# Model risk in a hedging perspective

Master thesis
Department of Mathematical Statistics
Royal Institute of Technology

by
Carl-Johan Johansson
Greger Sundqvist

**Abstract**

This thesis' aim is to develop a framework for model risk analysis when hedging a option position. The framework has been split up into parameter, assumption and market state model risk and applied to Black and Scholes pricing formula as an example.

The single most important parameter of Black and Scholes formula is the estimation of the underlying's volatility. The strongest and most risky assumptions are that volatility is constant and that the logreturns are normal distributed. Black and Scholes formula is able to handle different market states but estimation of volatility is harder during turbulent times. The framework is applicable on all models used to calculate hedge ratios and may be helpful when developing frameworks for measuring model risk in other contexts.

**Acknowledgements**

We would first of all like thank Krister Sundholm at Swedbank for his ideas and guidance during the work with this thesis. We would also like to thank Harald Lang at Royal Institute of Technology and Jan Röman at Swedbank for their helpful support.

Stockholm, January 2011

Carl-Johan and Greger

# Contents

# Chapter 1

# Introduction

Model risk occurs when modeling reality by doing assumptions in order to reduce the degrees of freedom. Knowledge of these simplifications is extra important as an issuer of options since there is no limit of the possible loss. This paper aims to investigate the effect that assumptions and parameter estimations have when hedging issued stock options with Black & Scholes. The hedging performance is evaluated by looking at the accumulated result of the portfolio during the life time of the option. Finding a way of measuring model risk is the first step to be able to monitor and limit it.

## 1.1 Problem formulation

The objective of this master thesis is to develop a framework for splitting model risk when hedging a stock option into its components and to find a way of measuring them.

### 1.1.1 Limitations

To narrow the scope of this thesis the following limitations have been done:

- analyzed model risk is in the context of a market maker issuing European options on stocks
- the model used to price the stock options is the Black and Scholes formula.

## 1.2 Methodology

When analyzing model risk in context of hedging an issued option the assumption that the formula describes the market price in a good manner is applied in this thesis. This assumption is obviously questionable when analyzing model risk even though the model already is widely used to this purpose. In other words the framework developed in this thesis measures the risk when using an arbitrary model to calculate hedge ratio, not its capability to price stock options consistent with the market price. However, a suggestion of how to setup simulations for testing a model's capability to price according to the market price is provided in Chapter 9. In this thesis the framework is applied on Black & Scholes formula.

The analysis of model risk will be divided into three different categories:

1. parameter model risk
2. assumption model risk
3. market state model risk

which are based on when they appear. Number one is risks when giving the chosen model improperly set input parameters. The second category is the risk implied by choosing a model, i. e. the simplifications made when describing the real world in order to create a model. Market state model risk is a measure of how the model handles different market states, for example turbulent contra ordinary times. Each has been dedicated its own section of this thesis.

In order to isolate the different components of model risk both Geometric Brownian Motions and bootstrapped historical data will be used. Modeling the stock price according to the assumptions of the pricing model enables the possibility to isolate and observe the model's sensitivity to badly estimated input parameters, and historical data provides data from the actual distribution of the stock price. The general idea is as shown in Figure 1.1 to plug in the model as the model box below, feed it with simulated stock trajectories, calculate and accumulate the result in SEK over the time period and finally calculate a risk measure. The gray boxes are independent of which model to apply the framework on, and perfect data depends on the assumptions of the model. A more thoroughly description of the simulation algorithm is provided in Section 4.1.

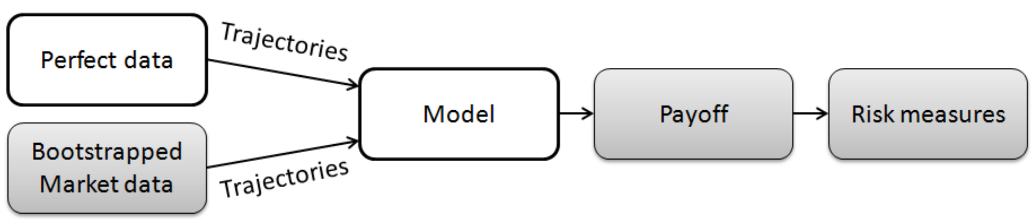


Figure 1.1: The general idea of the simulations.

# Chapter 2

# Theoretical background

## 2.1 Geometric Brownian Motion

The most common way to model a stock's trajectory is by a Geometric Brownian Motion (GBM) which is based on two assumptions[1]:

1. the relative expected return of the stock is assumed to be constant over time
2. the relative variability of the modeled returns of the stock is assumed to be constant for each time step.

Which leads to the GBM described by the stochastic differential equation

$$\begin{cases} \frac{dS}{S(t)} = \mu dt + \sigma dW(t) \\ S(0) = S_0 \end{cases} \iff \begin{cases} dS = \mu S(t) dt + \sigma S(t) dW(t) \\ S(0) = S_0 \end{cases} \tag{2.1}$$

where
$\mu$ = the expected rate of return
$S(t)$ = spot price at time $t$
$\sigma$ = the volatility
$W(t)$ = Wiener process.

By setting $Z(t) = \ln(S(t))$ and using Itô's lemma equation 2.1 may be solved as

$$dZ(t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW(t) \tag{2.2}$$

$$\int_t^{t+\Delta t} dZ = \left(\mu - \frac{\sigma^2}{2}\right) \int_t^{t+\Delta t} dt + \sigma \int_t^{t+\Delta t} dW(t) \tag{2.3}$$

[1] Hull 2009, p265

$$Z(t+\Delta t) - Z(t) = \left(\mu - \frac{\sigma^2}{2}\right)(t+\Delta t - t) + \sigma\left(W(t+\Delta t) - W(t)\right) \tag{2.4}$$

$$Z(t+\Delta t) - Z(t) = \left(\mu - \frac{\sigma^2}{2}\right)(t+\Delta t - t) + \sigma W(t) \tag{2.5}$$

which, expressed in spot prices and a standard normal distributed random variable $\epsilon$, equals

$$\ln\left(S\left(t+\Delta t\right)\right) - \ln\left(S\left(t\right)\right) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\epsilon\Delta t. \tag{2.6}$$

For simulating purposes $S\left(t+\Delta t\right)$ is expressed as

$$S\left(t+\Delta t\right) = S\left(t\right)\cdot e^{\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\epsilon\Delta t} \tag{2.7}$$

giving a time series of the stock:

$$S\left(0\right), S\left(\triangle t\right), S\left(2\triangle t\right), \ldots, S\left(T\right) \tag{2.8}$$

Properties worth mentioning are that both sides in equation 2.6 are normally distributed, while in equation 2.7 they are lognormal distributed.

## 2.2 The Black and Scholes framework

A European option is a contract that gives the buyer the right to buy the underlying asset of the contract from the issuer on a specified day for a price set at time zero. The contract can only be exercised at the day of maturity.

Today the standard model for valuation of European options on stocks is the Black and Scholes (B&S) formula which is derived from B&S equation obtained from equation 2.1 via Itô's lemma and seven assumptions[2].

1. The stock price follows the process given in equation 2.1 with $\mu$ and $\sigma$ constant.
2. Short selling of securities is permitted.
3. No transaction costs or taxes. All securities are fully divisible.
4. There are no dividends during the life of the derivative.
5. There are no risk free arbitrage opportunities.
6. Security trading is continuous and all strikes exists.

[2] Hull 2009, p286

7. The risk free rate of interest is constant and the same for all maturities and equal to the drift, $\mu$. The market actors are risk neutral.

By solving B&S equation for a European call option the B&S call option formula is acquired

$$C(S,t) = S\Phi(d_1) - Ke^{-r_f(T-t)}\Phi(d_2) \tag{2.9}$$

where

$$\begin{cases} d_1 = \frac{\ln\left(\frac{S}{K}\right)+\left(r_f+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = \frac{\ln\left(\frac{S}{K}\right)+\left(r_f-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \end{cases}$$

and

$\Phi(x)$ = the standard cumulative normal probability distribution for $x$

$K$ = strike price

$T$ = term to maturity

$r_f$ = risk free interest rate.

The corresponding solution for a put option is[3]

$$P(S,t) = Ke^{-r_f(T-t)}\Phi(-d_2) - S\Phi(-d_1). \tag{2.10}$$

Worth noticing is that the value of a long position will be positive during the lifetime of the contract, due to the fact that there is always a possibility that the option will have a value larger than zero at maturity.

[3] Hull 2009, p291

### 2.2.1 Greeks

The greeks are measures of the theoretical price sensitivity with respect to changes in the different parameters. The greeks according to the B&S framework are shown in Table 2.1[4].

| | Call | Put |
|---|---|---|
| delta ($\delta$) | $\frac{\partial C}{\partial S} = \Phi\left(d_1\right)$ | $\frac{\partial P}{\partial S} = \Phi\left(d_1\right) - 1$ |
| vega ($V$) | $\frac{\partial C}{\partial \sigma} = S\sqrt{T}\phi\left(d_1\right)$ | $\frac{\partial P}{\partial \sigma} = S\sqrt{T}\phi\left(d_1\right)$ |
| theta ($\Theta$) | $\frac{\partial C}{\partial t} = -\frac{S\phi(d_1)\sigma}{2\sqrt{T}} - r_f K e^{-r_f T}\Phi\left(d_2\right)$ | $\frac{\partial P}{\partial t} = -\frac{S\phi(d_1)\sigma}{2\sqrt{T}} + r_f K e^{-r_f T}\Phi\left(-d_2\right)$ |
| gamma ($\Gamma$) | $\frac{\partial^2 C}{\partial S^2} = \frac{\phi(d_1)}{S\sigma\sqrt{T-t}}$ | $\frac{\partial^2 P}{\partial S^2} = \frac{\phi(d_1)}{S\sigma\sqrt{T-t}}$ |
| rho ($\rho$) | $\frac{\partial C}{\partial r_f} = KTe^{-r_f T}\Phi(d_2)$ | $\frac{\partial P}{\partial r_f} = -KTe^{-r_f T}\Phi(-d_2)$ |

Table 2.1: Sensitivities according to the B&S framework.

where $\phi\left(x\right)$ is the standard normal probability density function for $x$.

## 2.3 Monte Carlo simulation

Monte Carlo simulation is a way of evaluating financial derivatives or strategies. The idea is to assume a mathematical model and adapt its parameters to the behavior of the underlying asset. When this is done random sampling from the model is performed to get a path of the underlying and a payoff of the instrument is calculated based on it. When $n$ paths have been sampled a mean of the discounted payoffs decides the price. The most important advantage using Monte Carlo simulation is that it may be used when the price not only depends on the spot price of the underlying at maturity but on its path. The negative side is that it is time consuming due to the fact that many simulated paths are necessary to increase the precision of the result.

The accuracy of the result depends on the number of trials which are carried out. Denote the mean and standard deviation of the Monte Carlo simulated prices of the instrument, $f$, by $\mu$ and $\sigma$ respectively. Then a 95% confidence interval of f is given by[5]

$$\mu - \frac{1.96\sigma}{\sqrt{n}} < f < \mu + \frac{1.96\sigma}{\sqrt{n}}. \tag{2.11}$$

[4]Hull 2009, p362-376
[5]Hull 2009, p430

## 2.4 The bootstrap method

One way of simulating a stock price over time is by choosing a model and trying to adapt it to the stock to be simulated, i. e. by a priori assumption regarding the distribution. This is called Monte Carlo simulation. Many times the assumption about the distribution is more or less wrong, for instance the assumption of normally distributed logreturns in the B&S framework. A method to simulate a stock price without doing any a priori assumptions about the distribution of the logreturns is non parametric bootstrap. The idea is that given a sample, which is a good approximation of the population, randomly drawing observations from it creating a large number of resamples on which some statistic is calculated. The relative frequency histogram of the statistic reveals a good estimate of its distribution.

The accuracy of the bootstrap method is dependent on whether the data which are bootstrapped are a series of independent and identically distributed (iid) random variables or a time series. Iid data causes no problem when bootstrapping but if not iid the dependence structure of a time series is lost when randomly drawing one observation at time from the sample. The oldest and most common way of implementing bootstrap when simulating time series is the block bootstrap method where each random draw picks a block of observations instead of only one[6].

Consider a sample of $k$ logreturns from the unknown probability distribution $\omega$. The procedure of the classic non parametric bootstrap is then as follows[7]:

1. create an empirical distribution function, $\Omega$, by giving all logreturns the same probability equal to $1/k$
2. Draw a random sample of size $l$ with replacement, this is called a resample
3. calculate the statistic of interest, for example the result of a portfolio from time $0$ to $T$, $R(0,T)$
4. repeat steps 2 & 3 $n$ times to get $n$ resamples
5. the relative frequency histogram of $R(0,T)$ is given by giving each $R_1(0,T), R_2(0,T), R_3(0,T), ..., R_n(0,T)$ probability $1/n$.

[6] Härdle, Horowitz and Kreiss 2003
[7] Efron and Tibshirani 1993, p47

## 2.5 Delta hedging

As issuer of an option there is no limit for the possible loss, due to this fact there is a need to control the risks involved. One way of doing this is by delta hedging which is based on the idea of being delta neutral. Or in other words insensitive to changes in the underlying of the contract. The portfolio delta is described as

$$\delta_p(S, t, w_s, w_c, w_p) = w_s + w_c\delta_c(S, t) + w_p\delta_p(S, t). \tag{2.12}$$

where $w_s, w_c, w_p$ are weights for positions in the stock, call and put options respectively. Delta of the underlying stock is per definition equal to one and deltas of put and call are calculated as in Table 2.1.

For an emitter the case is short positions in options which implies a certain delta position that can be hedged by the underlying stock. A more thorough description of delta hedging is found in Chapter 3.

## 2.6 Payoff structure

With a long position in a put or a call option the value at maturity is[8]

$$\begin{cases} C\left(S,T\right) = \max\left(0, S-K\right) \\ P\left(S,T\right) = \max\left(0, K-S\right) \end{cases} \tag{2.13}$$

The buyer has the choice whether to exercise the option or not. The issuer of the contract is bound to accept an exercise if the option holder requires it, which implies the following payout structure at maturity for the issuer

$$\begin{cases} C\left(S,T\right) = -\max\left(0, S-K\right) \\ P\left(S,T\right) = -\max\left(0, K-S\right) \end{cases} . \tag{2.14}$$

The value of a portfolio consisting of positions in call and put options, the underlying stock and a cash position may at any time be described as

$$V_p(S, t, w_s, w_c, w_p, \Omega) = w_s S(t) + w_c C(S, t) + w_p P(S, t) + \Omega\left(t\right) \tag{2.15}$$

where $\Omega$ represents the cash position. The cash part is calculated by

$$\Omega\left(t\right) = \Omega\left(t-\Delta t\right) e^{r_f \Delta t} + \left(w_s\left(t\right) - w_s\left(t-\Delta t\right)\right) S\left(t\right)\left(1-\xi\right) \tag{2.16}$$

[8] Hull 2009, p293

where $\xi$ is a transaction cost which has to be paid for each transaction. The result of the portfolio between two times is then

$$R\left(t-\Delta t,t\right)=V_p(S,t,w_s,w_c,w_p,\Omega)-V_p(S,t-\Delta t,w_s,w_c,w_p,\Omega). \tag{2.17}$$

Which gives the accumulated result over the period as

$$R(0,T)=\sum_{t=\Delta t}^{T}V_p(S,t,w_s,w_c,w_p,\Omega)-V_p(S,t-\Delta t,w_s,w_c,w_p,\Omega) \tag{2.18}$$

## 2.7 Risk measures

### 2.7.1 Value at risk

Value at risk (VaR) is a measure of risk that is supposed to summarize the risk of a financial asset to one single number. The concept of it may be expressed in one single statement:

*the loss of the coming n days will with probability p be less than or equal to $VaR_p$*

or in a more strict manner

$$P\left(L(t-n,t)\leq VaR_p\right)=p \tag{2.19}$$

where

$$L(t-n,t)=-R(t-n,t) \tag{2.20}$$

and $R(t-n,t)$ is the n days result of the portfolio defined in equation 2.18. The simplicity of it and the fact that it is used to calculate the minimum capital level of a financial institution are probably the main reasons to why its so widely used as risk measure. The obvious question is whether a single number is enough to describe the risk of a complicated portfolio. However, if this is accepted the setup of the calculations and the assumptions are critical for VaR's ability to measure risk.

In this paper the default setting of:

$n=1$

$p=0.99$

will be used. There are a number of different ways to calculate VaR but the two main approaches are historical and by Monte Carlo methods. The assumptions behind these two are conceptually very different. While historical assumes that the best way of describing the result is to look at the past for some appropriate time period the model based method tries to model the future result. The VaR calculations tied to this report will be performed using the historical approach to avoid an assumption of normal distribution of the accumulated result.

The historical VaR is calculated using the following algorithm[9]:

[9]Hult and Lindskog 2007

1. Sort the observations of daily results from the largest to the smallest, i. e. $l_1 > l_2 > l_3 > ... > l_m$ where $l_k$, $k = 1,\ 2,\ \dots\ ,\ m$ are observations of $L$.

2. Then
$$VaR_p = l_{[m(1-p)]+1} \tag{2.21}$$
where [x] is the largest integer less than or equal to x.

### 2.7.2 Expected shortfall

A problem with the VaR number is that it does not provide any information on how large the loss will be the last $1 - p$ percent of the days. Expected shortfall (ES) deals with this matter. ES gives an answer to the question:

*How much do I lose in average when the VaR number is breached?*

Mathematically the same thing is expressed as

$$ES_p = E\left[L \mid L > VaR_p\right]. \tag{2.22}$$

The historical approach of calculating $ES_p$ is

$$ES_p = \frac{\sum_{k=1}^{[n(1-p)]+1} l_k}{[n(1-p)]+1} \tag{2.23}$$

where $l_1 > l_2 > l_3 > ... > l_m$. $l_k$, $k = 1,\ 2,\ \dots\ ,\ m$ are observations of $L$.

# Chapter 3

# Delta hedging

This chapter is intended to improve understanding about delta hedging and to clarify how the result is created. Results and values are not used in the rest of the report or as a part of the framework.

## 3.1 Delta hedging in practice

During the period 2010-05-06 to 2010-09-14 the spot price of Ericsson B has developed as shown in the top graph of Figure 3.1[1]. Below are the B&S prices over the period for a call and put option with Ericsson B as underlying and strike SEK 80. At the bottom of Figure 3.1 the hourly result, $R(t-1,t)$, for a portfolio containing a short position of both a call and a put option, hedged with the underlying stock over the period is shown.

Worth noticing is how the options delta is affected by time. This is made visible by the small position needed in the underlying for being delta neutral in the early part of the period, in contrast to the position required towards the end. This is further explained in Section 3.2.

[1]Bloomberg 2010-10-21 (Market data)

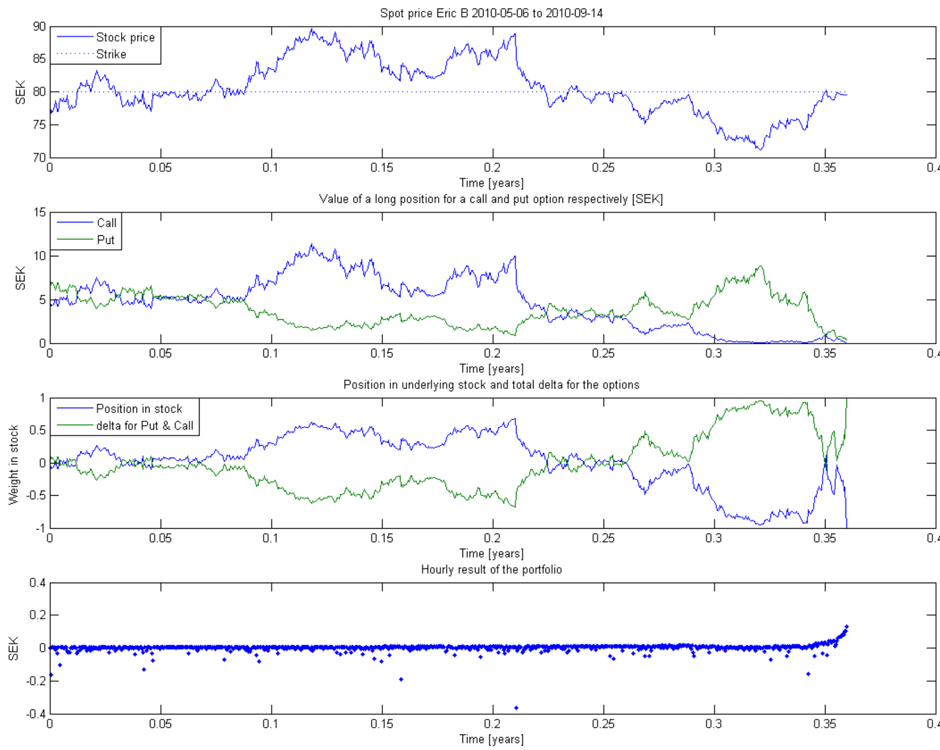


Figure 3.1: Hourly result of a portfolio with a short position in both a call and a put option. The hedge is adjusted every hour.

