Department of Mathematics and Physics

Black and Scholes

Seminar in MT1410 Analytical Finance 1

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Table of contents

Introduction

Fisher Black and Myron Schole's published their paper "the pricing of options and corporate liabilities" in 1973.

About the formula

The Black and Scholes's formula describes the value of a European option on an asset with no cash flows.
The inputs depend on:

1. Asset price
2. Strike price
3. Time to maturity
4. Risk-free interest rate
5. Volatility

The model is as referred to a continuous time model, and the stock price follows a lognormal probability distribution.

Assumptions

The Black-Schole's model requires a few assumptions:

A1. Frictionless markets i.e. there are no costs for transactions, no margin requirements, no short sale restrictions, no taxes etc.

A2. No counterparty risk i.e. participants have no chance of defaulting on a contract they undertake.

A3. Competitive markets i.e. market participants act as price takers.

A4. No arbitrage opportunities

A5. Trading takes place continuously in time.

This assumption is needed for the existence and uniqueness of the equivalent martingale probability distribution.

A6. The stock price follows a lognormal probability distribution and pays no dividends.

A7. Interest rates are constant.

The Greeks

To understand the Back and Schole's model, it is important to understand how the model's value changes when the input parameters change. These values changes for the model depending on the options stock price, strike price, time to maturity, volatility, and interest rate.

These parameters can change over the life of the option. So to find the change of these parameters we must find the sensitivity of the option's value to change in each of these variables. The Greeks are used for hedging purpose. Four of the five are named after letters in the Greek alphabet and that is the reason that they are called the Greeks.

This can be determined by taking the partial derivatives.

Delta

Delta for a European call option is given by

$$\Delta = \frac{\partial C}{\partial S} = N(d_1)$$

and for a European put option it is given by

$$\Delta = \frac{\partial P}{\partial S} = N(d_1) - 1$$

It is defined as the rate of change of the option price with respect to the price of the underlying asset.

Gamma

To continue we have the sensitivity of delta or the hedge ratio to changes in the value of the underlying asset. This is called gamma. It is defined by

Gamma for a European call or put option is given by

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}.$$

Where,

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2}$$

Theta

Further, the sensitivity of the option value to changes in the time to maturity is called theta and theta of a call is defined by

$$\Theta = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT} N(d_2)$$

Theta of a put is defined by

$$\Theta = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT} N(d_2)$$

A positive theta indicates that as the time to maturity comes close, the change in the value of the option is positive.

Vega

The sensitivity of the option value to changes in the volatility is called vega and vega for a European call or put option is given by

$$\upsilon = \frac{\partial C}{\partial \sigma} = S_0 \sqrt{T} N'(d_1) .$$

Rho

The sensitivity of the option value to changes in the interest rate is called Rho.

Rho for a European call option is given by

$$\rho = \frac{\partial C}{\partial r} = KTe^{-rT} N(d_2)$$

Rho for a put is given by

$$\rho = \frac{\partial P}{\partial r} = -KTe^{-rT} N(-d_2) .$$

The greeks for a call option

Delta is positive because it lies between 0 and 1, which make sense because a call is equivalent to a fractional position in the stock. With some probability less than one and greater than zero the option will end up as the stock being in-the-money.

As the stock price increases the call is more probable to end up in-the-money and that is why Gamma is positive.

As the time to maturity increases, the option becomes more valuable and since of that theta is negative.

As the time to maturity increases the variance of the stock's return increases making the option more valuable. Further, the distribution of the stock price at maturity spread out. Moreover, the option only benefits from stock prices larger than the strike and as the distribution spread out the probability of stock prices being in-the-money increases and because of that increasing the value of the option. This is the reason to why Vega is positive.

As the risk-free interest rate increases the present value of the strike price declines. This gives a positive Rho.

The greeks for a put option

Delta is negative because it lies between 0 and –1 because at maturity the stock is sold for the strike price only with some probability between 0 and 1.

Gamma is positive for the same reason as for a call option.

As the time to maturity increases the option becomes more valuable and since of that theta is negative.

Vega is positive for the same reason as for a call option.

As the risk-free interest rate increases the present value of the strike price declines and reduces the value of the option. Tha is why Rho is negative.

The Formula

Pricing a European call and put:

$$\Pi_C = SN(d_1) - Ke^{-rT}N(d_2)$$

$$\Pi_P = -SN(-d_1) + Ke^{-rT}N(-d_2)$$

Where;

Π_C = value of the call option

Π_P = value of the put option

S = stock price

K = strike price

T = time to maturity of the option in year

r = risk-free interest

N = normal cumulative function.

$N(d_1)$ = fraction whose value is between 0 and 1 determined by the price of the stock.

d_1 = derived from the following formula

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

σ = the volatility of the stock that is the statistical measurement on the change in stock price.

N(x) = $N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-y^2/2} dy$

References

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