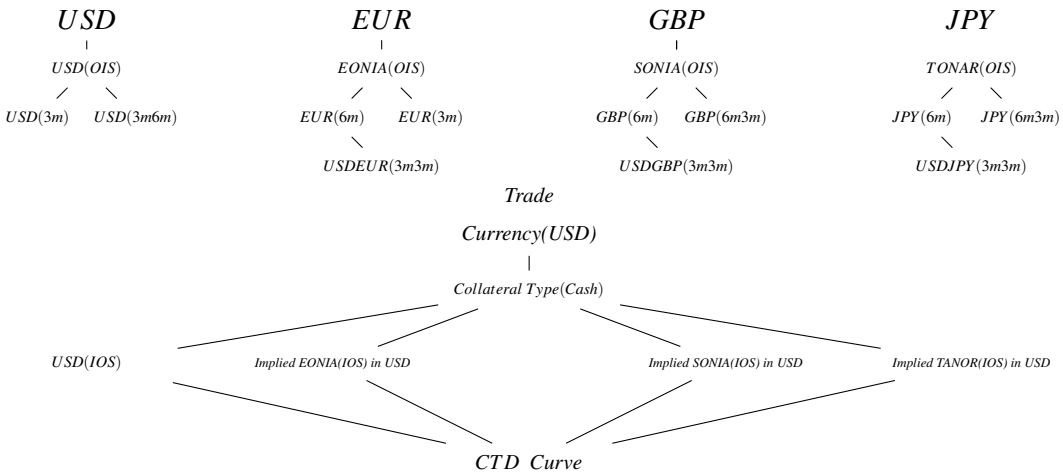

Figure 7.6: An Example of Multiple Currencies Bootstrapping Amounts

7.3 Curve Construction in Single Currency

Now, we are ready to go through the procedure of option pricing in a single currency framework. Let all option and forward contracts be collateralized in the domestic currency. To be able to price an option we need to find our discount factor and to find the discount factor we need to bootstrap a forward curve. Further, to be able to simulate our curve we need to introduce the most important calibration instruments to adjust the starting points of simulation. Let's look at some of such instruments and see how they work.

To, begin with, recall Corollary 7.2.1 and Equation (5.16) and we will go through three of the possible scenarios in this section. See Section 2.2, i.e., Market Instruments in [12].

7.3.1 Overnight Indexed Swap (OIS)

The overnight indexed swap (OIS) interferes itself by usual swap in the following feature. The floating rate of OIS is given by the daily compounded overnight rates. In the market, for a specific currency, the collateral rate is equal to overnight rate. That guides us to following relation [12, 9]

$$\mathrm{OIS}_n(t) = \frac{\sum_{i=1}^{n} \mathsf{E}^{\mathsf{Q}}\left[\exp\left(-\int_t^{T_i} c(s)ds\right)\left(\exp\left\{\int_{T_{i-1}}^{T_i} c(s)ds\right\}-1\right)\middle|\mathscr{F}_t\right]}{\sum_{i=1}^{n} \Delta_i \mathsf{E}^{\mathsf{Q}}\left[\exp\left(-\int_t^{T_i} c(s)ds\right)\middle|\mathscr{F}_t\right]},$$

or equivalently

$$\mathrm{OIS}_n(t) = \mathrm{OIS}(t, T_0, T_n) = \frac{v(t,T_0) - v(t,T_n)}{\sum_{i=1}^{n} \Delta_i v(t,T_i)}, \qquad 0 \le t \le T_0,$$

where the time-t market quote with starting time T_0 and maturity time T_n, is $\mathrm{OIS}_n(t) = \mathrm{OIS}(t, T_0, T_n)$. Moreover, Δ_i is the fraction of the **fixed leg** day count for the time interval (T_{i-1}, T_i).

Finally, note that, the right hand side of last equation is identical to the RHS of (5.16).

7.3.2 Interest Rate Swap (IRS)

An interest rate swap (IRS) can be described as an exchange of a fixed coupon bond and floating rate in a given frequency (tenor, i.e., ξ) for a predetermined period of time. The necessary condition between interest rate swap (IRS) with starting time T_0 and maturity time T_M and the LIBOR with tenor ξ, is given by

$$\mathrm{IRS}_m(t) = \frac{\sum_{j=1}^{m} \delta_j v(t,T_j) \mathsf{E}^{\mathsf{Q}^{T_j}}\left[L(T_{j-1}, T_j; \xi)\middle|\mathscr{F}_t\right]}{\sum_{j=1}^{m} \Delta_j v(t,T_j)}, \qquad 0 \le t \le T_0.$$

where, the time-t market quote with starting time T_0 and maturity time T_m, is $\mathrm{IRS}_m(t) = \mathrm{IRS}(t, T_0, T_m; \xi)$. Moreover, $L(T_{j-1}, T_j; \xi)$ is the LIBOR rate with the **floating leg** day count fraction δ_j whose tenor basis is ξ for the time interval (T_{j-1}, T_j).