The data in Figure 3.1 are based on hourly observations of S and the hedge is adjusted at each observation. Hedging this often is hard to achieve in the real world, and it would result in large transaction costs. Usually hedge adjustments are done more seldom as in Figure 3.2 or is triggered by certain events in the market.

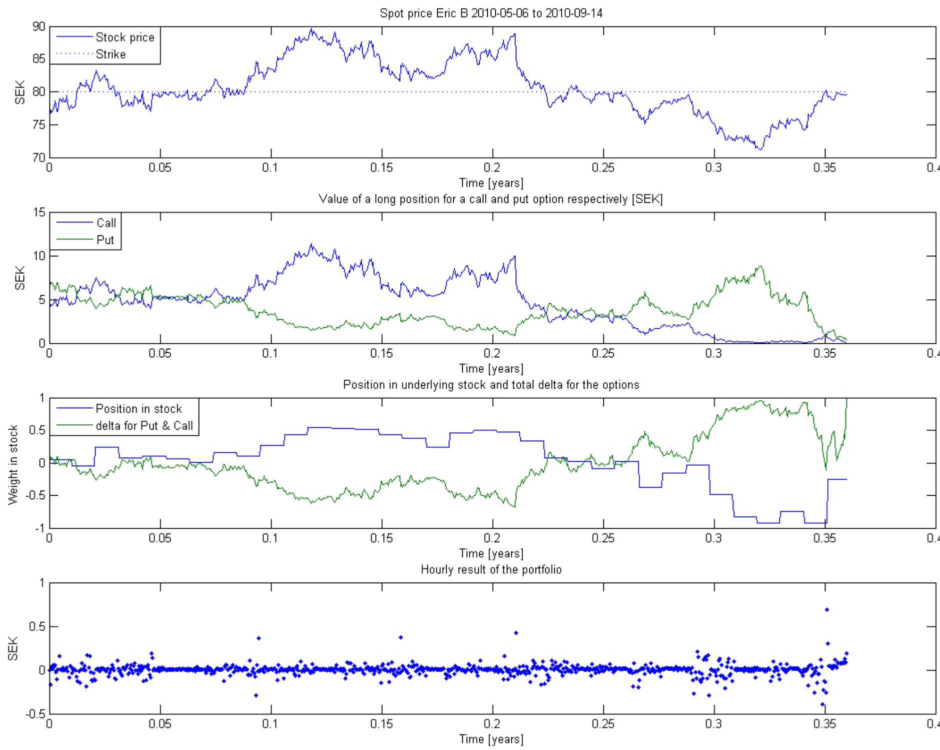


Figure 3.2: Same scenario as in Figure 3.1 but hedge adjustments every third day.

As Figure 3.2 shows the time between hedging is important, the longer these intervals get the less delta neutral the portfolio will become, which implies larger adjustments in the underlying when hedging. This also leads to a severe increase of the result's standard deviation, which can be seen in the bottom plot of Figure 3.2. A more detailed analysis of the consequence by different hedging intervals is found in Section 6.2.3.

## 3.2 Analysis of the portfolios hourly result

This section is dedicated to understanding of the hourly result, $R(t - \triangle t, t)$, when the underlying is simulated by a GBM. The hedged option position is one short call and one short put. In Figure 3.3 the hourly result of 100 000 trajectories are scattered. The red lines shows 1, 10, 90 and 99 percent percentiles of the hourly results for each time. Worth noticing is how many of the hourly returns which are concentrated around zero although it does not appear so at first glance.

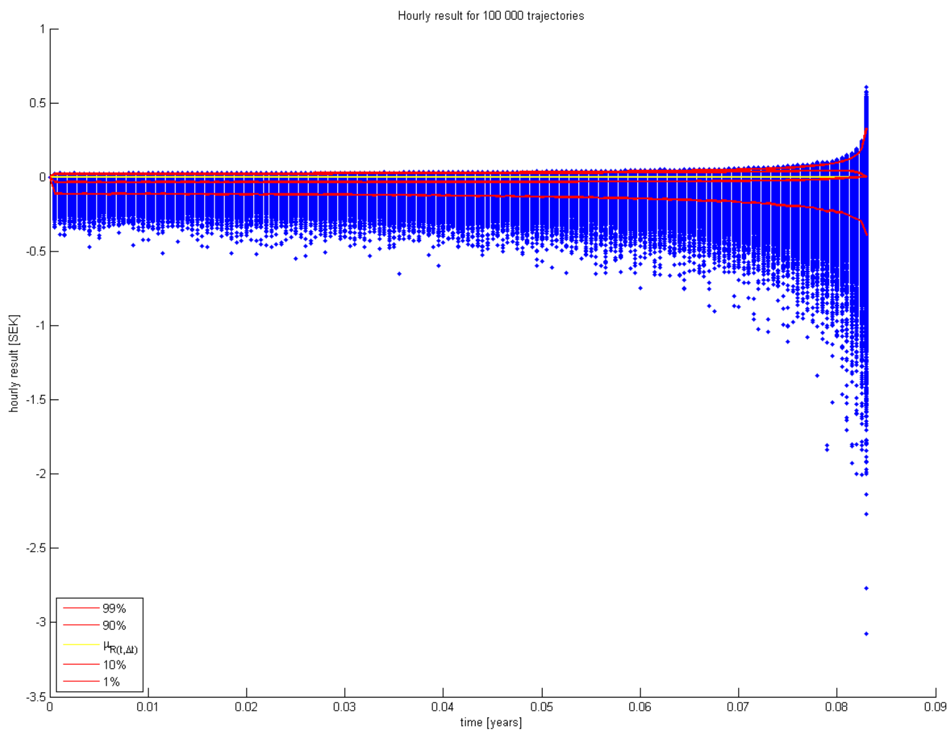


Figure 3.3: Scatter plot of hourly result over time for 100 000 GBM trajectories with underlying $\sigma = 0.295$ and $\mu = 0.035$.

In order to understand the the behavior of the daily result Figure 3.4 should be studied. It shows the portfolio's sensitivity to a change of one SEK in the underlying stock. The first obvious conclusion is that the largest impact on the result is obtained ATM close to maturity. Another fundamental insight is that when delta hedging a GBM with the correct parameters the daily result of a positive change in the underlying is always negative, all other parameters held constant. An explanation of this is provided below. The behavior that a one SEK increase at the money (ATM) results in larger impact on the result close to maturity than in the beginning is also worth pointing out, this is however not true for all spot prices. The above mentioned properties do as well apply to a decrease of the underlying stock.

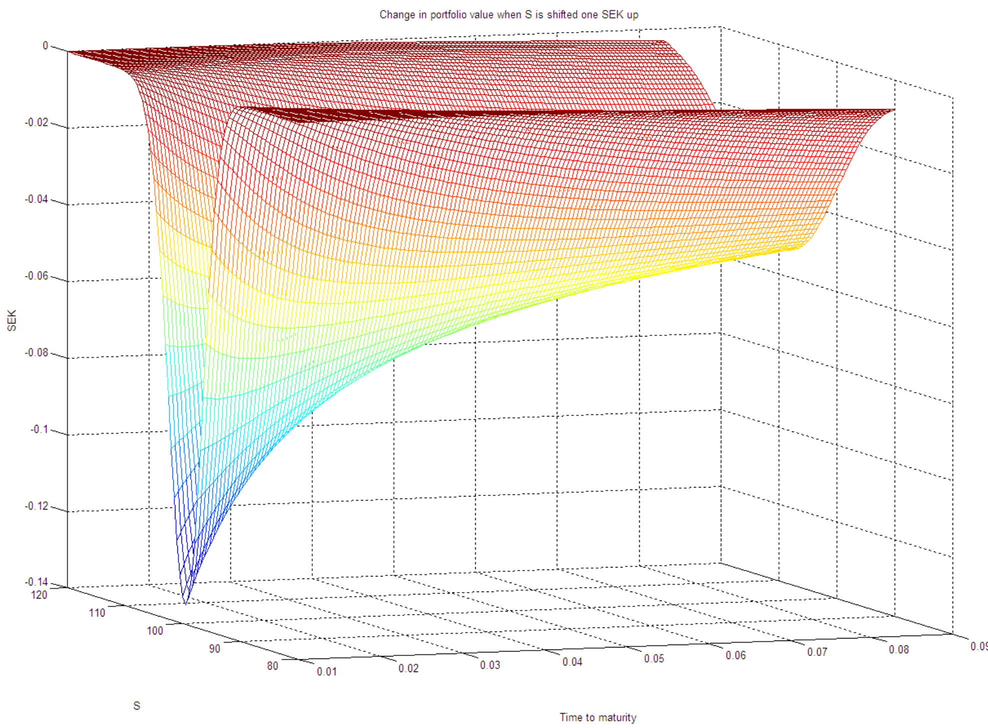


Figure 3.4: The portfolio's sensitivity to a raise in S of one SEK.

The option price is divided into two components, intristic value and time value. The intristic value is defined as the value of the option if it was exercised today. The time value represents the possibility that the underlying spot will be in or more in the money at maturity[2]. In other words Figure 3.4 shows the change in intristic value at different times and at different spot prices.

[2]Hull 2009, p186

To clarify the fact that the change in intristic value is always negative Figure 3.5 will be used. Deltas for the put, call and a portfolio consisting of short positions in both the put and call are plotted as lines, and the stock position of the portfolio as a dotted line. The strike of the options is SEK 100. Notice that the portfolio delta is a strictly decreasing function over $S$ causing the negative change in intristic. For instance consider one hedging occasion when the spot price is SEK 110, then the stock position of the portfolio will be 0.76 until the next hedging occasion. But if the spot price increases by two SEK then the position is too small causing that the options value will decrease more than the increase of the stock position. In other words the result is negative, the same arguments may be applied on a decrease of the stock price. When starting at a spot price of SEK 90 then a similar reasoning is applicable keeping in mind that a short stock position's value is decreased when the stock price increases.

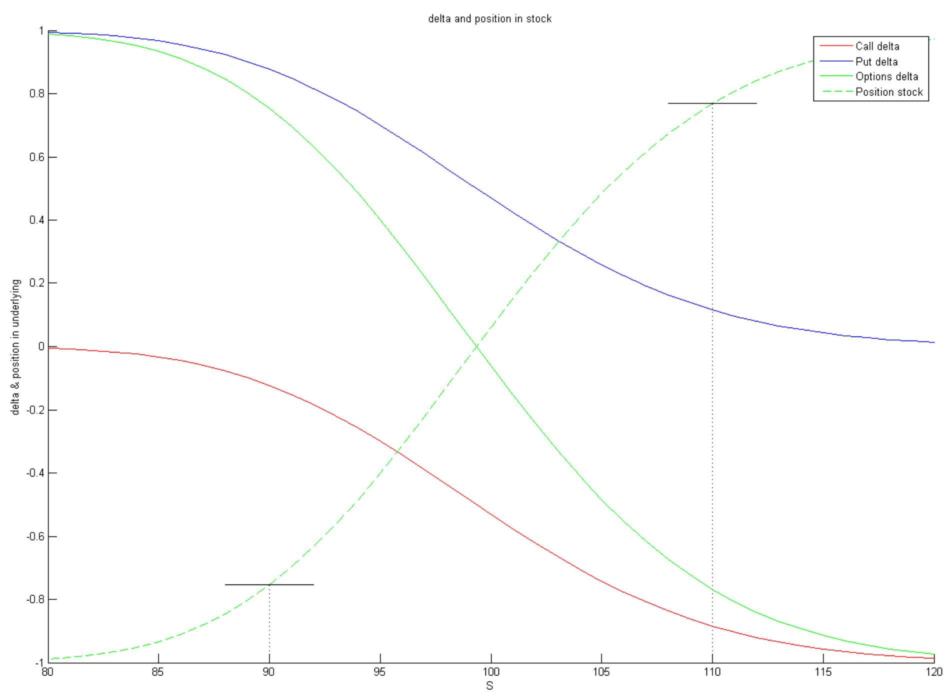


Figure 3.5: Plot of the delta and stock position of the portfolio.

This far, understanding of what is causing the negative daily results in Figure 3.3 has been developed. However the positive part is still to be explored, and by the reasoning above it is probably originating from the time value of the option. In Figure 3.6 the change in value of one short call and put option, due to one hour closer to maturity for some different times and spot prices, is plotted. The behavior is the same as of the intristic value, the increase of the result is highest ATM close to maturity and has a negative slope over time for spot price SEK 90.

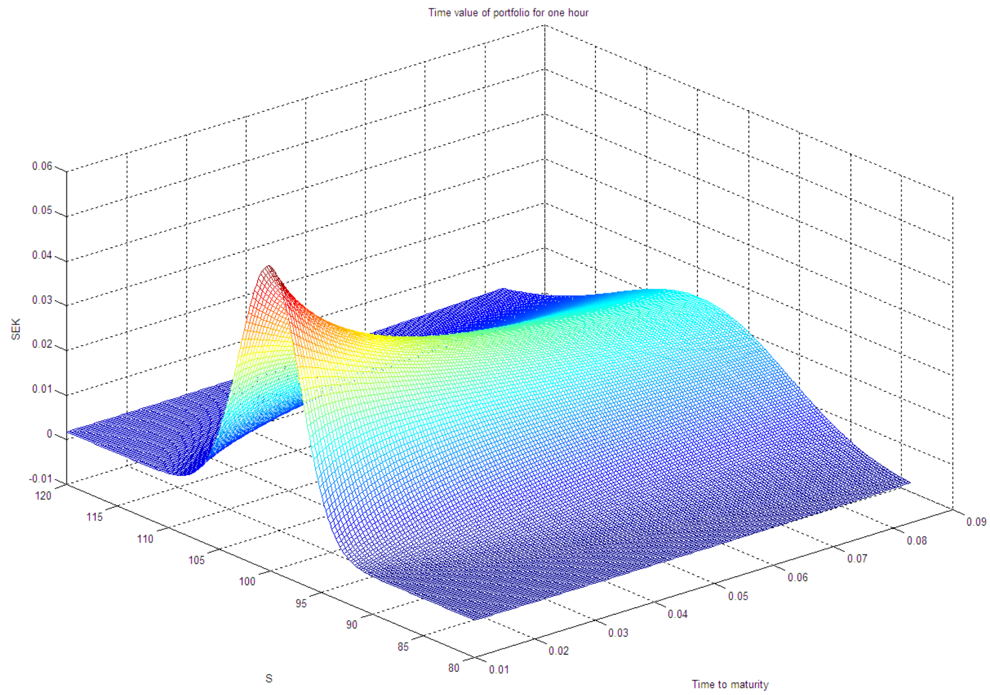


Figure 3.6: The change in time value of a portfolio from one hour to another.

In Figure 3.7 a red line describes the portfolio's theta ATM. As may be seen the theta, or time value change, ATM of the portfolio follows the upper frontier closely which leads to the conclusion that most of the positive daily results of the portfolio is caused by change in time value. The last small difference up to the most extreme positive results are probably interest rate on a short position in the stock.

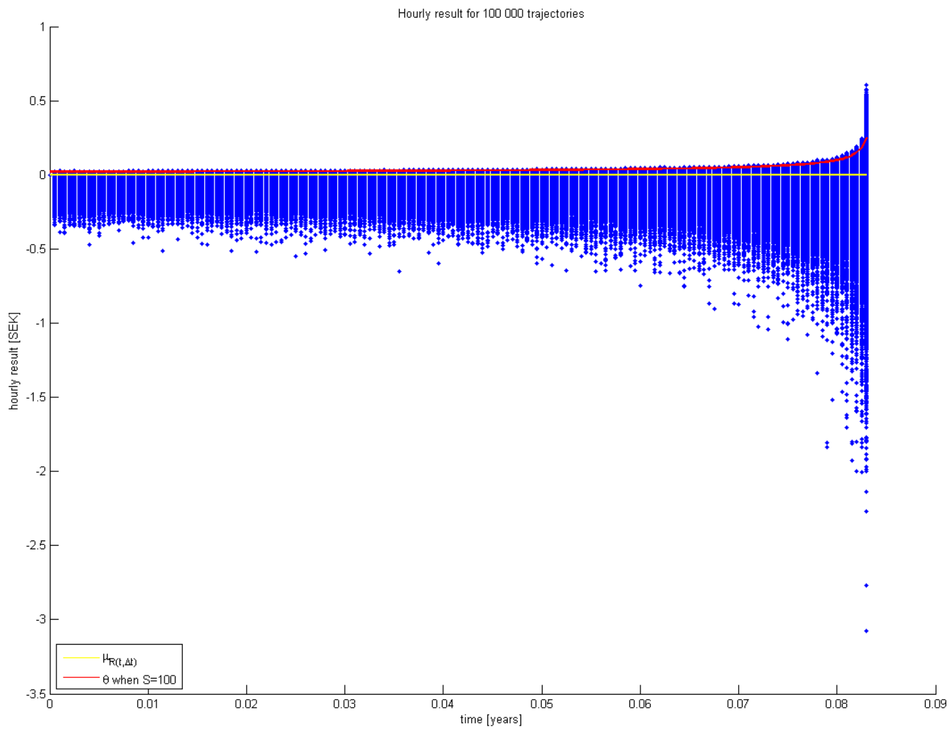


Figure 3.7: Scatter plot of hourly result over time for 100 000 GBM trajectories with underlying $\sigma = 0.295$ and $\mu = 0.035$.

## 3.3 Continuous security trading

Assumption number six of the B&S framework, that trading in the option's underlying stock is continuous, is often unproblematic since options are emitted mostly on stocks of large companies. However, if an option is emitted on a more obscure stock where only a few transactions take place this may cause losses due to:

- adjusting the hedge may be more expensive
- a less traded stock is more likely to have very heavy tailed logreturns which increases the error of assuming a normal distribution.

If trading in the underlying stock is occasional or non existing the market is said to be illiquid.

# Chapter 4

# Method

A way of splitting model risk into its components and measuring them is by Monte Carlo and bootstrap based simulation. Monte Carlo is used in this thesis to evaluate the model when the assumption of the distribution of logreturns is correct. Doing this opens the possibility to isolate and measure the model's sensitivity to a badly estimated input parameter. Bootstrapping is used to measure the risk originating from the assumption of the distribution of the logreturns and non constant volatility among other factors.

## 4.1 Simulation algorithm

The simulations follows the concept stated in the introduction of this paper, that is as shown in Figure 4.1. Category one is a Monte Carlo based GBM and category two bootstrapped trajectories from market data.

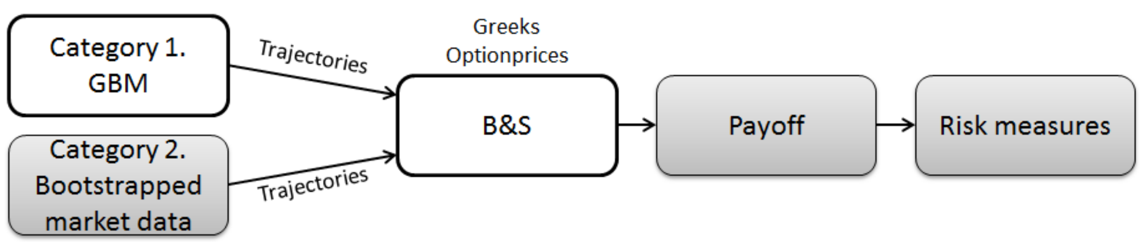


Figure 4.1: Description of the methodology of the simulations.

The simulations are set up according to the following algorithm:

1. simulate a trajectory as a GBM or bootstrapping market data (Sections 2.1 - 2.4)
2. calculate the options' prices and deltas for each time step with the B&S formula (Section 2.2)

3. adjust the position in the stock to cancel the options delta at each time step (Section 2.5)
4. calculate the accumulated result of the portfolio from time zero to $T$ (Section 2.6)
5. repeat steps 1 to 4 $n$ times
6. calculate mean, standard deviation, $VaR_{99\%}$ and $ES_{99\%}$ on the $n$ accumulated results (Section 2.7)
7. repeat step 1 to 6 $m$ times
8. calculate mean of the $m$ statistics calculated in step 7.

Throughout Chapters 5 to 8 a standard setup of the parameters for simulations and pricing will be used, with the exception of the parameter which is investigated.

## 4.2 Framework

Throughout the report the following assumptions from the B&S framework will be considered true:

- Short selling is permitted
- There are no arbitrage opportunities
- The stock pays no dividends.

If the stock would pay dividends they may be handled by simply subtracting the discounted dividends from the spot price.

### 4.2.1 Default parameters

In order to make the parameters as realistic as possible and facilitate comparison the drift and volatility of the GBM was set equal to the mean and standard deviation of the logreturns of Ericsson B. If nothing else is mentioned the following parameters will be used throughout the rest of the report:

$\mu_{GBM} = \hat{\mu}_{EricB} = 0.035$ per year

$\sigma_{GBM} = \hat{\sigma}_{EricB} = 0.295$ per year

$S(0) = 100$ SEK

$T = {}^{1}\!/_{12}$ years

$K = 100$ in percent of $S(0)$, that is ATM

$r_f = \mu_{GBM}$

$\sigma_{B\&S} = \sigma_{simulations} =$ the simulated standard deviation

$w_c = -1$

$w_p = -1$

$\triangle t = \frac{1}{250 \cdot 8}$ years $= 1h$

*hedgestep* $= \triangle t =$ the time interval between hedge adjustments.

The put and call options with the parameters above has the following values at time zero:

$$\begin{cases} P(S,t) = P(100,0) = 3.2478 \\ C(S,t) = C(100,0) = 3.5390 \end{cases} \tag{4.1}$$

which implies a premium to the market maker of SEK 6.7867 at time zero.

A more thorough description of the different volatilities may be necessary. When the GBM is simulated the volatility is set to $\sigma_{GBM}$ but since the trajectories are rather short the standard deviation of the simulated trajectory may differ and is denoted $\sigma_{simulations}$. The volatility used when pricing and calculating greeks is denoted $\sigma_{B\&S}$. $\hat{\sigma}_{EricB}$ is the standard deviation of Ericsson B estimated from the actual time series.

### 4.2.2 Number of simulations

$VaR_{99\%}$ will be calculated based on the entire number of trajectories, $n$. To decrease the standard deviation of $VaR_{99\%}$ and $ES_{99\%}$, simulations of $n$ trajectories will be carried out $m$ times. Which yields that $VaR_{99\%}$ and $ES_{99\%}$ is based on a sample of size $mn$ trajectories.

| No of simulations, $n$ | | 25 000 | 50 000 | 75 000 | 100 000 |
|---|---|---|---|---|---|
| $\mu_{m=100}$ | *R(0,T)* | -0.00348 | -0.00352 | -0.00345 | -0.00335 |
| | $VaR_{99\%}$ | 0.61727 | 0.61855 | 0.61779 | 0.61921 |
| | $ES_{99\%}$ | 0.85486 | 0.85776 | 0.85753 | 0.85788 |
| $\sigma_{m=100}$ | *R(0,T)* | 0.00152 | 0.00104 | 0.00086 | 0.00074 |
| | $VaR_{99\%}$ | 0.01304 | 0.01021 | 0.00784 | 0.00662 |
| | $ES_{99\%}$ | 0.02026 | 0.01550 | 0.01176 | 0.01014 |
| *Absolute error* $m = 10$ | *R(0,T)* | 0.00094 | 0.00065 | 0.00053 | 0.00046 |
| | $VaR_{99\%}$ | 0.00808 | 0.00633 | 0.00486 | 0.00411 |
| | $ES_{99\%}$ | 0.01256 | 0.00961 | 0.00729 | 0.00628 |
| *Relative error* (%) $m = 10$ | *R(0,T)* | 27.06 | 18.38 | 15.45 | 13.71 |
| | $VaR_{99\%}$ | 1.31 | 1.02 | 0.79 | 0.66 |
| | $ES_{99\%}$ | 1.47 | 1.12 | 0.85 | 0.73 |

Table 4.1: Calculated errors at 95% level of confidence of *R(0,T)*, $VaR_{99\%}$ and $ES_{99\%}$ when $m = 100$ for varying $n$. The trajectories were simulated as GBMs.

To estimate confidence intervals at level 95% of *R(0,T)*, $VaR_{99\%}$ and $ES_{99\%}$ mean and standard deviation was calculated based on a sample of $m = 100$ to get better numbers. This was done with the default parameters set earlier in this section and resulted in the figures shown in Table 4.1. Trajectories were simulated as GBMs. The absolute errors shown are based on simulations where $m = 10$ which were calculated according to

$$absolute\,error = \frac{1.96\sigma_{m=100}}{\sqrt{10}} \tag{4.2}$$

and are shown in the same table. Relative error is defined as $Absolute\,error/\mu_{m=100}$. Note that the standard deviation of the simulated result is approximately twice the size for $n = 25\ 000$ compared to $n = 100\ 000$ since $\sigma_{R(0,T)} \sim \frac{1}{\sqrt{n}}$ [1]. Confidence intervals are given by

$$\mu_{m=100} - absolute\,error < E\left[R\left(0,T\right)\right] < \mu_{m=100} + absolute\,error \tag{4.3}$$

When simulations are done throughout this thesis the setup of $n = 100\ 000$ and $m = 10$ will be considered as a part of the default setting. This is since the errors may be considered as sufficiently small for the kind of analysis carried out. Especially errors in the risk measures $VaR_{99\%}$ and $ES_{99\%}$ which most of the analysis will be based on are small. Another factor is the time to do the computations which are at a practical maximum with the resources given using these settings. The relation between $m$ and $n$ has shown to be unimportant, i. e. double $m$ and divide $n$ by two gives the same errors. Note that these calculations are applied only at the default setting and should only be considered as a hint when this is changed.

To relate the absolute error to something, the premium that a market maker gets at time zero is SEK 6.7867 according to equation 4.1. In other words the absolute errors of $ES_{99\%}$ and $VaR_{99\%}$ are only 0.06% and 0.09% of the premium respectively.

[1] Hull 2009, p430

The analog confidence intervals when bootstrapping trajectories using the default parameters are shown in Table 4.2 below. As expected, the absolute standard errors of $ES_{99\%}$ and $VaR_{99\%}$ are larger than when simulating trajectories as a GBM, 0.13% and 0.26% of the market maker's premium respectively.

| No of simulations, $n$ | | 100 000 |
|---|---|---|
| $\mu_{m=100}$ | $R(0,T)$ | -0.01440 |
| | $VaR_{99\%}$ | 1.10473 |
| | $ES_{99\%}$ | 1.65801 |
| $\sigma_{m=100}$ | $R(0,T)$ | 0.00131 |
| | $VaR_{99\%}$ | 0.01460 |
| | $ES_{99\%}$ | 0.02908 |
| Absolute error $m = 10$ | $R(0,T)$ | 0.00081 |
| | $VaR_{99\%}$ | 0.00905 |
| | $ES_{99\%}$ | 0.01802 |
| Relative error (%) $m = 10$ | $R(0,T)$ | 5.63 |
| | $VaR_{99\%}$ | 0.81 |
| | $ES_{99\%}$ | 1.09 |

Table 4.2: Calculated errors at 95% level of confidence of $R(0,T)$, $VaR_{99\%}$ and $ES_{99\%}$ when $m = 100$ for bootstrapped trajectories.

### 4.2.3 Contract parameters

Contract parameters are set when the deal between the market maker and the customer is entered and are known. In this section risks of a variety of contracts are measured.

#### 4.2.3.1 Strike

As Figure 4.2 shows, the strike of the contract will not have any significant effect on the expected return, $\mu_{R(0,T)}$, due to the delta hedging. However the standard deviation of the expected return is dependent on the different strikes, and with this follows that $VaR_{99\%}$ and $ES_{99\%}$ will be affected as well. A change of the strike will not affect these measurements in a linear way, but instead they reach their maximum when the strike is a bit above ATM. This is explained by the larger need of adjustments of the hedge, gamma, when the spot price of the underlying is close to the strike. Close to the strike a small change in the stock price implies a less delta neutral position than the same change further away from the strike. When the position is less delta neutral the changed spot price will affect the result more. Worth noticing is that $VaR_{99\%}$ is lower than one standard deviation for cases where the strike is set far from the current spot price, this is explained by the fact that $VaR_{99\%}$ is calculated historically as defined in Chapter 2.

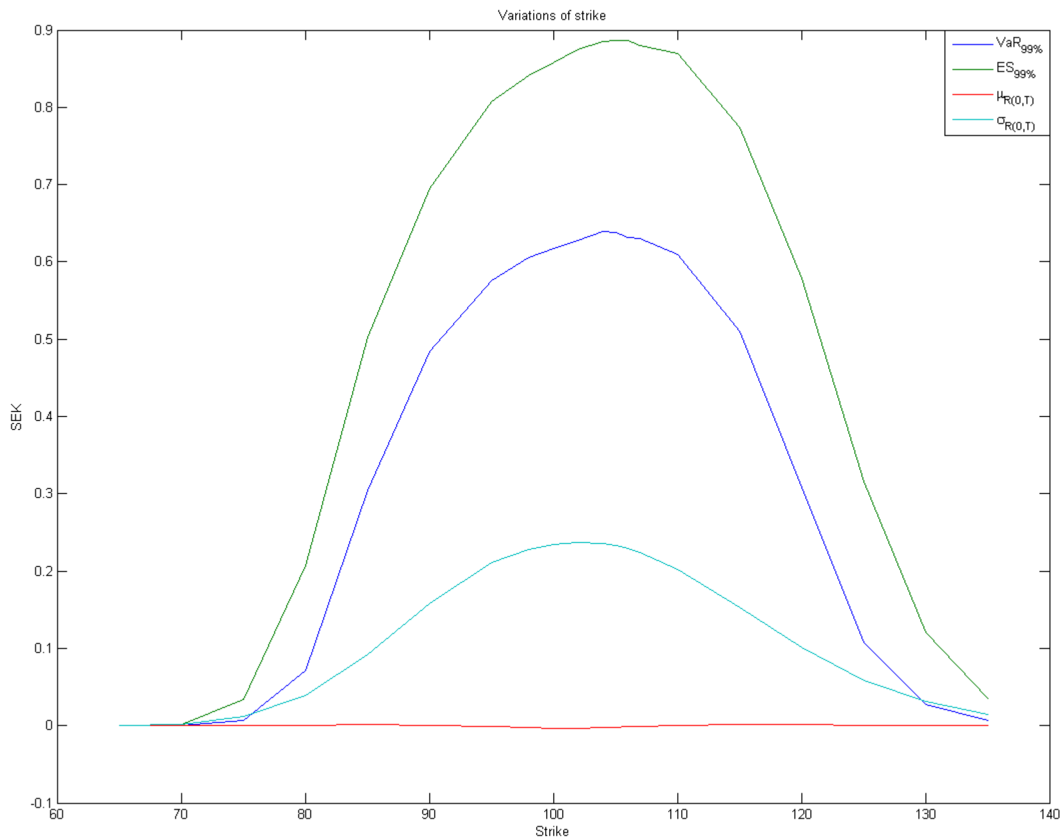


Figure 4.2: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ of the result of the portfolio depending on the strike.

#### 4.2.3.2 Term to maturity

Term to maturity is a contract parameter which is defined as the time to maturity at time zero, i. e. $T$. As Figure 4.3 shows, the length of a contract does not have any significant effect on the outcome of the results.

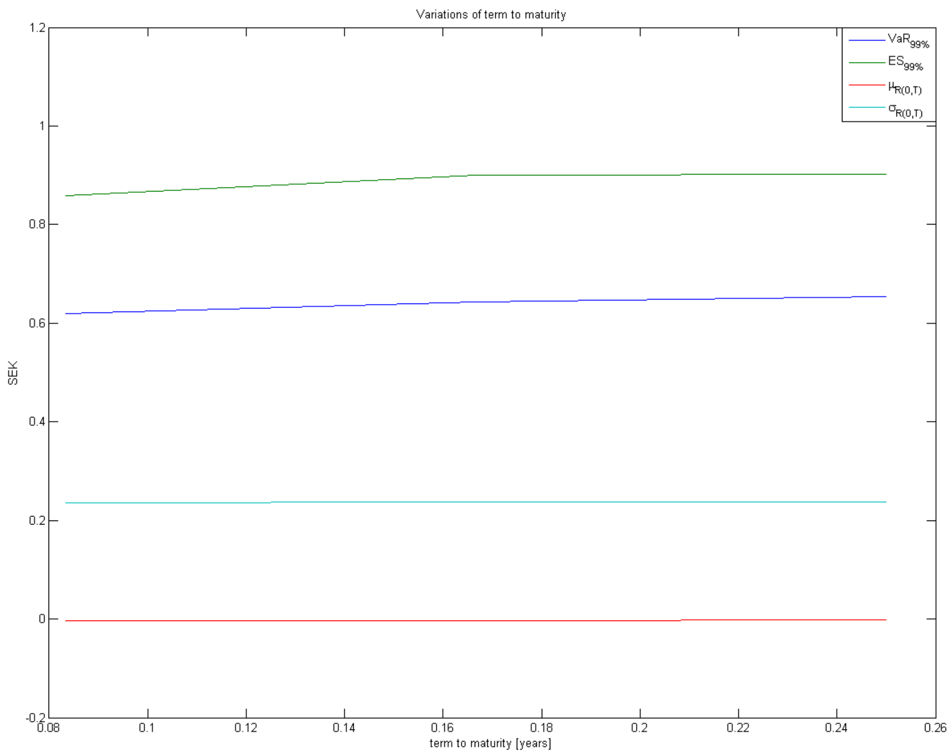


Figure 4.3: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ of the portfolio's result depending on T.

Although contracts with a longer term to maturity will result in small increases for $\sigma_{R(0,T)}$, $VaR_{99\%}$ and $ES_{99\%}$ the size of these changes are almost negligible. A parallel can be drawn to Figure 3.3 which illustrates that $\mu_{R(t-\triangle t,t)}$ is zero. In the theoretical world the B&S formula will generate a price for the options so that $\mu_{R(0,T)}$ equals zero. This together with delta hedging yields that $\sigma_{R(0,T)}$ can be considered small, and as it turns out in Figure 4.3 the B&S formula is robust for a longer term to maturity.

#### 4.2.3.3 Reference values

In Table 4.3 the risks of some contracts using the default settings are presented as reference values for coming comparisons.

| Reference data | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 0.00070 | -0.00337 | 0.00064 | 0.00143 | -0.00283 | -0.00054 |
| $\sigma_{R(0,T)}$ | 0.15744 | 0.23482 | 0.20267 | 0.10077 | 0.23722 | 0.23192 |
| $VaR_{99\%}$ | 0.48376 | 0.61792 | 0.60995 | 0.30797 | 0.64229 | 0.66172 |
| $ES_{99\%}$ | 0.69479 | 0.85743 | 0.87017 | 0.57903 | 0.90037 | 0.93608 |

Table 4.3: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ of the result of the portfolio for some K and T.

### 4.2.4 View of illiquidity

When transactions in a stock only appears occasionally, i. e. is not continuously traded the stock is said to be illiquid. This may cause problem when hedging since the transaction may be delayed or even impossible to carry out. However, if given an enough advantageous price a counter party will react on the opportunity. From this point of view the illiquidity may be modeled as paying some extra when buying and getting some less than the last price when selling the stock. Note that this way of looking at illiquidity is analog to a transaction cost, $\xi$, defined in Section 2.6.

Another view of illiquidity is to focus on the fact that time between transactions is longer but when one is carried out the probability of a large jump in the stock price is larger. This behavior may be seen when comparing histograms of logreturns of Ericsson B, Figure 6.2, and Black earth farming (BEF), Figure 6.5. In comparison, BEF is more heavy tailed than Ericsson B since many logreturns are equal to zero due to illiquidity but when a change appears it is larger. As expected this behavior is more likely to appear when looking at hourly data than daily since even most illiquid companies have some transactions every day.

# Chapter 5

# Parameter model risk

The model's risk originating from bad estimation of input parameters may behave very different from time to time. In this section the model's sensitivity with respect to each parameter is investigated and measured in terms of $ES_{99\%}$ and $VaR_{99\%}$. The parameters have been divided into:

- Pricing parameters
- Hedging interval

based on when they are set by the stock option market maker.

When a pricing parameter is incorrectly estimated the calculations are made in line with the assumption of an arbitrage free market, this implies that over time the options are traded at the correct price according to the B&S formula. Otherwise, arbitrage opportunities would arise. When a parameter is badly set by the market maker the volatility used for calculating delta is different from the simulated volatility. If delta is calculated with a different volatility than $\sigma_{B\&S}$, which is the correct one, then it is denoted $\sigma_{hedge}$. The notation of risk free rate will be analog, i. e. if the risk free rate used to calculate the hedge is different from $r_f$ then it is denoted $r_{f-hedge}$.

## 5.1 Pricing parameters

Pricing parameters are set by the market maker in order to valuate the option and calculate greeks. These parameters are unknown but estimated to describe the market. An interesting factor from the model risk point of view is the sensitivity to badly set pricing parameters.

### 5.1.1 Volatility

The sensitivity of the B&S formula to a badly set volatility parameter, $\sigma_{hedge} \neq \sigma_{B\&S} \approx \sigma_{GBM}$, is shown in Figure 5.1. Notice that the risk in terms of $VaR_{99\%}$ and $ES_{99\%}$ are smallest around the underlying's actual volatility, i.e. $\sigma_{hedge} = \sigma_{GBM} = 0.295$. Another thing worth pointing out is that the risk originating from setting the volatility used for hedging too low is larger than setting it too high.

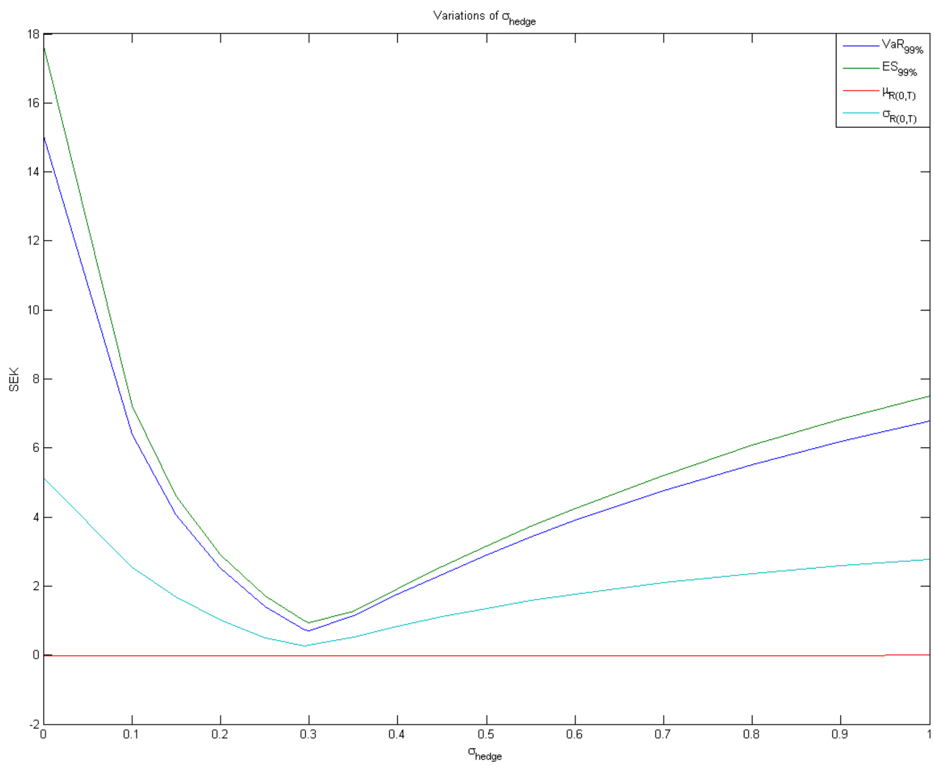


Figure 5.1: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio depending on a badly set volatility $\sigma_{hedge}$ of the underlying.