7.3.3 Tenor Swap (TS) and Basis Spread

A floating versus floating swap whose floating rates are determined by the LIBOR floating rate with different tenors on one side and the LIBOR floating rate with different tenors plus a fixed spread on the other side is called tenor swap (TS). It is common to add the spread on the top of the LIBOR with shorter tenor (see Figure 7.1). As it depicted in Figure 7.1, a 6-month LIBOR is exchanged with a 3-month floating rate plus the spread. The following represents the necessary condition for tenor swap [12]

$$\sum_{i=1}^{n} \delta_i v(t, T_i) \left(\mathsf{E}^{\mathsf{Q}^{T_i}} \left[L(T_{i-1}, T_i; \xi_S) \middle| \mathscr{F}_t \right] + \mathrm{TS}(t) \right) = \sum_{j=1}^{m} \delta_j v(t, T_j) \mathsf{E}^{\mathsf{Q}^{T_j}} \left[L(T_{j-1}, T_j; \xi_L) \middle| \mathscr{F}_t \right]$$

where the time-t value of tenor swap basis spread with starting time T_0 and maturity time T_n, is denoted by $\mathrm{TS}(t) = \mathrm{TS}(t, T_0, T_n; \xi_S, \xi_L)$. Short tenor is denoted by ξ_S and ξ_L is long tenor, maturity dates T_n and T_m are set to be the same and i and j stand for different payment frequencies.

Remark 7.3.1. To see the procedure for the construction of a curve in multiple currencies framework we refer to [9].

7.4 Constructing the Curves Using HJM Framework

So far, we have almost gone through all the necessary subjects to make our forward curve. Now, we need to introduce the dynamics of our discounting factor which in this chapter is collateral rate as well as the dynamics of basis spread. Then, we can use the HJM framework to simulate our forward curve. Let's see how the dynamics and SDEs of collateral and basis spread look like.

7.4.1 Dynamics of Collateral Rate

To begin with, recall (2.7), the continuous forward collateral rate $c(t, T)$ is (see Section 2.3 in [12])

$$c(t, T) = -\frac{\partial}{\partial T} \ln v(t, T), \qquad t \leq T,$$

on the equivalent form

$$v(t,T) = \exp\left\{-\int_t^T c(t,s)ds\right\}, \qquad t \leq T, \tag{7.1}$$

where the spot rate is $c(t,t) = c(t)$. Now, let the the dynamics of the forward collateral rate under the risk-neutral probability (Money Market) measure Q satisfies the following SDE

$$dc(t,s) = \mu(t,s)dt + \sum_{i=1}^{d} \sigma_i(t,s)dW_i^{\mathsf{Q}}(t), \tag{7.2}$$

or equivalently

$$dc(t,s) = \mu(t,s)dt + \boldsymbol{\sigma}(t,s)^\top d\boldsymbol{W}^{\mathsf{Q}}(t),$$

where

- $\{\boldsymbol{W}^{\mathsf{Q}}\}$ is a d-dimensional Brownian motion under Q,
- μ is a scalar function and represents the drift coefficient,
- $\boldsymbol{\sigma}$ is a d- dimensional diffusion vector.

Applying Itô formula to (7.1) we obtain (Equation 2.13 in [12])

$$\frac{dv(t,T)}{v(t,T)} = \left(c(t) - \int_t^T \mu(t,S)ds + \frac{1}{2}\left\|\int_t^T \boldsymbol{\sigma}(t,s)ds\right\|^2\right)dt - \sum_{i=1}^{d}\left(\int_t^T \sigma_i(t,s)ds\right)dW_i^{\mathsf{Q}}.$$

Considering the fact that $v(t,T) = \mathsf{E}^{\mathsf{Q}}\left[\exp\left(-\int_t^T c(s)ds\right)\middle|\mathscr{F}_t\right]$ represents the default-free collateralized discount bond, we can conclude that the drift coefficient of $v(t,T)$ must be equal to $c(t)$. Thus, we must have

$$\mu(t,s) = \sum_{i=1}^{d} \sigma_i(t,s)\left(\int_t^s \sigma_i(t,u)du\right),$$

substituting μ in (7.2) we will get

$$dc(t,T) = \sum_{i=1}^{d}\left(\sigma_i(t,s)\int_t^s \sigma_i(t,u)du\right)dt + \sum_{i=1}^{d}\sigma_i(t,s)^\top dW_i^{\mathsf{Q}}(t).$$

7.4.2 Dynamics of Basis Spread

To find the dynamics of basis spread $\mathrm{BS}(t,T;\xi)$, we firstly need to go through the dynamics of LIBORs with different tenors. The original idea of the LIBOR market models (LMM) was used by Mercurio [25]. He modeled the spread basis SDE using the forward expectations of

LIBORs. The other approach was taking the corresponding spot process in account published in [10, 11]. We will consider the model which was developed in [12] which is the same as Mercurio's scheme, but it is developed with explicitly separated spread process.