Table A.1 shows some contracts' sensitivity to a badly set hedging volatility. When comparing $K$ = [100 110] for $T = 1/12$ the behavior of $VaR_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ may seem a bit odd since when standard deviation increases $VaR_{99\%}$ decreases. The explanation to this is a combination of the mean's effect on $VaR_{99\%}$ and the fact that historical $VaR_{99\%}$ is calculated.

The same table with the resulting difference relative to the reference values in Table 4.3 due to the shift is shown in Table 5.1. As expected $\sigma_{R(0,T)}$, $VaR_{99\%}$, and $ES_{99\%}$ generally increases when hedging with a volatility set too high. This combined with contracts that run over a longer period will result in that the risk in the portfolio continues to grow even higher over time.

The seemingly large shifts for $\mu_{R(0,T)}$ in Table 5.1 may be a bit misleading. Since $\mu_{R(0,T)}$ is very close to zero even a small absolute change will yield a large shift relatively.

| $\sigma_{hedge} = 1.2\sigma_{GBM}$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 487% | -199% | 110% | 177% | -194% | -491% |
| $\sigma_{R(0,T)}$ | 219% | 136% | 211% | 238% | 205% | 281% |
| $VaR_{99\%}$ | 43% | 92% | 52% | -18% | 135% | 127% |
| $ES_{99\%}$ | 8% | 55% | 15% | -52% | 85% | 74% |

Table 5.1: Change in $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio due to a shift $\sigma_{hedge} = 1.2\sigma_{GBM}$ for some K, T.

### 5.1.2 Risk free rate

The consequences of a badly set risk free rate is displayed in Figure 5.2. The only risk free rate that was changed from the predefined in Section 4.2 was the one used when calculating delta. Calculating delta with the risk free rate set too high implies too high delta and when using a smaller risk free rate than the correct one the resulting delta is too low. The plot shows the accumulated result when varying $r_{f-hedge}$. As expected the standard deviations and risk measures are smallest when it is close to $r_f = 0.035$, which is the correct one.

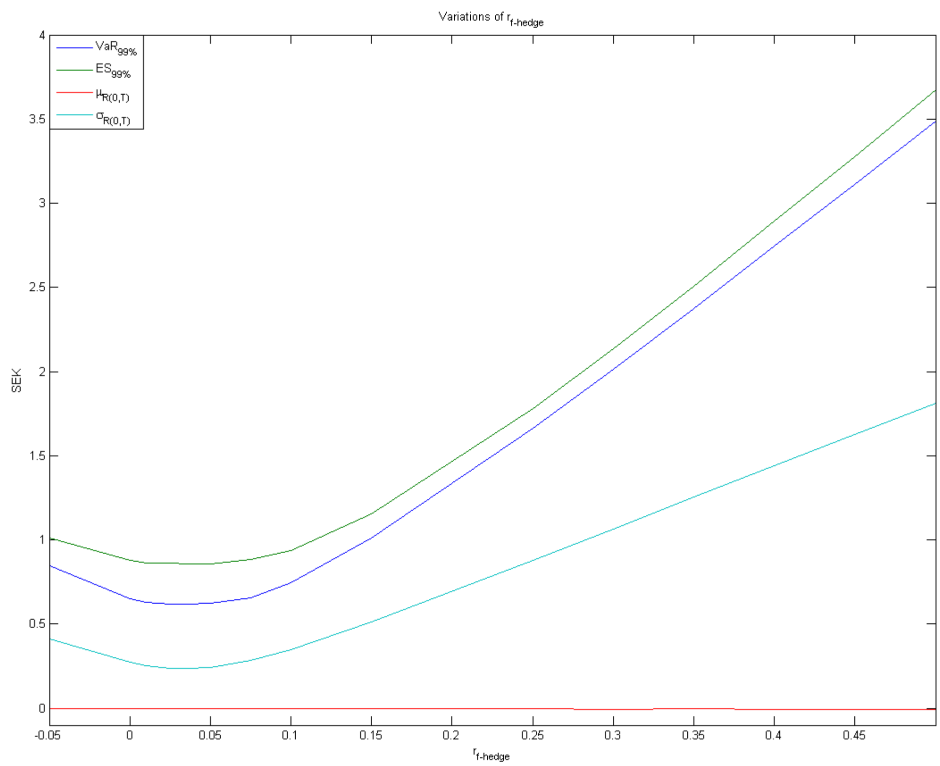


Figure 5.2: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio depending on a badly set risk free rate $r_{f-hedge}$.

Table A.2 shows properties of the accumulated result when using a 20% too high risk free rate, and Table 5.2 shows the change from the reference values in Table 4.3. The result is that the highest $VaR_{99\%}$ and $ES_{99\%}$, which occurs when $K = [100\ 110]$, are slightly decreased or unchanged, and for $K = 90$ the risks are experiencing a raise. Worth noticing is that for higher strikes $VaR_{99\%}$ and $ES_{99\%}$ actually decreases. Term to maturity affect the result in a way that setting $r_{f-hedge}$ badly during a longer time period increases the change in $VaR_{99\%}$ and $ES_{99\%}$.

| $r_{f-hedge} = 1.2r_f$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $^1/_{12}$ | $^1/_{12}$ | $^1/_{12}$ | $^1/_{12}$ | $^2/_{12}$ | $^2/_{12}$ |
| $\mu_{R(0,T)}$ | 12% | -7% | -22% | -7% | 31% | 6% |
| $\sigma_{R(0,T)}$ | 1% | 1% | 0% | 0% | 2% | 1% |
| $VaR_{99\%}$ | 4% | 0% | -2% | -5% | 0% | -4% |
| $ES_{99\%}$ | 2% | 0% | -1% | -3% | -1% | -4% |

Table 5.2: Change in comparison to the reference values in Table 4.3 when $r_{f-hedge} = 1.2r_f$ for some K and T in percent.

## 5.2 Hedging interval

In Figure 5.3 the *hedgestep's* effect on the accumulated result is plotted. If hedgestep equals $2\triangle t$ then the hedge is revaluated every second $\triangle t$ and so on. Due to computational limitations no plot for hedge interval smaller than one $\triangle t = 1$h was possible to carry out for the default setting but the result is as expected decreasing to zero as $\triangle t$ declines. When *hedgestep* grows the other extreme is hedging at time zero and just leaving the position until time $T$.

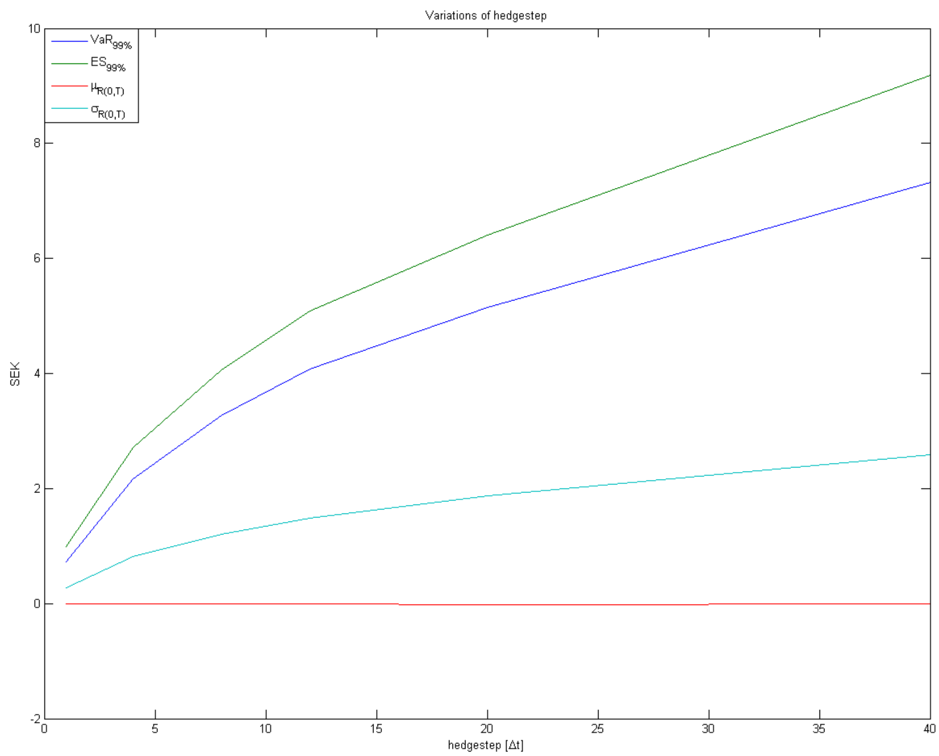


Figure 5.3: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio depending on the time intervals between hedging.

For comparison purposes the absolute results and the change relatively the reference values in Table 4.3 for the standard setup of contracts were calculated. The numbers are showed in Table A.3 and 5.3 respectively. The result of hedging more seldom is obviously an increased risk in terms of $VaR_{99\%}$ and $ES_{99\%}$ but the relative increase is smaller for contracts with longer term to maturity. Another thing worth pointing out is that the larger *hedgestep* increases the risk measures of strikes far from *S(0)* rather than close to.

| $hedgestep = 1.2\triangle t$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 391% | -271% | 199% | 259% | -191% | -349% |
| $\sigma_{R(0,T)}$ | 42% | 27% | 41% | 38% | 27% | 37% |
| $VaR_{99\%}$ | 42% | 29% | 40% | 43% | 26% | 33% |
| $ES_{99\%}$ | 38% | 25% | 36% | 41% | 21% | 29% |

Table 5.3: Change in $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when $hedgestep = 1.2\triangle t$ compared to the reference values.

## 5.3 Conclusion

Setting the risk free rate used when calculating delta wrong raises the risks when hedging with the B&S formula. As expected, when the same error is done during a longer time period the effect on the risks is increased as well. The conclusion is that the model's sensitivity to a badly set risk free rate is rather small, and the fact that estimating risk free rate is fairly easy makes the parameter non critical.

The outcome of hedging with a badly set volatility is higher standard deviation of the result and larger risks in the portfolio. The risk is greater if the volatility is set too low rather than too high, and it will continue to build up over time. The result's sensitivity to a badly set volatility is much larger than to setting the risk free rate wrong. Worth noticing is that for high strikes the risk may even decrease when setting $\sigma_{hedge}$ too high.

The choice of hedging interval is another result affecting factor which is determined by the market maker. When delta hedging the choice is simply about how much the market maker is willing to risk by not adjusting the hedge continuously, which will not be rational due to transaction costs. However, right now transaction costs are disregarded but still continuous hedging will not be possible. Increased interval between hedging increases the risk measures, especially for setups with strikes far from *S(0)*.

# Chapter 6

# Assumption model risk

In this section the object is to test the assumptions of the model and measure the risk originating from each of them. The best way of doing this is by bootstrapping market data. Unfortunately bootstrapping makes it impossible to isolate the impact of each factor, e.g. it is impossible to measure the effect of an illiquid underlying without distortion caused by non normally distributed logreturns. However, the simulation method in this section is varying between bootstrapping and Monte Carlo. Assumption model risk is divided into:

- non normal distributed logreturns
- illiquid underlying
- transaction cost

## 6.1 Data Selection

Market data for the stocks of Ericsson B and Black earth farming serves as examples of underlyings with different properties. Note that the GBM parameters are estimated from Ericsson B to simplify comparison.

### 6.1.1 Ericsson B

Ericsson B works as a foundation for data simulation and analysis in this thesis for several reasons. Primarily, it is very liquid and as close to continuously traded as the Swedish stock exchange permits. Secondly, it is possible to divide the historical prices into different market states. The selection of data are hourly observations of the last trade between 2010-05-06 and 2010-09-14[1], displayed in Figure 6.1.

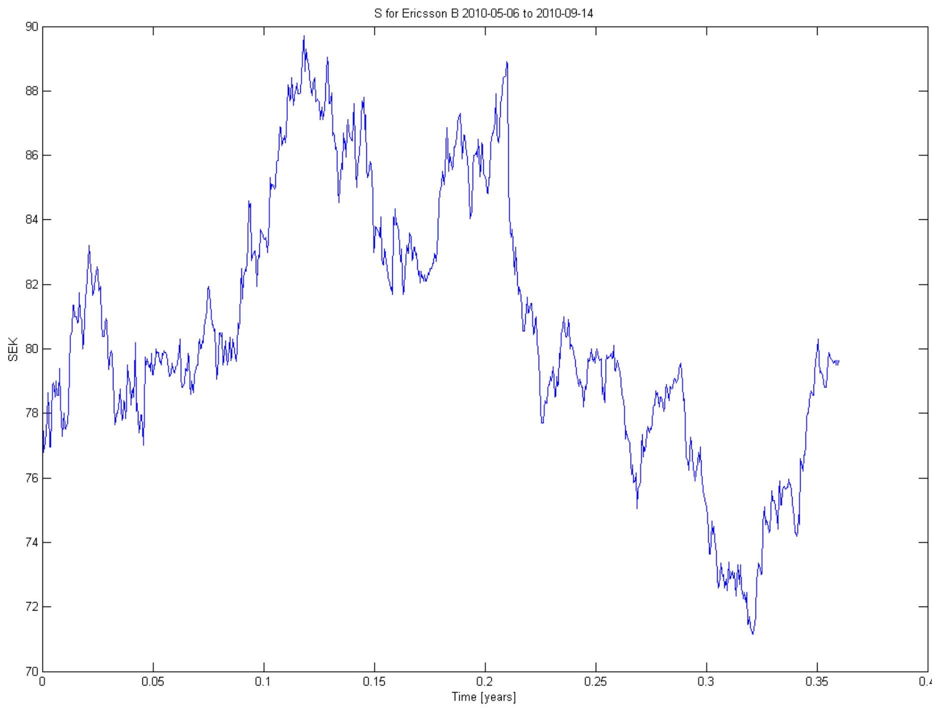


Figure 6.1: Time series Ericsson B.

[1] Bloomberg 2010-10-28 (Market data)

The logreturns are displayed in Figure 6.2 together with a fitted normal distribution. The QQ plot of Ericsson B's logreturns, in Figure 6.3, confirms the non normal property.

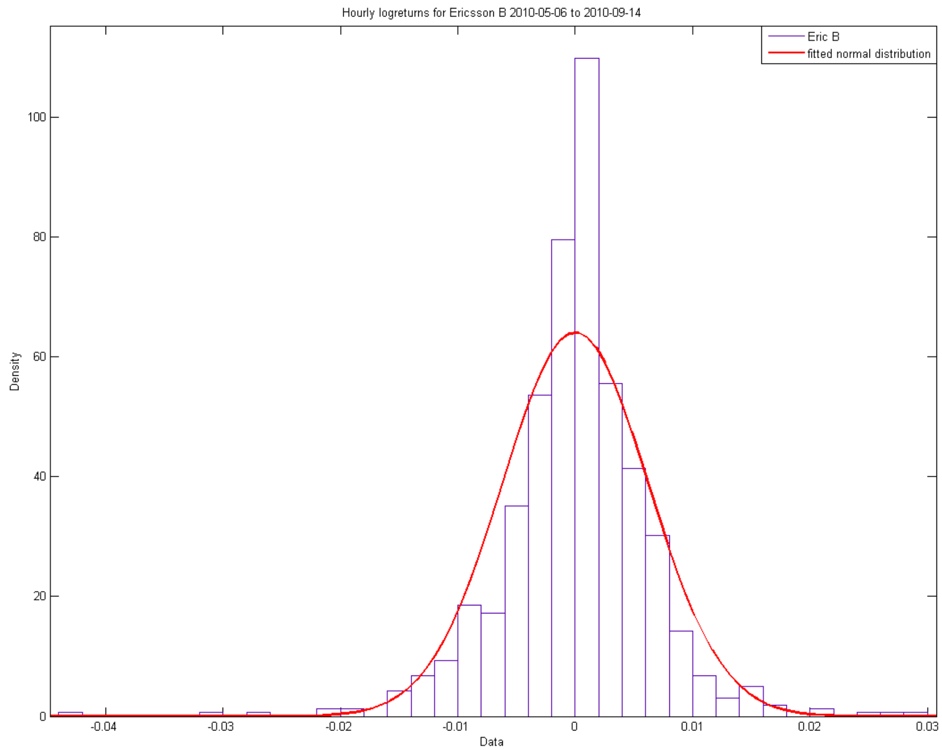


Figure 6.2: Hourly logreturns of Ericsson B with a fitted normal distribution.

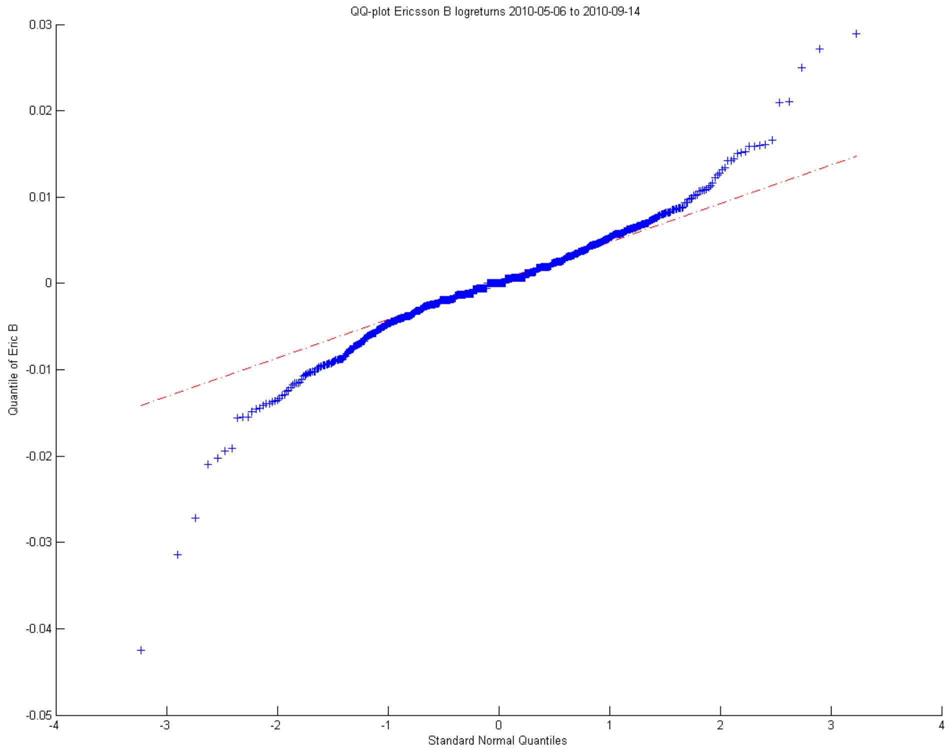


Figure 6.3: QQ plot over hourly logreturns of Ericsson B.