Now, let the collateralized forward LIBOR, i.e., L^c and overnight indexed swap(IOS), i.e., L^{OIS} under forward-neutral probability measure Q^T satisfies

$$\begin{aligned} L^c(t,T_{i-1},T_i;\xi) &= \mathsf{E}^{\mathsf{Q}^{T_i^c}}\left[L(T_{i-1},T_i;\xi)\middle|\mathscr{F}_t\right], \\ L^{\text{OIS}}(t,T_{i-1},T_i) &= \mathsf{E}^{\mathsf{Q}^{T_i^c}}\left[\frac{1}{\delta_i}\left(\frac{1}{v(T_{i-1},T_i)}-1\right)\middle|\mathscr{F}_t\right] = \frac{1}{\delta_i}\left(\frac{v(t,T_{i-1})}{v(t,T_i)}-1\right) \end{aligned}$$

Further, let the LIBOR-OIS basis spread process be as follows

$$\text{BS}(t,T_i;\xi) = L^c(t,T_{i-1},T_i;\xi) - L^{\text{OIS}}(t,T_{i-1},T_i).$$

Since $\text{BS}(t,T_i;\xi)$ under the collateralized forward measure Q^{T^c} is martingale, we can form the basis spread SDE by

$$\frac{d\text{BS}(t,T;\xi)}{\text{BS}(t,T;\xi)} = \sum_{i=1}^{d}\sigma_{BS_i}(t,T;\xi)dW_i^{T_c}(t),$$

where σ_{BS} is $d-$dimensional volatility function of BS or other state variables. Now, we change our forward-neutral probability measure form Q^{T^c} to risk-neutral (Money Market) probability measure Q. Using Maruyama-Girsanov's theorem the Brownian motions in Q^{T^c} and Q, have following relation (see page 7 in [12])

$$dW^{\mathsf{Q}^{T_c}}(t) = \left(\int_t^T \sigma(t,u)du\right)dt + dW^{\mathsf{Q}},$$

finally the d-dimensional SDE of BS under probability measure Q is given by

$$\frac{d\text{BS}(t,T;\xi)}{\text{BS}(t,T;\xi)} = \sum_{i=1}^{d}\left(\sigma_{BS_i}(t,T;\xi)\int_t^T \sigma_i(t,u)du\right)dt + \sum_{i=1}^{d}\sigma_{BS_i}(t,T;\xi)dW_i^{T_c}(t).$$

It is important to keep in mind that the specification of process $\{BS\}$ involves all the relevant tenors (i.e., 1m, 3m, 6m and so forth) in the market.

To see the results of curve construction in a single currency, we end this part by adding following figures which are taken from [9].

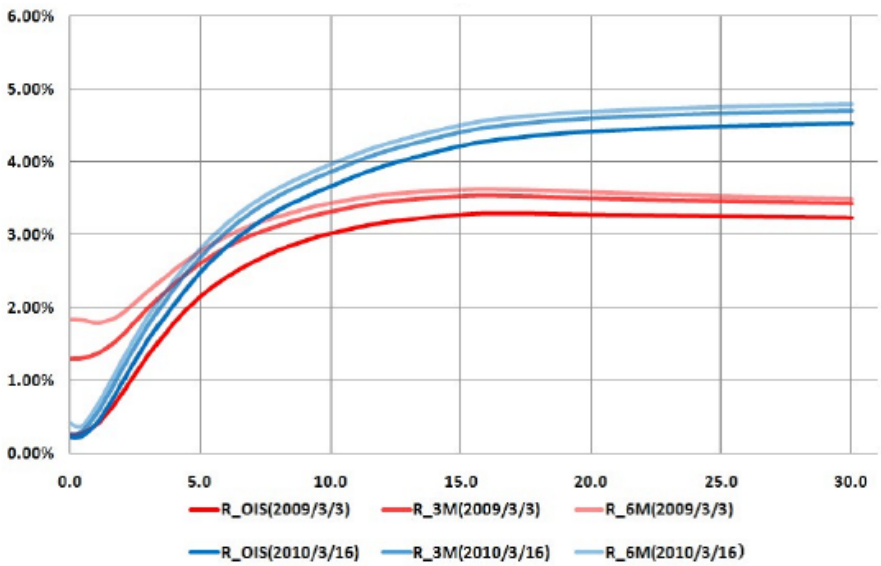

Figure 7.7: USD Swap Curves (Figure 3 in [9]).