### 6.1.2 Black earth farming

Assumption 6 i section 2.2 reads "Security trading is continuous..." which is a very strong assumption for some stocks. There are many listed stocks where trades are done a couple of times a day at best. To test delta hedging of a short option position where the underlying is illiquid, market data for BEF has been used. The time series in Figure 6.4 shows hourly observations of the spot price between 2010-04-27 and 2010-11-09[2].

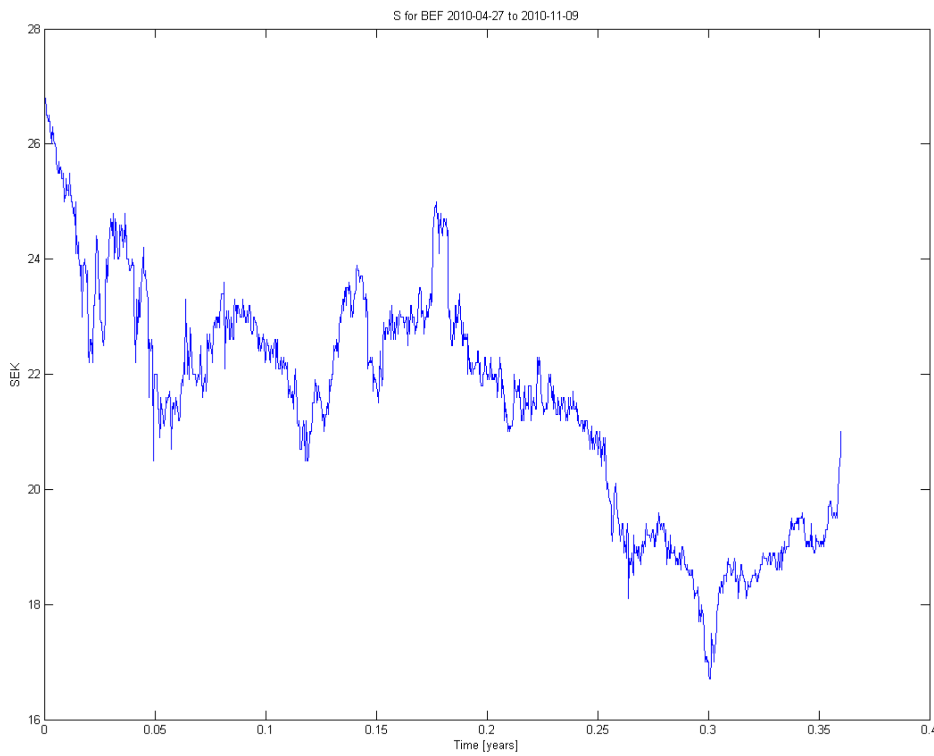


Figure 6.4: Time series BEF.

[2]Bloomberg 2010-11-09 (Market data)

The main reason for choosing this stock is that in approximately 27% of the hourly observations the price has not changed from the previous and that the traded volume is very low. Figure 6.5 shows the hourly logreturns for the same period which has been used in Section 6.3 to simulate bootstrap trajectories for illiquid underlyings.

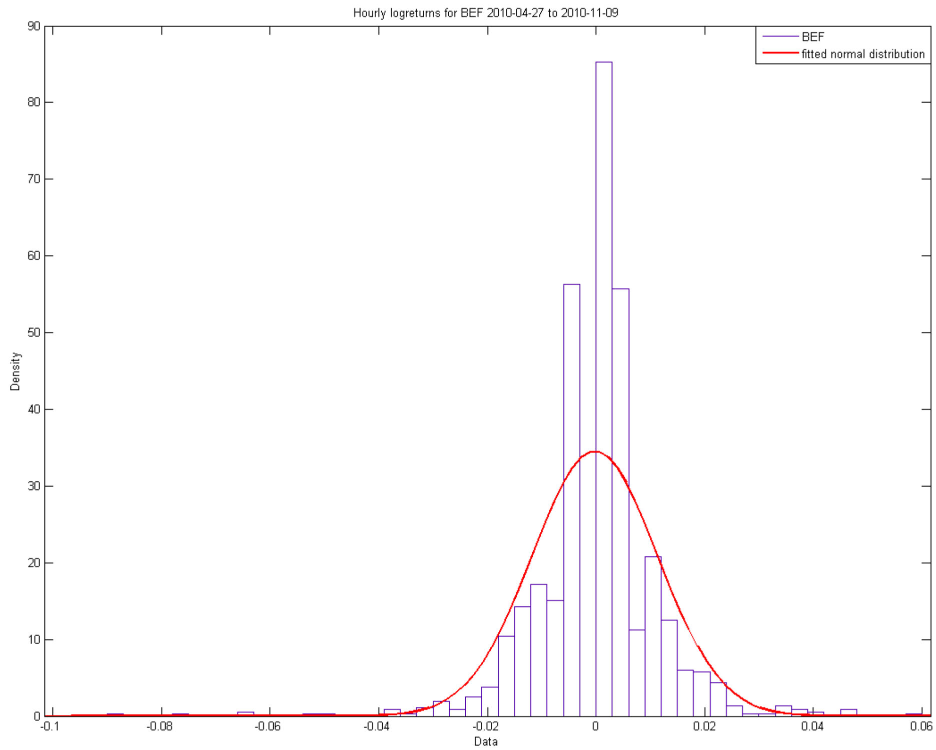


Figure 6.5: Hourly logreturns BEF with a fitted normal distribution.

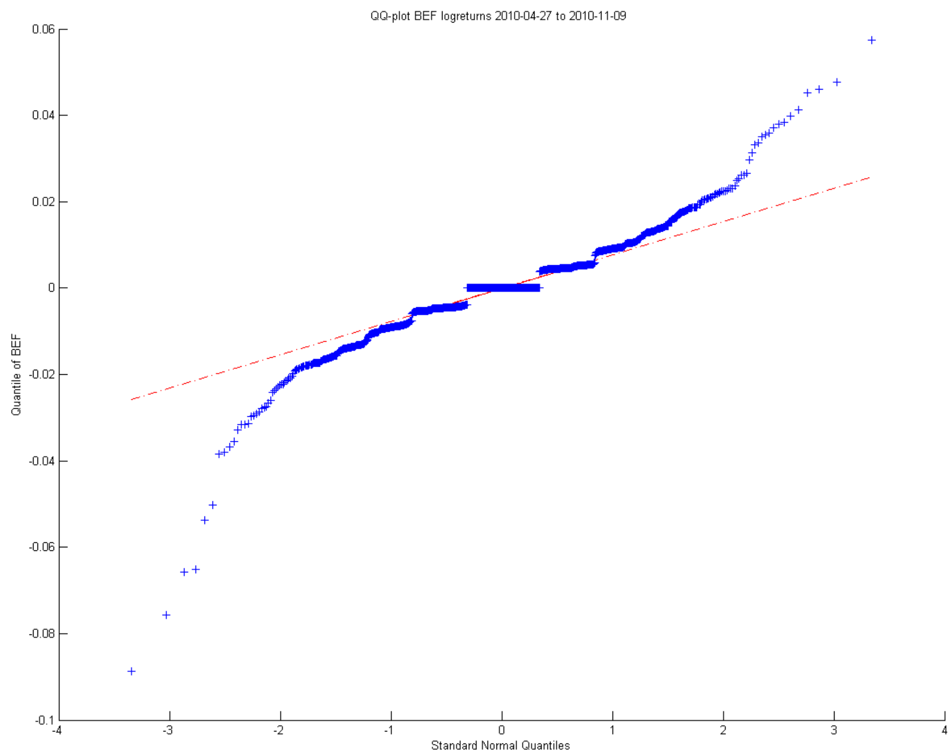


Figure 6.6: QQ plot over hourly logreturns of BEF.

The QQ plot in Figure 6.6 shows that the logreturns for BEF is more non normal distributed than Ericsson B. There are obviously securities which are much more illiquid, these are however often hard to get reliable market data for. The logreturns for both Ericsson B and BEF are, as Figures 6.2 and 6.5 shows, much more heavy tailed than their corresponding fitted normal distributions.

## 6.2 Non normal distributed logreturns

The risk caused by the assumption of normal distributed logreturns is measured by bootstrapping Ericsson B time series on hourly basis. Since the assumption tested is the normal distributed logreturns a bootstrap algorithm assuming iid logreturns is sufficient. Figure 6.2 shows how the hourly logreturns of Ericsson B differs from the normal distribution, the following simulations will measure the risk of this assumption. Table 6.1 shows the simulated risks when bootstrapping the logreturns instead of simulating a GBM. Worth pointing out is that the risk in terms of $VaR_{99\%}$ and $ES_{99\%}$ are largest for strikes around *S(0)* and that the effect of a larger $T$ is rather small, but larger for strikes different than *S(0)*. Notice that the default parameter setting of the GBMs is estimated from the time series of Ericsson B.

| Bootstrapped reference data | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 0.01395 | -0.01369 | -0.00725 | 0.00299 | -0.00870 | -0.01247 |
| $\sigma_{R(0,T)}$ | 0.26282 | 0.37790 | 0.32672 | 0.16191 | 0.39298 | 0.38794 |
| $VaR_{99\%}$ | 0.81354 | 1.10207 | 1.01799 | 0.48250 | 1.14323 | 1.17906 |
| $ES_{99\%}$ | 1.29069 | 1.64916 | 1.56458 | 0.94888 | 1.71092 | 1.78057 |

Table 6.1: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ of the result of the portfolio for some K and T.

Table 6.2 shows the change in risks relatively the reference values in Table 4.3 for some contracts and strikes. The relative increase is largest where the absolute risks are highest.

| Relative change due to non normal distribution | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 1 905% | -306% | -1 240% | 109% | -207% | -2 194% |
| $\sigma_{R(0,T)}$ | 67% | 61% | 61% | 61% | 66% | 67% |
| $VaR_{99\%}$ | 68% | 78% | 67% | 57% | 78% | 78% |
| $ES_{99\%}$ | 86% | 92% | 80% | 64% | 90% | 90% |

Table 6.2: Relative change from reference data of GBM, Table 4.3, of $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ of the result when bootstrapping non normal logreturns.

### 6.2.1 Contract parameters

#### 6.2.1.1 Strike

Figure 6.7 shows how $VaR_{99\%}$ and $ES_{99\%}$ depends on the options' strike price. The behavior is the same as when the trajectories are GBMs but the risks has almost doubled. The negative accumulated result around $K = 100$ observed when simulating GBMs has decreased even more.

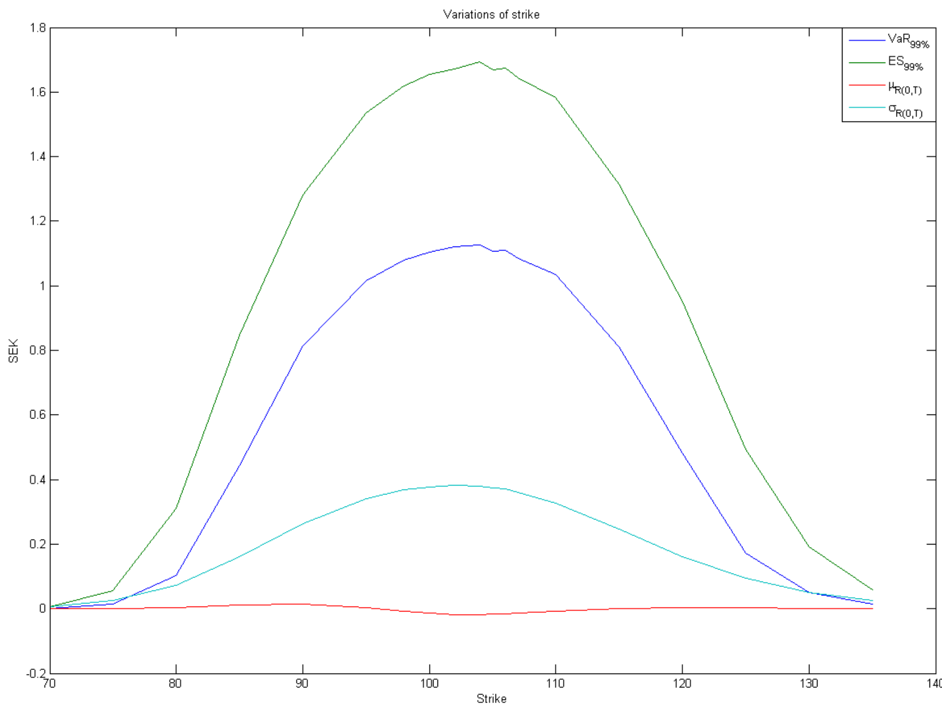


Figure 6.7: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio depending on the strike.

#### 6.2.1.2 Term to maturity

Figure 6.8 shows the same behavior as when the trajectories are GBMs, i. e. the risks are independent of the contract's term to maturity. However, all the risk measures have almost doubled.

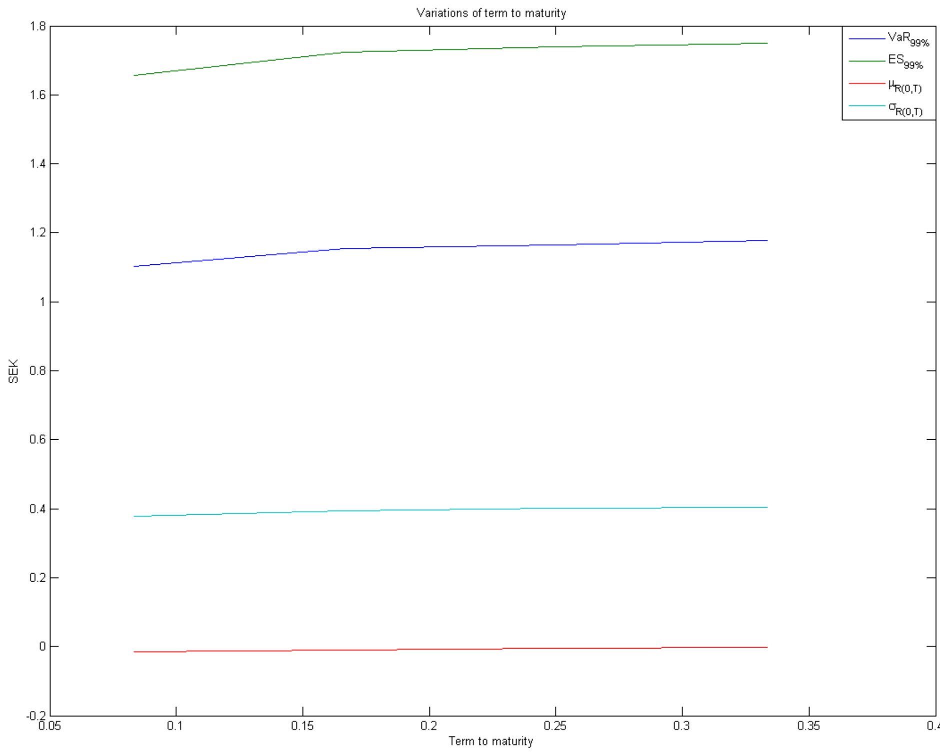


Figure 6.8: Risk measures depending on different terms to maturity, T.

### 6.2.2 Pricing parameters

#### 6.2.2.1 Volatility

Figure 6.9 shows the risks if setting the volatility used to calculate the hedge ratio improperly. The behavior is the same as when the trajectories are GBMs and the numbers almost the same. However, when $\sigma_{hedge}$ is close to the correct one the risks are higher when bootstrapping data than when using a GBM.

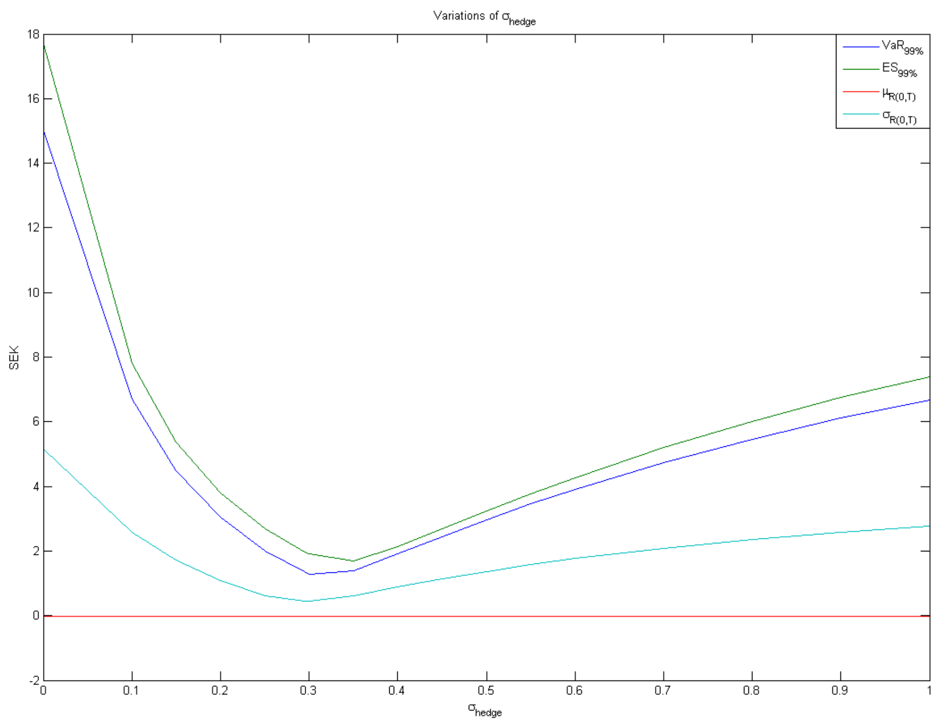


Figure 6.9: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio depending on a badly set volatility $\sigma_{hedge}$ of the underlying.

Table A.4 shows the risks when hedging a bootstrapped time series of Ericsson B with 20% too high volatility in the delta calculation for some different contracts. The risks are larger than when doing the same thing on GBMs as in Table A.1 but with the same dynamics.

Table 6.3 shows that when hedging a bootstrapped time series the increase of risk, in terms of $VaR_{99\%}$ and $ES_{99\%}$, is smaller due to an incorrectly set volatility in the delta calculation. However, the smaller relative increase is from the higher initial level for bootstrapped data in Table 6.1.

| $\sigma_{hedge} = 1.2\hat{\sigma}_{EricB}$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $^1/_{12}$ | $^1/_{12}$ | $^1/_{12}$ | $^1/_{12}$ | $^2/_{12}$ | $^2/_{12}$ |
| $\mu_{R(0,T)}$ | -226% | -55% | 623% | 692% | -51% | 400% |
| $\sigma_{R(0,T)}$ | 111% | 70% | 120% | 143% | 104% | 147% |
| $VaR_{99\%}$ | 3% | 29% | 5% | -38% | 52% | 42% |
| $ES_{99\%}$ | -15% | 4% | -16% | -46% | 16% | 5% |

Table 6.3: Change in $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio due to a shift $\sigma_{hedge} = 1.2\hat{\sigma}_{EricB}$ for some K, T.

#### 6.2.2.2 Risk free rate

The outcome when hedging with a badly set risk free rate, for an underlying based on non normal logreturns, is similar to the simulations from the GBM model in Section 5.1.2. The risks are however, according to Figure 6.10, enhanced over all tested scenarios. This implies that delta hedging using the B&S formula is not optimal for underlyings with non normal logreturns.

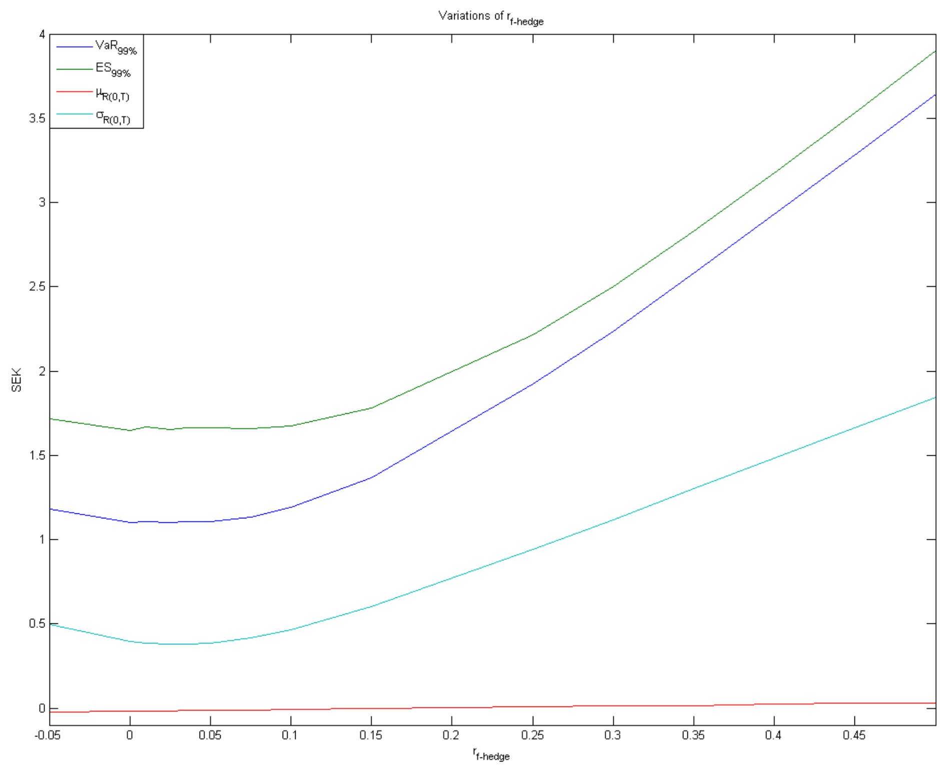


Figure 6.10: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio depending on a badly set risk free rate $r_{f-hedge}$.

Table 6.4 shows that the relative changes from Table 6.1 when $r_{f-hedge} = 1.2r_f$ are negligible. Except for $\mu_{R(0,T)}$ of options with longer term to maturity, which is increased.

| $r_{f-hedge} = 1.2r_f$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 1% | 0% | 7% | 1% | 26% | 21% |
| $\sigma_{R(0,T)}$ | 0% | 1% | 0% | 1% | 2% | 1% |
| $VaR_{99\%}$ | 2% | 0% | 0% | -2% | 1% | -3% |
| $ES_{99\%}$ | 1% | 1% | -1% | -1% | 0% | -2% |

Table 6.4: Change in comparison to the bootstrapped reference values in Table 6.4 when $r_{f-hedge} = 1.2r_f$ for some K and T in percent.

### 6.2.3 Hedging interval

The effect of increased time between each adjustment of the hedge is higher risk which is shown in Figure 6.11. As expected the risks and standard deviations are going down to zero as the time between hedging occasions decreases.

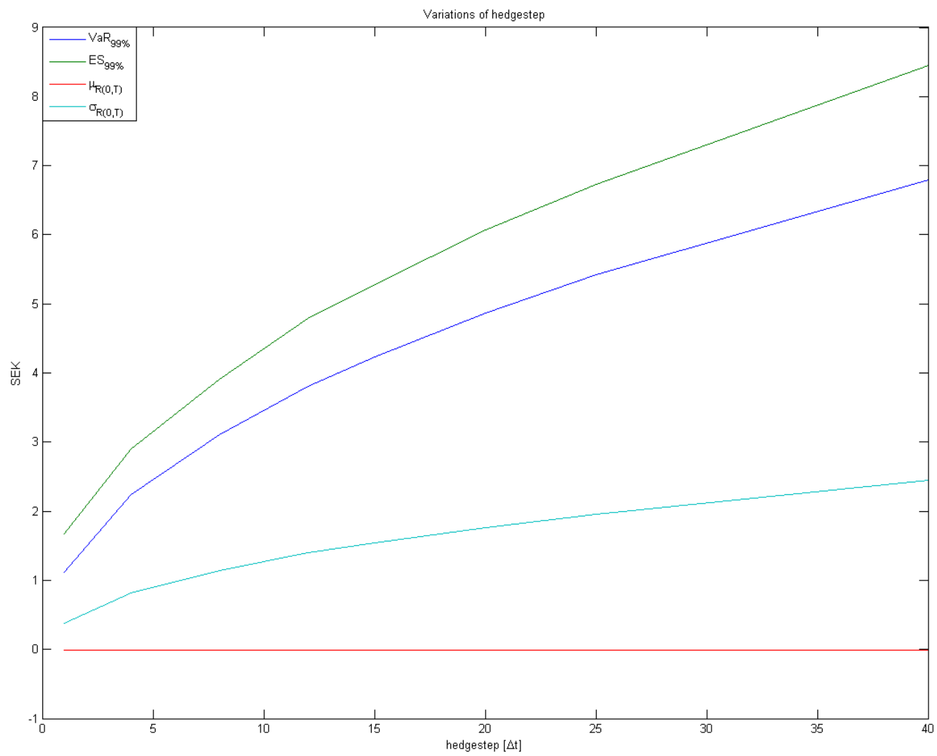


Figure 6.11: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio depending on the time intervals between hedge adjustments.

## 6.3 Illiquid underlying

Two different methods have been used to simulate an illiquid stock. In the first part the underlyings trajectories have been generated by bootstrapped data from BEF. This is to show the effect of non normal distributed logreturns. In the second part a transaction cost has been used instead. The transaction cost can be seen as a way of simulating an illiquid underlying, i. e. a spread between bid and ask prices when market activity in a certain security is low. Then a buyer of the security would have to bid a higher price, seen as a premium for the poor liquidity, for the transaction to be made, likewise a seller would have to ask for a lower price to be able to sell the security.

### 6.3.1 Non normal distributed logreturns

Figures 6.5 and 6.6 show that the logreturns for BEF are much more heavy tailed than a normal distribution. This data have been used to bootstrap trajectories for an illiquid underlying. Table A.6 shows the outcome for $\mu_{R(0,T)}$, $\sigma_{R(0,T)}$, $VaR_{99\%}$ and $ES_{99\%}$ for the standard setup of scenarios. As expected the risks have increased as the logreturns got less normal distributed. This is confirmed by Table 6.5 which shows the relative changes for the same scenarios but with bootstrapped trajectories from BEF instead. The general trend is that the risks increases over all and more when the strike is not ATM but does decrease slightly with a longer time to maturity.

| Bootstrapped data from BEF | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | -101% | -190% | -364% | -344% | -200% | -138% |
| $\sigma_{R(0,T)}$ | 168% | 107% | 135% | 299% | 108% | 113% |
| $VaR_{99\%}$ | 164% | 110% | 122% | 315% | 109% | 105% |
| $ES_{99\%}$ | 150% | 110% | 120% | 232% | 110% | 106% |

Table 6.5: Relative change in $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when underlying trajectories are built up by data bootstrapped from BEF compared to Ericsson B.

A more quantitative way of studying the effect of non normal distributed underlyings is displayed in Figure 6.12 where $VaR_{99\%}$ for the portfolio has been calculated for 75 underlyings sorted by average hourly traded volume in SEK. Even if the majority of all the companies generates fairly low $VaR_{99\%}$, it seems as there are some sort of threshold around a turnover of SEK 10 000 000 per hour above which $VaR_{99\%}$ with every underlying gets stabilized. An explanation for this is that less transactions implies higher probability that the underlying has a more heavy tailed distribution. The mid cap company resulting in $VaR_{99\%} \approx 19$SEK is HQ Bank which had a hourly loss of 82% when their license where withdrawn which increases the $VaR_{99\%}$ and $ES_{99\%}$ dramatically.

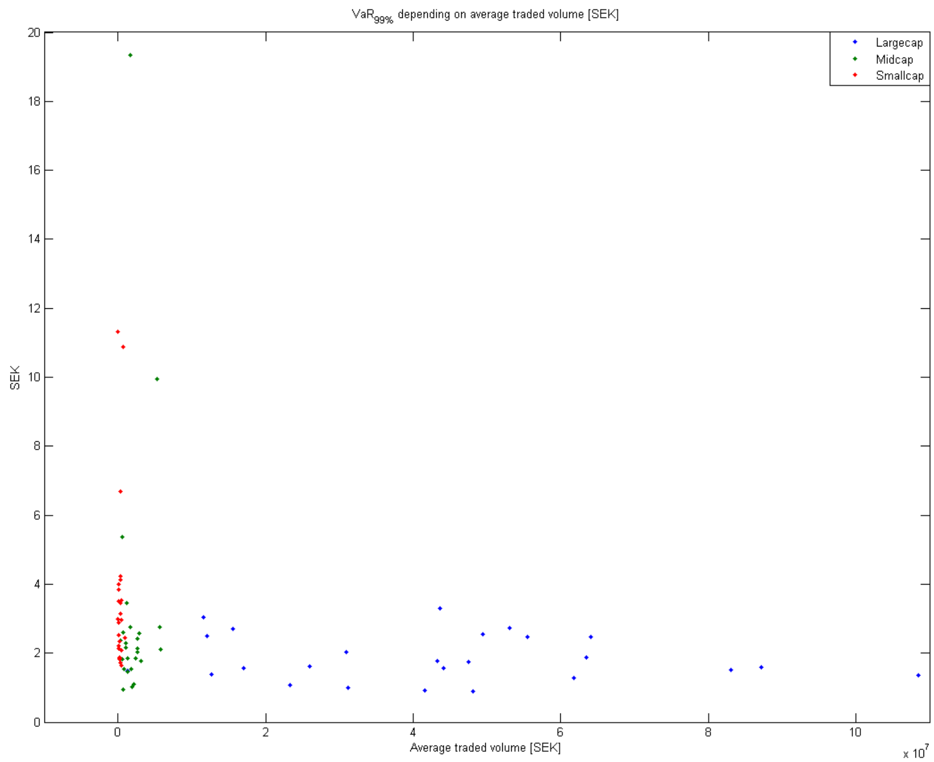


Figure 6.12: $VaR_{99\%}$ when hedging options on 75 underlyings with varying liquidity.

### 6.3.2 Liquidity premium or transaction cost

Transaction cost is a fee paid for a transaction to be executed, in this case it is used to simulate a spread between bid and ask prices i. e. a liquidity premium. At an actual market place the size of this spread varies over time and securities. In the simulations made for Table A.7 a constant premium of two percent have been used. This takes the risks up to very high levels overall. Worth noticing is that for $K = 120$, $\mu_{R(0,T)}$ is close to zero which is explained by the fact that fewer transactions is needed for being delta neutral, which also serves to explain the low risks relatively lower strikes.

The simulations does not take in consideration the effect that the hedge transaction would have on $S$. That would result in a loop where every hedge transaction made would change $S$ and therefore make the portfolio non delta neutral implying a need of a new hedge transaction changing the price and so on.

In most parts of this thesis liquidity premium has been excluded but as Figure 6.13 shows, the risks increases and the expected return decreases linearly as the transaction costs grows. Due to the large amounts of transactions necessary for being delta neutral in every time step the liquidity premium dominates the outcome of the simulations.

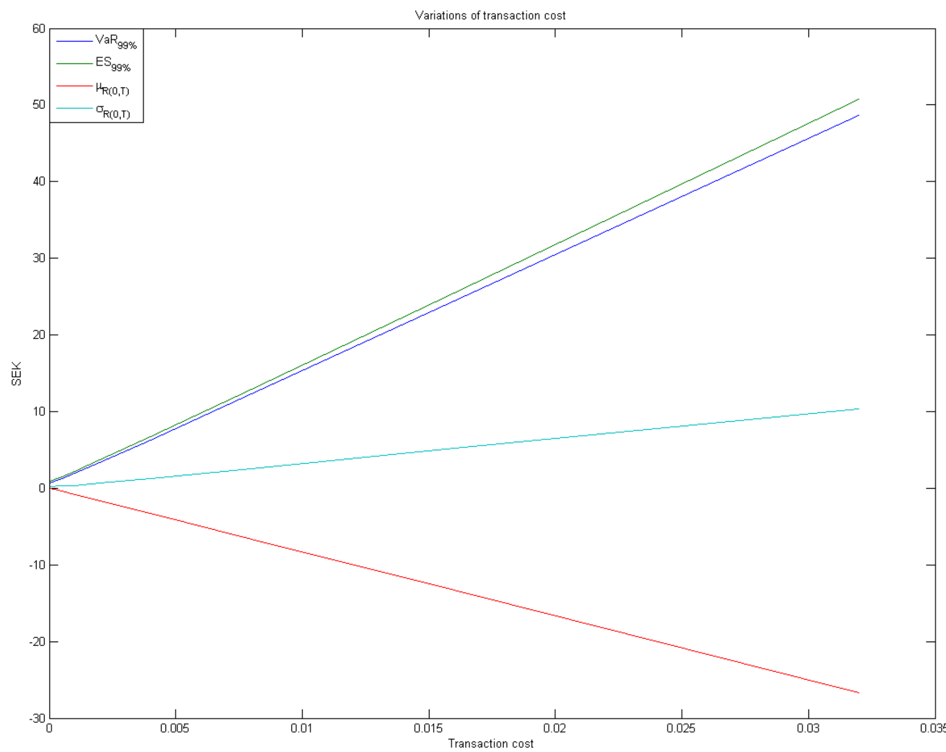


Figure 6.13: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio depending on the transaction cost.

Figure 6.13 shows that for every transaction made when hedging money is lost. This implies that when operating at a market with transaction costs another strategy than being delta neutral in every time step is necessary.

Figure 6.14 shows how $\mu_{R(0,T)}$, $\sigma_{R(0,T)}$, $VaR_{99\%}$ and $ES_{99\%}$ changes when hedging less often. In this case with a liquidity premium set to one percent. Changing the hedge at around every twelfth hour results in minimum for $VaR_{99\%}$ and $ES_{99\%}$, while $\mu_{R(0,T)}$ keeps getting less negative with longer steps between hedging.

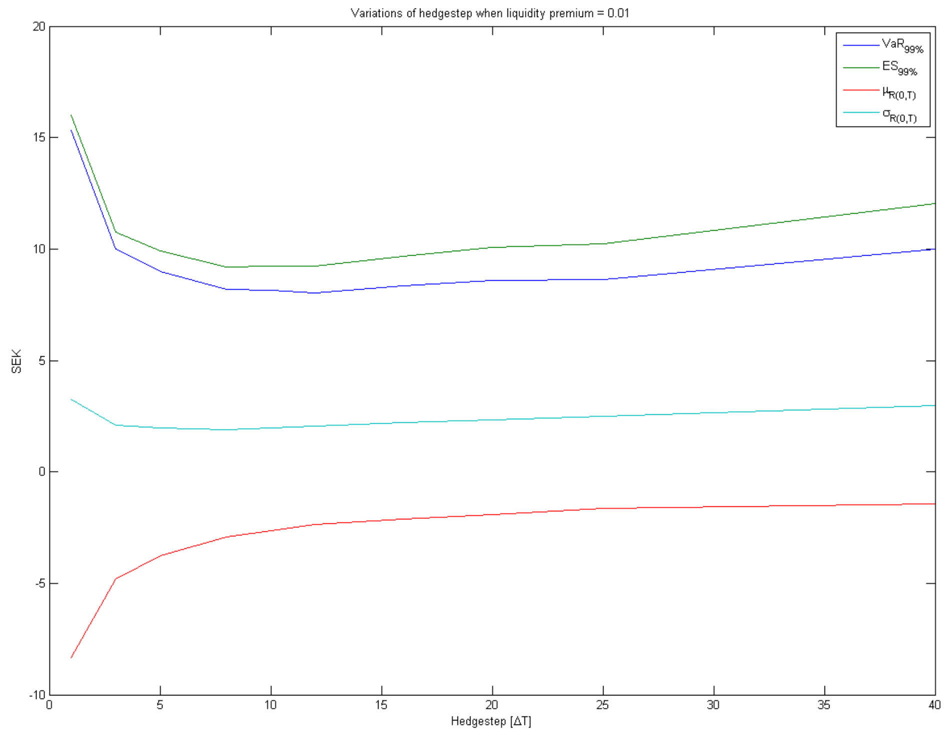


Figure 6.14: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio depending on hedgestep with a liquidity premium of 1%.

The same principle applies for Figure 6.15 where $VaR_{99\%}$ is plotted just as in Figure 6.14 but for liquidity premiums of 0.5%, 1%, 2% and 4% respectively. The figure does not only show that $VaR_{99\%}$ increases with higher liquidity premium but also that the minimum $VaR_{99\%}$ for each case occurs with longer hedging intervals. If the liquidity premium is high enough there will in general be more profitable not to hedge at all or use another strategy than delta hedging.

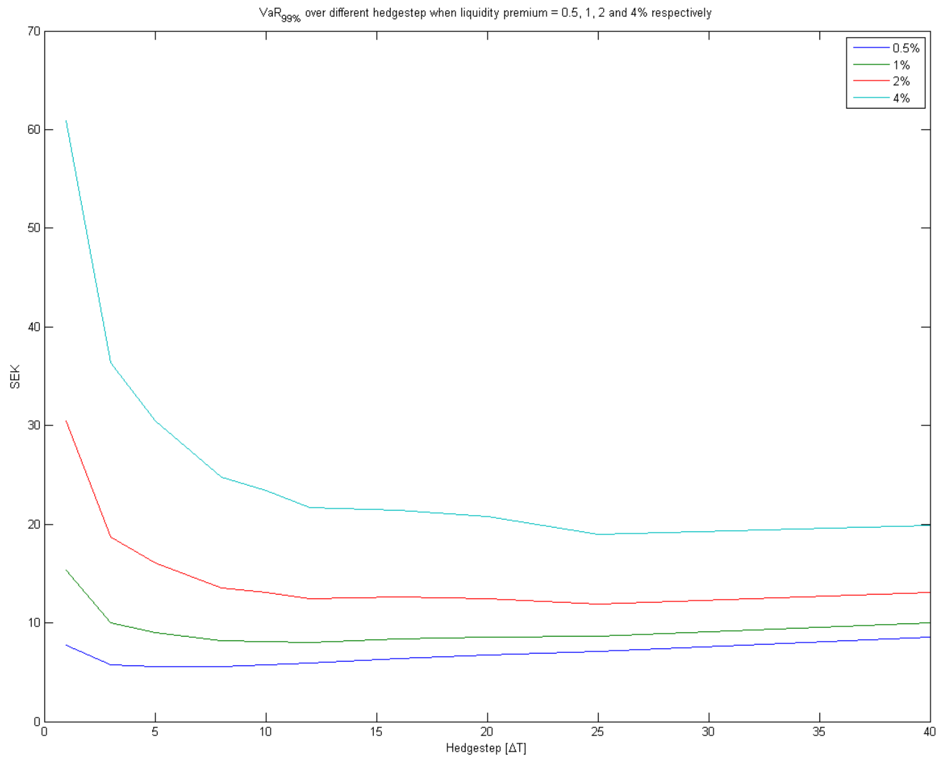


Figure 6.15: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio depending on *hedgestep* with a liquidity premium of 1%.

## 6.4 Conclusion

When delta hedging time series with logreturns drawn from the actual distribution of a stock the risks dynamics are the same as when hedging a GBM. The numbers of $VaR_{99\%}$ and $ES_{99\%}$ are generally higher when hedging time series from the true distribution but the consequence of an incorrectly set pricing parameter is much larger, this may be seen in Figures 6.9 and 6.10 where the difference from their equivalent GBM plots are hardly noticed except for close the correct value. This is as expected since the theoretically minimum risk when delta hedging is never reached on real time series due to the non normal distribution property of the logreturns. The earlier mentioned figure shows that the risk increases much faster when setting $\sigma_{hedge}$ too low rather than too high.

The effect of increasing *hedgestep* is the expected, i. e. a higher risk. The presence of a liquidity premium or transaction cost will heavily affect the expected return and the risk of the portfolio. If the size of this premium is large the effects it creates will overshadow all other factors. Furthermore there will be no incitement to issue options at the price the B&S formula suggests since $\mu_{R(0,T)}$ is negative.

The outcome of the analysis may be summarized as: the risk caused by the assumption of normal distributed logreturns is small in relation to the risk of setting the volatility parameter incorrectly, especially too low. If the spread between bid and ask prices are too large the liquidity premiums that has to be paid will erase all possible profit and rise the risk of the portfolio to very high levels.