Figure 7.7, illustrates the USD zero rate curves of Fed-Fund rate, 3m and 6m LIBORS in 2009-03-03 and 2010-03-16. The zero rate curves of IOS rate, 3m and 6m LIBOR for JPY and EUR in 2010-03-16 are depicted in Figure 7.8.

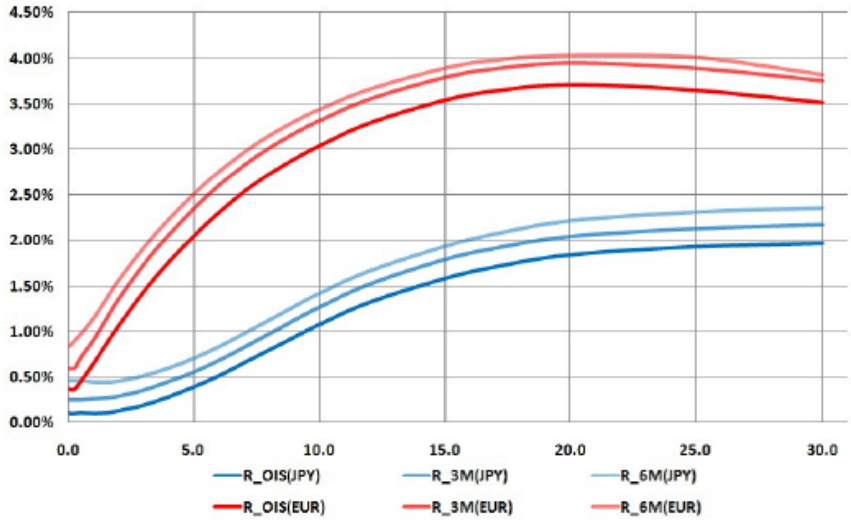

Figure 7.8: JPY and EUR Swap Curves (Figure 5 in [9]).

Chapter 8

Conclusion

In the conclusion of our work, we will summarize the reflection of our objectives in this thesis. In this manner, we go through our objectives and explain shortly how we have achieved them during our work.

The topic of this thesis was about the evaluation of interest rate derivatives. In Chapter 1, after an introduction about the vital role of interest rates in microeconomics and macroeconomics points of view and everyone's daily life, we presented our objectives in this work. We identified and formulate the problem and stated a procedure with several steps to answer our questions and achieve our goals which was pricing interest rate derivatives. Later in Chapter 2, we started describing the type of interest rates and their fundamental roles in the financial market. Further, we went through the fundamental definitions, financial terms, simple examples and elementary mathematics. After that, we started expanding our knowledge in financial mathematics and stochastic calculus in Chapter 3. We went through stochastic differential equations, pricing derivatives under risk-neutral evaluation and forward-neutral evaluation. During this procedure, we considered some examples and stochastic volatility models.

After searching, collecting, evaluating and interpreting our financial mathematics tools and knowledge to price a derivative, we try to identify, formulate and solve the other problem which was pricing interest rate derivatives especially when the interest rates are stochastic. Additionally, the negative interest rates have been introduced to the market recently. So, in Chapter 4 we looked at term structure models which can help us to deal with stochastic rates. Moreover, some of the term structure models which give either negative or positive interest rates were mentioned and their advantage and disadvantage compared and contrasted. In chapter 5, we started presenting the price formulas for some interest rate derivatives. We looked at the most popular interest rate derivatives in the market and discussed about the negative and positive price processes. In chapter 6, the evaluation of forward curve came in our attention and we looked at the HJM framework and the LMM model to evaluate the forward curves and consequently estimate the discount factor in pricing of interest rate derivatives.

Finally in Chapter 7, we focused on the new framework in the financial market. That was collateral agreement and collateral rate which have been popular from and after financial crisis in 2007 and 2008. The methodology and the advantages of collateral agreement were discussed and we went through the evaluation of the forward curve using collateral rate. Finally, the role of multiple currencies and the term cheapest-to-deliver were discussed briefly.