# Chapter 7

# Market state model risk

Financial time series are heteroscedastic, i. e. has varying variance over time. Factors causing this behavior are extreme kurtosis and auto correlated squared observations. Extreme kurtosis is when most of the variance origins from extreme but not very frequent time periods of deviations[1].

In this paper market state model risk is defined as how well the model handles times where the market is turbulent contra ordinary times. Another view of the same measure is how the model handles a very volatile underlying asset contra a less volatile during ordinary times.

## 7.1 GBM

This section measures the theoretical risks during different market states in the meaning volatile or less volatile market and high or low interest rate.

[1] Ruiz, E. and Pascual, L., 2002

### 7.1.1 Volatility

In this simulation all parameters were set to default except $\sigma_{GBM}$ which were varied from zero to 100%. Figure 7.1 shows how $\mu_{R(0,T)}$, $\sigma_{R(0,T)}$, $VaR_{99\%}$ and $ES_{99\%}$ are dependent on the volatility of the underlying hedging with the correct volatility. The graph shows that there is a linear relation between volatility of the underlying and the risk measures while $\mu_{R(0,T)}$ is fairly constant around zero.

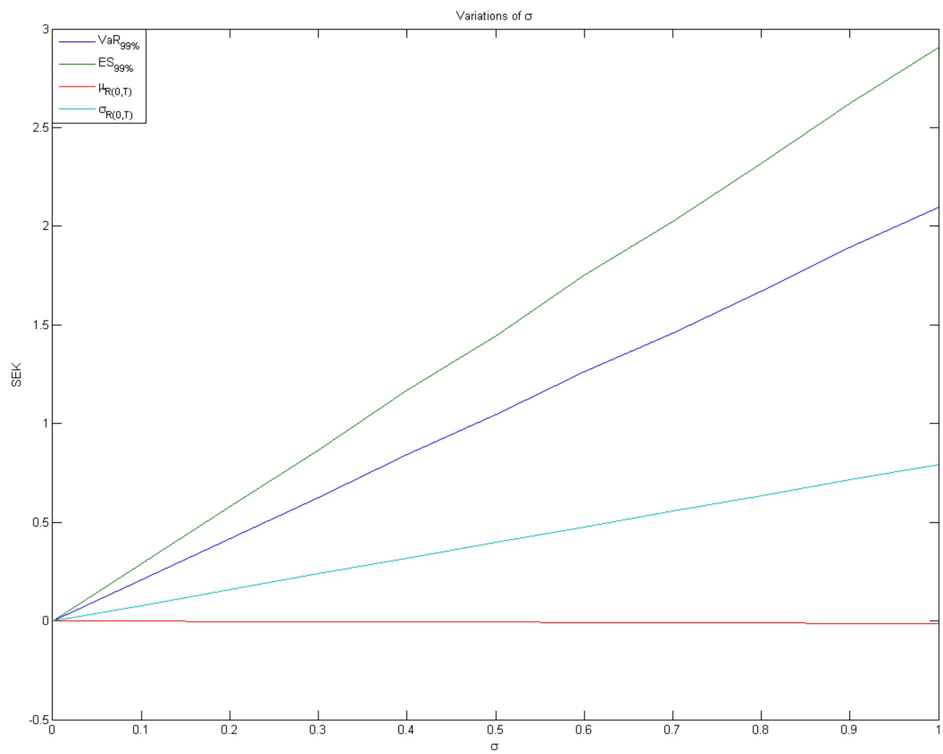


Figure 7.1: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio depending on the volatility of the underlying stock.

As may be seen in Table 7.1 the change of $\mu_{R(0,T)}$, $\sigma_{hedge}$, $VaR_{99\%}$ and $ES_{99\%}$ increases with higher strikes when the volatility of the underlying is set 20 % higher than the default number of $\sigma_{GBM}$. In contrary to this a change in term to maturity will not have any significant effect.

| $\sigma_{GBM} = 1.2\hat{\sigma}_{EricB}$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 49% | -48% | -157% | 85% | -6% | -204% |
| $\sigma_{R(0,T)}$ | 9% | 31% | 41% | 82% | 31% | 36% |
| $VaR_{99\%}$ | 9% | 29% | 36% | 102% | 29% | 34% |
| $ES_{99\%}$ | 8% | 29% | 34% | 70% | 28% | 32% |

Table 7.1: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when $\sigma_{GBM} = 1.2\hat{\mu}_{EricB}$ for some K, T.

Table A.8 reveals that the absolute risks are still higher for the strike $K = 110$ than for $K = 100$ as seen generally earlier in this thesis.

### 7.1.2 Risk free rate

This simulation's purpose is to clarify the behavior of the risks if the market's risk free rate is raised a lot. Figure 7.2 shows when $r_f$ is varied from zero up to 50% and the outcome is that it hardly matters as long as it is correctly set in the pricing model.

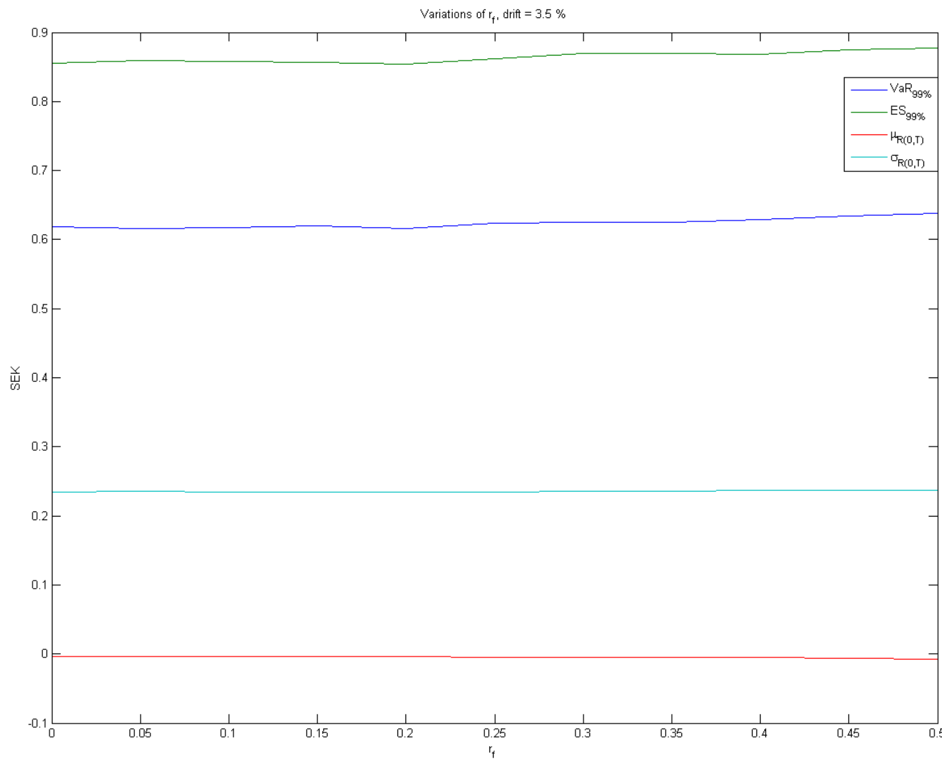


Figure 7.2: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio depending on the risk free rate.

Table 7.2 confirms what Figure 7.2 showed earlier but enables comparison to a 20 % increase in other factors. The absolute risks due to a 20% change in $r_f$ are shown in Table A.9.

| $r_f = 1.2\hat{\mu}_{EricB}$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 12% | -5% | -23% | -3% | 7% | 53% |
| $\sigma_{R(0,T)}$ | 0% | 0% | 0% | 0% | 0% | 0% |
| $VaR_{99\%}$ | 1% | 1% | 0% | 1% | 0% | 0% |
| $ES_{99\%}$ | 1% | 1% | -1% | 1% | -1% | 0% |

Table 7.2: Change in $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when $r_f = 1.2\hat{\mu}_{EricB}$ compared to the GBM reference values.

## 7.2 Bootstrapped market data

The properties stated in the beginning of Chapter 7 are lost when bootstrapping a time series following the algorithm stated in Section 2.4, i. e. drawing logreturns randomly to build a trajectory. To take heteroscedasticity into account the method chosen in this paper is to look at a long time series of Ericsson B and select shorter time periods where the volatility is assumed to be constant. The last step is to bootstrap market data from each of these shorter time periods to see how the risks depends on the different market states.

Figure 7.3 shows a 20 years long time series of Ericsson B on daily basis in red. In the background in blue a moving historical standard deviation is plotted. What might surprise is that the peaks of the two do not match in time, this is due to the fact that the volatility is relative. Figure 7.4 does make the heteroscedastic property obvious since the large logreturns appears in clusters around some periods of time.

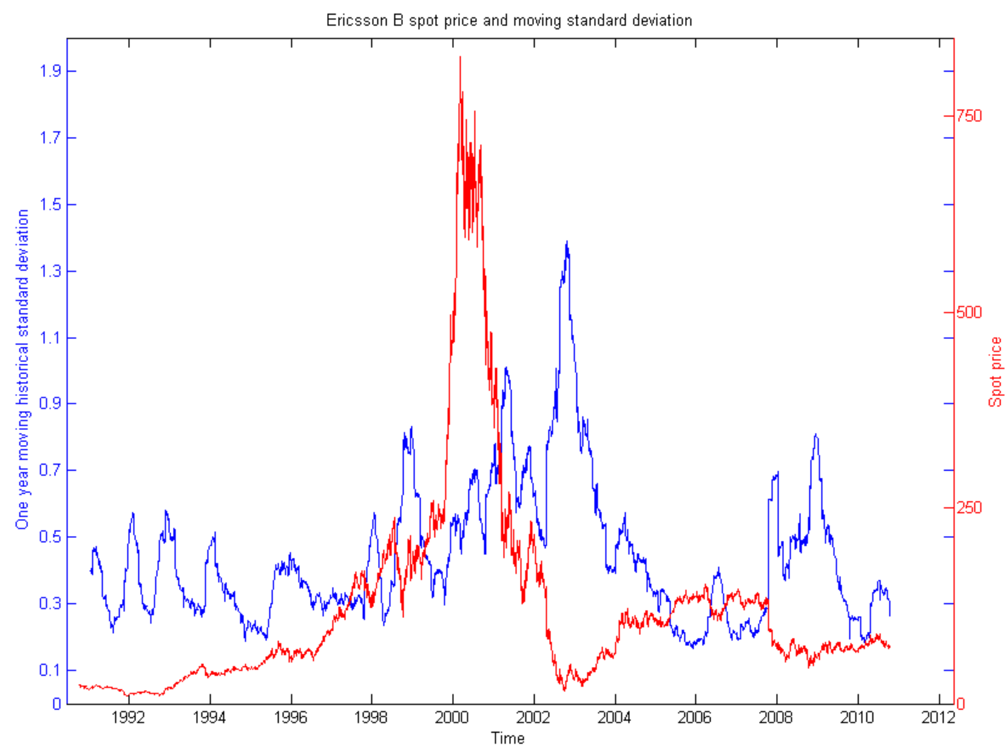


Figure 7.3: 20 years time series of Ericsson B as blue and moving one year historical volatility calculated on 60 days market data.

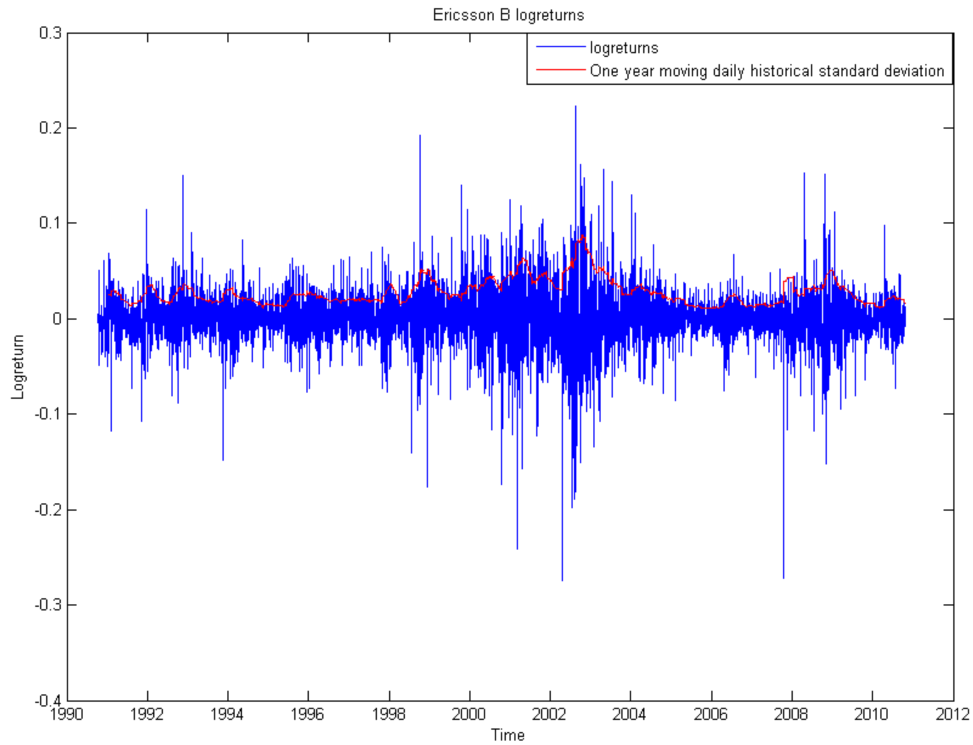


Figure 7.4: 20 years logreturns of Ericsson B as blue and moving one year historical volatility based on 60 days market data.

The market states defined for the analysis are:

- normal: $\sigma_{1year\,historical} < 0.5$ (70.2 % of the time historically)
- turbulent: $0.5 \leq \sigma_{1year\,historical} < 0.7$ (18.1 % of the time historically)
- extra ordinary: $0.7 \leq \sigma_{1year\,historical}$ (11.7 % of the time historically)

and in addition a historically worst case which is the time period with highest one year historical volatility.

The market states are defined only with respect to volatility, not risk free interest, since Section 7.1.2 made obvious that the impact of it is small provided that it is correctly set. Since the risk free interest is a parameter easy to set the assumption is reasonable.

Results of the simulations are shown in Table 7.3 where the notation of $\sigma \nearrow 0.5$ shows that the volatility is 0.5 increasing and $\sigma \searrow 0.5$ decreasing. The simulations are carried out as the standard setup with exception of $\triangle t = 8$ hours, i. e. adjustments of the hedge once every day. The end date in the table is the last of the 60 days of market data that has been used to calculate the yearly historical volatility. As expected the risks increases with the market volatility.

| Market state | Yearly volatility | End date | $\mu_{R(0,T)}$ | $\sigma_{R(0,T)}$ | $VaR_{99\%}$ | $ES_{99\%}$ |
|---|---|---|---|---|---|---|
| Normal | $\sigma \approx 0.44$ | 1995-12-21 | -0.1358 | 0.8965 | 2.0416 | 2.4950 |
| Turbulent | $\sigma \nearrow 0.5$ | 2008-08-27 | -0.1289 | 1.2658 | 3.1946 | 4.2966 |
| Turbulent | $\sigma \searrow 0.5$ | 2004-04-06 | -0.3391 | 1.1975 | 3.2351 | 4.0696 |
| Extra ordinary | $\sigma \nearrow 0.7$ | 2001-01-18 | -0.0973 | 1.4796 | 3.5811 | 4.7796 |
| Extra ordinary | $\sigma \searrow 0.7$ | 2001-10-26 | -0.0815 | 1.6129 | 3.6396 | 4.6008 |
| Hist. worst case | $\sigma \approx 1.38$ | 2002-10-25 | -0.3920 | 2.8219 | 6.7628 | 8.6163 |

Table 7.3: $\mu_{R(0,T)}$, $\sigma_{R(0,T)}$, $VaR_{99\%}$ and $ES_{99\%}$ for different market states.

### 7.2.1 Volatility based on historical estimation

The problem with volatility is that it is impossible to predict, instead it is often estimated on historical data. The obvious question is then *how good is this estimate?* The upper graph in Figure 7.5 shows a moving yearly volatility based on 60 trading days observation both in the past, $\sigma_{60d\,historical}$, and forward, $\sigma_{60d\,forward}$. The lower plot shows the difference between them. This reveals how accurate the volatility is estimated for a 60 days long option based on the past 60 trading days.

With an infinite time series the average error would be equal to zero, and with the observed values in Figure 7.5 it is -0.0013. However the largest errors for a 60 day period are more than $\pm$ 0.5. Set in perspective this would be the same as reading the risks in Figure 6.9 for $\sigma_{hedge}$ equal to zero or 0.8, which then yields a $VaR_{99\%}$ at SEK 15 or 5.5, compared to less than SEK 2 with the correct volatility. These numbers are however only accurate in that specific case.

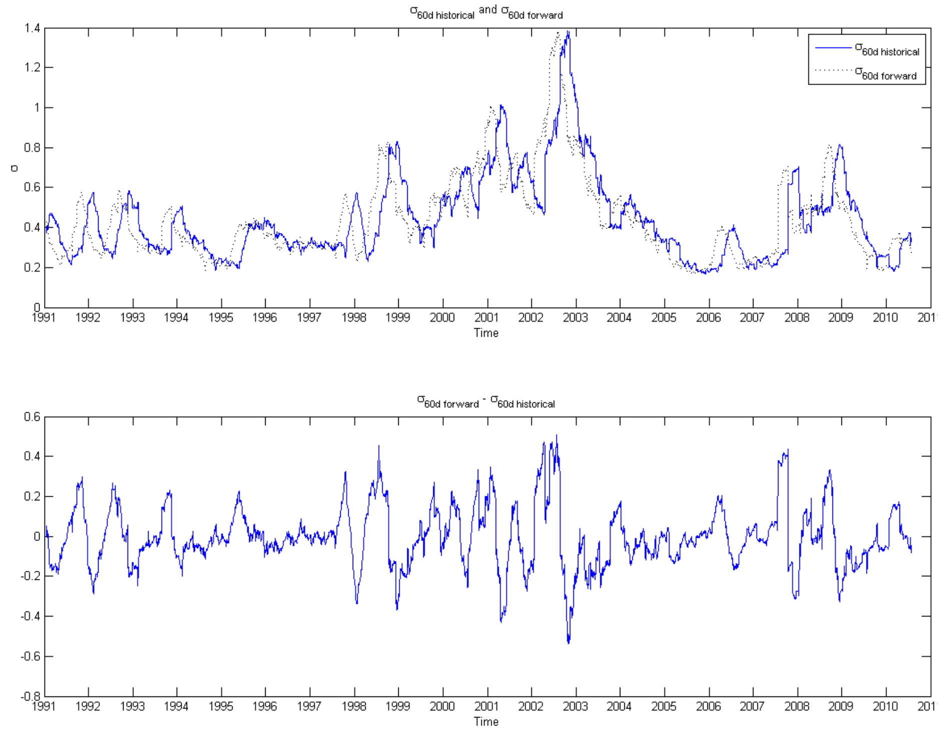


Figure 7.5: $\sigma_{60d\,historical}$, $\sigma_{60d\,forward}$ of Ericsson B and the difference between them.

## 7.3 Conclusion

The GBM simulations showed that in an ideal world the risks depends linearly on the market volatility and almost not at all on the risk free rate. When bootstrapping market data from different periods of time the result changed a bit but the linearity could to some extent serve as a rule of thumb regarding the behavior of the risks. However, the precision of this approximation is strongly dependent on the distribution of the logreturns which varies a lot over time.

The conclusion stated above is under the assumption that the pricing volatility is correctly set and it is reasonable to think that it is harder to set it correctly during more turbulent market states. In other words it is likely that the risks are at higher levels when the market is turbulent. The B&S framework is able to handle different market states with a roughly linear increase of risks when market volatility goes up. However higher attention has to be paid at setting the volatility parameter correctly, especially not too low since that would lead to the behavior showed in Figure 6.9 which is rapidly growing unknown risks.

# Chapter 8

# Additivity

This far the objective has been to isolate and measure the impact of a shift in each input parameter in the B&S formula, but what about if two are shifted at the same time? The purpose of this section is to establish whether the results due to shifted parameters are additive, i. e. is

$$\Psi_{p1}\left(0,T\right)+\Psi_{p2}\left(0,T\right)=\Psi_{p1\;\&\;p2}\left(0,T\right) \tag{8.1}$$

true? Where $\Psi_{p1}\left(0,T\right)$ is the change in risks when parameter $p1$ is shifted.

### 8.0.1 Incorrect $\sigma_{hedge}$ and $r_{f-hedge}$

$\Psi_{\sigma_{hedge}=1.2\sigma_{GBM}\;\&\;r_f=1.2\mu_{EricB}}$ in Table A.10 should be compared to $\Psi_{\sigma_{hedge}=1.2\sigma_{GBM}}$ and $\Psi_{r_f=1.2\mu_{EricB}}$ calculated from Tables A.2, A.1 and 4.3. If

$$\Psi_{\sigma_{hedge}=1.2\sigma_{GBM}\;\&\;r_f=1.2\mu_{EricB}}-\left(\Psi_{r_f=1.2\mu_{EricB}}+\Psi_{\sigma_{hedge}=1.2\sigma_{GBM}}\right) \tag{8.2}$$

is close to zero it is likely that $r_{f-hedge}$ and $\sigma_{hedge}$ are additive. This is hard to prove, but since Table 8.1 shows small differences between shifting $r_{f-hedge}$ and $\sigma_{hedge}$ together and summing the separate shifts they seem to be additive.

| $\Psi_{\sigma_{hedge}=1.2\sigma_{GBM}\;\&\;r_f=1.2\mu_{EricB}}-\left(\Psi_{r_f=1.2\mu_{EricB}}+\Psi_{\sigma_{hedge}=1.2\sigma_{GBM}}\right)$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | -0.00119 | 0.00017 | -0.00037 | 0.00078 | 0.00142 | 0.00028 |
| $\sigma_{R(0,T)}$ | -0.01203 | -0.00111 | 0.01187 | 0.00688 | -0.00321 | 0.02626 |
| $VaR_{99\%}$ | -0.03090 | 0.00497 | 0.02814 | 0.01847 | 0.02684 | 0.07792 |
| $ES_{99\%}$ | -0.02968 | 0.00750 | 0.02779 | 0.01875 | 0.03991 | 0.08328 |

Table 8.1: Difference in $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when $\Psi_{\sigma_{hedge}=1.2\sigma_{GBM}\;\&\;r_f=1.2\mu_{EricB}}$ compared to $\Psi_{r_f=1.2\mu_{EricB}}+\Psi_{\sigma_{hedge}=1.2\sigma_{GBM}}$.

### 8.0.2 Incorrect $\sigma_{hedge}$ and *hedgestep*

The shifted values for $\Psi_{\sigma_{hedge}=1.2\sigma_{GBM}\ \&\ hedgestep=1.2\triangle t}$ in Table A.11 should be compared to $\Psi_{\sigma_{hedge}=1.2\sigma_{GBM}}$ and $\Psi_{hedgestep=1.2\triangle t}$ calculated from Tables A.1, A.3 and 4.3. The differences between them are the values in Table 8.2. Which indicates non additivity in this case since for $\sigma_{R(0,T)}$, $VaR_{99\%}$ and $ES_{99\%}$

$$\Psi_{\sigma_{hedge}=1.2\sigma_{GBM}\ \&\ hedgestep=1.2\triangle t} < \left(\Psi_{hedgestep=1.2\triangle t} + \Psi_{\sigma_{hedge}=1.2\sigma_{GBM}}\right)$$

applies. The risks do not get as large when setting $\sigma_{hedge}$ and *hedgestep* wrong as the sum of risk changes when setting them wrong respectively.

| $\Psi_{\sigma_{hedge}=1.2\sigma_{GBM}\ \&\ hedgestep=1.2\triangle t} - \left(\Psi_{hedgestep=1.2\triangle t} + \Psi_{\sigma_{hedge}=1.2\sigma_{GBM}}\right)$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | -0.00481 | 0.01001 | -0.00141 | -0.00358 | 0.00681 | 0.00221 |
| $\sigma_{R(0,T)}$ | -0.21696 | -0.28348 | -0.27620 | -0.13589 | -0.28850 | -0.30841 |
| $VaR_{99\%}$ | -0.66859 | -0.75825 | -0.83129 | -0.43716 | -0.77155 | -0.84932 |
| $ES_{99\%}$ | -0.93423 | -1.02596 | -1.15048 | -0.80384 | -1.04192 | -1.16811 |

Table 8.2: Difference in $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when $\Psi_{\sigma_{hedge}=1.2\sigma_{GBM}\ \&\ r_f=1.2\mu_{EricB}}$ is compared to $\Psi_{hedgestep=1.2\triangle t} + \Psi_{\sigma_{hedge}=1.2\sigma_{GBM}}$ .

### 8.0.3 Conclusion

The conclusion can then be drawn that $\sigma_{hedge}$ and $r_{f-hedge}$ are likely to be additive. Combining $\sigma_{hedge}$ and *hedgestep* on the other hand results in risks that are smaller than the sum of separate shifts. Which indicates non additivity.

# Chapter 9

# Discussion

There are two ways of looking at a model's performance

1. business perspective. The model should price the option as fair as possible to give the market maker an advantage at the market

2. accounting and risk perspective. The model should price the option as close to the market price as possible to reflect the value of the option position if it had to be exited today.

Looking at today's market the difficulty is how to set the volatility parameter. Category two sets the volatility to the implied volatility and category one uses other methods for estimation.

To some extent the framework used in this thesis does check the B&S formula's ability to price non normal distributed trajectories fairly. This since when bootstrapping the mean of the accumulated result is close to zero. In order to test a model's ability to price consistent with the market price instead the following method is suggested, in this case stated for the B&S formula on a stock option:

1. collect market data for $S_{stock}$, $r_f$ and $S_{option}$ for a certain period of time which includes a known dividend payment

2. estimate $\sigma$ from $S_{stock}$ time series

3. calculate the theoretical option price, $\Pi_{option}$, for each time step

4. calculate the *pricing error* $= -\left|\Pi_{option} - S_{option}\right|$ for each time step

5. calculate VaR on the pricing errors of the time steps

The difficulty of the above stated method is to find an option which is sufficiently liquid over the entire period of time. Note that this method does as well test the models ability to price when dividends are paid.

In this thesis the model's ability to estimate the market's actual delta is measured and using the above stated method the pricing ability is hopefully captured. However there are a number of greeks left in Table 2.1 which the model may handle varyingly good. It seems reasonable to assume that if the model prices the option with small *pricing errors* over time, then it handles the greeks sufficiently good from an accounting and risk perspective. However, knowing that for instance the volatility parameter is adjusted often. Performance testing of the model's calculations of greeks will probably be necessary.

In Section 5 VaR and ES numbers are calculated on the market maker's result from time zero to maturity of the stock option when a pricing parameter is set 20% too high. The outcome of this is not a proper VaR number since the factor of how wrong an actual trader sets for instance the volatility parameter varies over time and trader. The VaR numbers calculated to estimate the risk related to parameter setting in this thesis would more properly be described as a conditional VaR and be expressed mathematically as

$$P\left(L\left(t-n,t\right) \leq VaR_p \mid \sigma_{hedge} = 1.2\sigma_{B\&S}\right) = p. \tag{9.1}$$

These numbers serves to compare risks between different models and pricing parameters. To get a measure of the actual risk originating from estimation of parameters the behavior of the traders has to be investigated and the conditional VaR only considered as a hint. When trying to use the conditional VaR to this purpose the choice of the shift, here 20%, is crucial since the error will vary over estimation occasions and most likely vary over different market states. For instance, estimation of volatility is more likely to be hard during times when market volatility is increasing swiftly than during times of more constant market volatility. To increase the relevancy, further investigations of the distribution of the parameter setting error have to be done, the mean of this error would hopefully be close to zero. This provides possibility to simulate setting of the parameter instead of setting it to a fixed number, here 20%. Parameter setting may at first be considered as operational risk but it is actually model risk since different models has varying sensitivity to parameters which are varyingly hard to estimate. In other words it is a part of the process of choosing a valuation model.

The additivity tests done in Section 8 are to be considered as a first investigation meant to understand the behavior of the risks rather than proving additivity. In order to prove additivity a more mathematically strict method has to be applied. However, the method chosen in this thesis serves its purpose.

When bootstrapping time series in order to test a model's model risk attention has to be paid to which time series to choose. As an example, look at HQ Bank in Figure 6.12 which had $VaR_{99\%} \approx$ SEK 19 caused by a large daily loss of 82%. That $VaR_{99\%}$ number is the outcome of an extreme event for the company rather than a measure of how the model handles non normal logreturns. In other words it is a risk originating from the stock chosen to issue options on rather than a model risk and should therefore not be used to this purpose.

# Chapter 10

# Conclusion

According to the problem formulation the report is split into three components.

1. Parameter model risk. The single most risk increasing factor is a badly set volatility when hedging. The risks increases more if the volatility is set too low rather than too high. Second is the interval between hedging occasions, the risk increases as this interval grows. Hedging with an improperly set risk free rate has less effect, moreover, it is easier to estimate.

2. Assumption model risk. The risks in the portfolio will increase as the logreturns get less normal distributed. However the worst outcome of simulations with improperly set hedging parameters are approximately the same when the underlying is bootstrapped from real market data as when the trajectories are GBM generated. The presence of a liquidity premium will overshadow the other factors if the premium is large enough.

3. Market State model risk. When hedging a trajectory with normal distributed logreturns the risks will increase linearly with the volatility. That behavior may to some extent also serve as a rule of thumb on bootstrapped market data. In addition to this it is harder to estimate the volatility during turbulent times.

In general it could be said that when hedging, volatility is the most important parameter, this is also the one that is hardest to estimate. Hedging with both improperly set volatility and risk free rate results in risks the size of the sum of shifting them separately, suggesting that the two parameters are additive. Another thing worth pointing out is that when hedging with an improperly set parameter the risks will continue to grow with the term to maturity. When looking at simulations where a fixed shift to a parameter has been applied the risk measures are conditional on that specific shift, this is sufficient when comparing models and hopefully limiting.

The approach described in Figure 1.1 is general when analyzing model risk from a hedging perspective. It is just to plug in another model than the B&S formula since the separation on parameter, assumption and market state model risk is applicable to all models calculating hedge ratio. With another model follows other assumptions and parameters, which may imply modifications of the perfect data generation etc. The B&S formula is widely used today for valuation and sensitivity calculation of European stock options and may be seen as a benchmark model when setting limits.

# Chapter 11

# References

Efron, B. and Tibshirani, R.J. (1993), *An Introduction to the Bootstrap.* New York*:* Chapman and Hall*,* ISBN 0-412-04231-2

Hull, J. (2009), *Options, futures and other derivatives* (seventh edition). New Jersey: Pearson Education, ISBN: 0-13-601586-7

Hult, H. and Lindskog, F. (2007), *Mathematical modeling and statistical methods for risk management,* Stockholm: KTH mathematical statistics, lecture notes

Härdle, W., Horowitz, J. and Kreiss, J-P. (2003), *Bootstrap Methods for Time Series.* International Statistical Review Vol. 71, No. 2 , pp. 435-459

Ruiz, E. and Pascual, L. (2002), *Bootstrapping Financial Time Series.* Journal of Economic Surveys, 16: 271–300. doi: 10.1111/1467-6419.00170

# Appendix A

# Output from simulations

| $\sigma_{hedge} = 1.2\sigma_{GBM}$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 0.00408 | -0.01006 | 0.00134 | 0.00395 | -0.00832 | -0.00321 |
| $\sigma_{R(0,T)}$ | 0.50227 | 0.55365 | 0.63124 | 0.34090 | 0.72309 | 0.88323 |
| $VaR_{99\%}$ | 0.69359 | 1.18364 | 0.92722 | 0.25365 | 1.50694 | 1.50375 |
| $ES_{99\%}$ | 0.74790 | 1.32519 | 1.00451 | 0.27996 | 1.66858 | 1.62427 |

Table A.1: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when $\sigma_{hedge} = 1.2\sigma_{GBM}$ for some K, T on GBM trajectories.

| $r_{f-hedge} = 1.2r_f$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 0.00078 | -0.00362 | 0.00049 | 0.00133 | -0.00195 | -0.00051 |
| $\sigma_{R(0,T)}$ | 0.15878 | 0.23633 | 0.20319 | 0.10086 | 0.24300 | 0.23500 |
| $VaR_{99\%}$ | 0.50260 | 0.61732 | 0.59807 | 0.29318 | 0.63954 | 0.63342 |
| $ES_{99\%}$ | 0.71213 | 0.85649 | 0.85886 | 0.56232 | 0.88779 | 0.90240 |

Table A.2: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when $r_{f-hedge} = 1.2r_f$ for some K, T on GBM trajectories.

| $hedgestep = 1.2\triangle t$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 0.00342 | -0.01250 | 0.00190 | 0.00513 | -0.00824 | -0.00244 |
| $\sigma_{R(0,T)}$ | 0.22303 | 0.29851 | 0.28576 | 0.13938 | 0.30024 | 0.31671 |
| $VaR_{99\%}$ | 0.68499 | 0.79726 | 0.85503 | 0.44083 | 0.80783 | 0.87873 |
| $ES_{99\%}$ | 0.95727 | 1.07393 | 1.18151 | 0.81525 | 1.08884 | 1.20600 |

Table A.3: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when $hedgestep = 1.2\triangle t$ for some K, T on GBM trajectories.

| $\sigma_{hedge} = 1.2\hat{\sigma}_{EricB}$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | -0.01759 | -0.02116 | 0.03791 | 0.02368 | -0.01314 | 0.03737 |
| $\sigma_{R(0,T)}$ | 0.55394 | 0.64324 | 0.71951 | 0.39288 | 0.80082 | 0.95656 |
| $VaR_{99\%}$ | 0.83503 | 1.42435 | 1.06976 | 0.30000 | 1.73282 | 1.67268 |
| $ES_{99\%}$ | 1.09886 | 1.71964 | 1.31242 | 0.51136 | 1.98149 | 1.87340 |

Table A.4: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when $\sigma_{hedge} = 1.2\hat{\sigma}_{EricB}$ for some K, T on bootstrapped trajectories.

| $r_{f-hedge} = 1.2r_f$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 0.01406 | -0.01373 | -0.00677 | 0.00302 | -0.00643 | -0.00988 |
| $\sigma_{R(0,T)}$ | 0.26250 | 0.38106 | 0.32785 | 0.16342 | 0.39948 | 0.39092 |
| $VaR_{99\%}$ | 0.83234 | 1.10531 | 1.01765 | 0.47409 | 1.15657 | 1.14616 |
| $ES_{99\%}$ | 1.29845 | 1.67272 | 1.55314 | 0.93603 | 1.71463 | 1.73696 |

Table A.5: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when $r_{f-hedge} = 1.2r_f$ for some K, T on bootstrapped trajectories.

| Bootstrapped data from BEF | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | -0.00019 | -0.03977 | -0.03370 | -0.00728 | -0.02608 | -0.02974 |
| $\sigma_{R(0,T)}$ | 0.70410 | 0.78357 | 0.76706 | 0.64546 | 0.81562 | 0.82681 |
| $VaR_{99\%}$ | 2.15125 | 2.31279 | 2.25931 | 2.00375 | 2.39205 | 2.41534 |
| $ES_{99\%}$ | 3.23028 | 3.46480 | 3.44626 | 3.14897 | 3.58671 | 3.67153 |

Table A.6: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when underlying trajectories are build up by data bootstrapped from BEF for some K, T.

| $\xi = 0.02$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | -8.55191 | -16.62950 | -8.11736 | -0.00324 | -23.50992 | -17.57139 |
| $\sigma_{R(0,T)}$ | 5.21428 | 6.50808 | 8.23782 | 3.85756 | 8.93676 | 11.80945 |
| $VaR_{99\%}$ | 23.56350 | 30.47482 | 29.81574 | 20.17322 | 42.08583 | 43.70535 |
| $ES_{99\%}$ | 25.19453 | 31.68908 | 31.62599 | 24.07470 | 43.64166 | 45.62425 |

Table A.7: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when $\xi = 0.02$ for some K, T on GBM trajectories.

| $\sigma_{GBM} = 1.2\hat{\sigma}_{EricB}$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 0.00104 | -0.00499 | -0.00036 | 0.00265 | -0.00302 | -0.00165 |
| $\sigma_{R(0,T)}$ | 0.17188 | 0.30745 | 0.28603 | 0.18389 | 0.30969 | 0.31455 |
| $VaR_{99\%}$ | 0.52943 | 0.80017 | 0.82824 | 0.62325 | 0.82562 | 0.88627 |
| $ES_{99\%}$ | 0.75148 | 1.10975 | 1.16605 | 0.98589 | 1.15170 | 1.23994 |

Table A.8: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when $\sigma_{GBM} = 1.2\hat{\mu}_{EricB}$ for some K, T on GBM trajectories.

| $r_f = 1.2\hat{\mu}_{EricB}$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 0.00078 | -0.00354 | 0.00049 | 0.00138 | -0.00264 | -0.00025 |
| $\sigma_{R(0,T)}$ | 0.15778 | 0.23497 | 0.20260 | 0.10096 | 0.23667 | 0.23245 |
| $VaR_{99\%}$ | 0.48839 | 0.62104 | 0.60887 | 0.31093 | 0.64022 | 0.66123 |
| $ES_{99\%}$ | 0.69848 | 0.86318 | 0.86466 | 0.58246 | 0.89201 | 0.93341 |

Table A.9: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when $r_f = 1.2\hat{\mu}_{EricB}$ for some K, T on GBM trajectories.

| $\Psi_{\sigma_{hedge}=1.2\sigma_{GBM}\ \&\ r_f=1.2\mu_{EricB}}$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 0.00228 | -0.00677 | 0.00019 | 0.00321 | -0.00318 | -0.00236 |
| $\sigma_{R(0,T)}$ | 0.33414 | 0.31924 | 0.44095 | 0.24710 | 0.48844 | 0.68066 |
| $VaR_{99\%}$ | 0.19777 | 0.57009 | 0.33354 | -0.05063 | 0.88874 | 0.89166 |
| $ES_{99\%}$ | 0.04078 | 0.47433 | 0.15082 | -0.29704 | 0.79555 | 0.73778 |

Table A.10: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when $\sigma_{hedge} = 1.2\sigma_{GBM}$ and $r_f = 1.2\mu_{EricB}$ for some K, T on GBM trajectories.

| $\Psi_{\sigma_{hedge}=1.2\sigma_{GBM}\ \&\ hedgestep=1.2\triangle t}$ | | | | | | |
|---|---|---|---|---|---|---|
| $K$ | 90 | 100 | 110 | 120 | 100 | 110 |
| $T$ | $1/12$ | $1/12$ | $1/12$ | $1/12$ | $2/12$ | $2/12$ |
| $\mu_{R(0,T)}$ | 0.00199 | -0.00918 | 0.00119 | 0.00407 | -0.00692 | -0.00290 |
| $\sigma_{R(0,T)}$ | 0.35090 | 0.33387 | 0.43813 | 0.24361 | 0.49761 | 0.65962 |
| $VaR_{99\%}$ | 0.22623 | 0.60473 | 0.34101 | -0.05065 | 0.90093 | 0.87145 |
| $ES_{99\%}$ | 0.07615 | 0.51574 | 0.16537 | -0.28767 | 0.81513 | 0.72608 |

Table A.11: $VaR_{99\%}$, $ES_{99\%}$, $\mu_{R(0,T)}$ and $\sigma_{R(0,T)}$ for the portfolio when $\sigma_{hedge} = 1.2\sigma_{GBM}$ and $hedgestep = 1.2\triangle t$ for some K, T on GBM trajectories.