As we can see in our daily life, the role of evaluating interest rates is not just related to financial mathematics. This evaluation process can involve and affect different groups of specialists and individuals. As we discussed it in the introduction, the government's macroeconomics policies, international market, international trading, individual's economy and lots of other examples can be involved in this area of studies. A good example of such claim is about the retirement salary. Lots of retired people are not satisfied with their pension salary and that is mainly due to the low accumulated interest rates to their monthly saving comparing with the inflation rate during several decades. As a result, lots of pension fund companies offer their clients a higher rate of return arguing that they would invest their clients' monthly salary in a risk-less market portfolio with higher rate of return.

For future work, I have two main objectives to go through. First is to simulate the forward curves using some of the term-structure models and try to use normal and log-normal volatilities in corresponding simulations. Second is to price some derivatives using the estimated forward rates.

Glossary

Bond Yield Discount rate which, when applied to all cash flows of a bond, causes the present value of the cash flows to equal the bond's market price. 16

Bootstrap Method A procedure for calculating the zero-coupon yield curve from market data. 18

Cap Rate The rate determining payoffs in an interest rate cap. 24

CDS Credit Default Swap, is similar to an insurance contract which even cover the underlying asset which are not owned. 101

Collar *See* Interest Rate Collar. 26

Collateralization A system for posting collateral by one or both parties in a derivatives transaction. 100

CSA Credit Support Annex. 100

CTD Cheapest To Deliver. 106

Currency Swap A swap where interest and principal in one currency are exchanged for interest and principal in another currency. 24

Day Count A convention for quoting interest rates. 107

EONIA Euro Over Night Index Average. 102

Floor *See* Interest Floor Rate. 25

Floor Rate The rate in an interest rate floor agreement. 25

Floorlet One component of a floor. 25

Forward Rate Rate of interest for a period of time in the future implied by today's zero rates. 16

Haircut Discount applied to the value of an asset for collateral purposes. 102

IDSA International Swaps and Derivative Association. 100

Interest Rate Cap An option that provides payoff when a specified interest rate is above a certain level. The interest rate is a floating rate that is reset periodically. 24

Interest Rate Collar A combination of interest rate cap and interest rate floor. 26

Interest Rate Derivative A derivative whose payoffs dependent on future interest rates. 30

Interest Rate Floor An option that provides a payoff when an interest rate is below a certain level. The interest rate is a floating rate that is rest periodically. 25

Interest Rate Option An option where the payoff is dependent on the level of interest rate. 30

Interest Rate Swap An exchange of a fixed rate of interest on a certain notional principal for a floating rate of interest on the same notional principal. 14

IRS See interest rate swap. 107

LIBOR London Interbank Offered Rate. The rate offered by banks on Eurocurrency deposits (i.e., the rate at which a bank is willing to lend to other banks). 15

Money Market Account An investment that is initially equal to one unit of money and at time t, increases at the very short-term risk-free interest rate prevailing at that time. 26

MTA Minimum zero Transfer Amount. 103

Notional Principal The principal used to calculate payments in an interest rate swap. The principal is "notional" because it is neither paid nor received. 21

O/N Over Night. 100

OIS See overnight indexed swap. 16

OTC See over the counter market. 100

Over-The-Counter Market A market where traders deal by phone. The trader are usually financial institutions, corporations and fund managers. 24

Overnight Indexed Swap Swap where a fixed rate for a period (usually one month up to a year) is exchanged for the geometric average of the overnight rates during the period. 10

Par Yield The coupon on a bond that makes its price equal to principal. 16

Premium The price of an option. 74

Principal The par or face value of a debt instrument. 16

Repo Repurchase agreement. A procedure for borrowing money by selling securities to a counterparty and agreeing to buy them back later at a slightly higher price. 103

Repo Rate The rate of interest in a repo transaction. 15

Riksbanken Sweden's central Bank. 103

Risk-Neutral Valuation The valuation of an option or other derivative assuming the world is risk-neutral. Risk-neutral valuation gives the correct price for a derivative in all worlds, not just in a risk-neutral world. 17

Risk-Neutral World A world where investors are assumed to require no extra return on average for bringing risks. 17

SONIA Sterling Over Night Index Average. 102

Spot Interest Rate See zero-coupon interest rate. 47

Swap An agreement to exchange cash flows in the future according to a prearranged formula. 21

Swap Rate The fixed rate in an interest rate swap that causes the swap to have a value of zero. 76

Swaption An option to enter into an interest rate swap where a specified fixed rate is exchanged for floating. 30

Treasury Rate Are the instruments used by a government to borrow in its own currency. 15

TS Tenor Swap. 108

Yield A return provided by an investment. 28

Zero-Coupon Interest Rate The interest rate that would be earned on a bond that provides no coupons. 16

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