Poisson process and Jump Diffusion

Table of Content:

1. Poisson process

- A **Poisson process** is a stochastic process which is defined in terms of the occurrences of events in some space. A stochastic process N(t) is a Poisson process if,
- (i) N (0) = 0,
- (ii) N (t) – N (s) counts the number of events which occur in the interval (s, t],
- (iii) N (s) ≤ N (t) if s ≤ t,
- (iv) N(t) – N(s) is independent of all events occurring outside the time interval (s,t],
- (v) The probability of the number of events in some subinterval (s, t] is given by

$$P[(N(t) - N(s)) = n] = \frac{e^{-\lambda(t-s)}[\lambda(t-s)]^{n}}{n!}; \quad x = 1,2,\ldots$$

- The inter-arrival and first arrival times in the time interval (0, t] are continuous exponential distributed random variables and the probability density function is:

$$P(\tau) = \lambda e^{-\lambda t}$$

- The probability of the first jump occurring in the time interval (0, s] is:

$$P(\tau < s) = \int_0^s \lambda e^{-\lambda t} dt = 1 - e^{-\lambda s}$$

- And also

$$P(N = 0) = \frac{(\lambda t)^0 e^{-\lambda s}}{0!} = e^{-\lambda s} = P(\tau > s)$$

The Memoryless Property

$$P(\tau > t + s \mid \tau > s) = \frac{P(\tau > t + s)}{P(\tau > s)} = e^{-\lambda s}$$

- This illustrates the Markov property, the fact that the process has not jump until time s does not direct the probability of the future jumps.

The stationary increment property

- By (v) and (i), we see that each random variable $N(\mathrm{t}) - N(\mathrm{s})$ has the same distribution as $N(\mathrm{t} - \mathrm{s})$, so that

$$P\ (\ N(t)\ -\ N(s) = n\ ) = P\ (\ N(t\text{-}s) = n\ )$$

The independent increment property

- If s < t < u are three distinct time instants, then by (iv) N(u) − N(t) is independent of N(t) − N(s), so

$$P\,(\,N(u\,)\,-N(t) = m\,,\,N(t) - N(s) = n\,)$$
$$= P\,(\,N(u) - N(t) = m\,)\,P\,(\,N(t) - N(s) = n\,)$$

How a Poisson process is mathematically characterized

- The value of a standard Poisson process after a time t is expressed as

$$N(t) = N(0) + \sum_{s<t} [N(s) - N(s^-)]$$

- In a more useful form, the process can be expressed as $dN(t)$ which means the change in the Poisson process over time dt. By the Memoryless Property, the value of $dN(t)$ at any time t does not depend on the history of the process.

$$p(\tau < dt) = 1 - e^{-\lambda dt} = 1 - \lambda dt + \frac{1}{2}(\lambda dt)^2 - \ldots \sim \lambda dt$$

- Since dt is very small, $dN(t)$ can be thought as a random variable that increase by one over a time step dt with probability $1 - \lambda dt$ or zero with probability λdt .

2. Jump Diffusion

- If X is a stochastic diffusion process that can jump as well then it is called jump diffusion:

$$dX = A(t, X)dt + B(t, X)dW + C(t, X)dN$$

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- The first two terms are the usual drift that have been used extensively to model stock prices in finance. *W* is a Wiener process, The last term introduces the possibility of a jump occurring. *'dN'* constitutes a standard Poisson process. The jumps of *X* occur when the Poisson process jumps. The scaling by *C(x,t)* allows the jump size to vary.

- To derive the *SDE* for a function *F(X)*, we should consider the process *X* as the sum of *2* processes:

$$dX^c = A(t, X)dt + B(t, X)dW$$

$$dY = C(t, X)dN$$

- Then, to consider the Taylor series expansion of *F(x)* by first considering the contribution from the continuous process and then the jump process:

$$dF = \frac{dF}{dX} dX^c + \frac{1}{2}\frac{d^2F}{dX^2}\left(dX^c\right)^2 + \left[F\left(X + C\left(t, X\right)\right) - F\left(X\right)\right] dN$$

- The last term arises from the jump component. *[x+C(x,t)]* denotes the value of the process *x* just after a jump. The majority of the times the last term is zero because *dN=0.*

3. Example

Are stock returns different over weekends? -- A jump diffusion analysis of the “Weekend Effect”

3.1 Description

- The volatility of stock returns over weekends is much smaller than could be predicted from intraweek volatility. In short, holding stocks over weekends gives low and perhaps negative returns, but also provides relatively low risk.
- We find that a jump diffusion model is superior to a simple diffusion model, and that the jump diffusion model of stock returns provides strong support for the weekend effect.

3.2 The Simple Diffusion Model

- The standard model of stock returns is the diffusion model.

$$dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t)$$

α is the drift rate,

σ is the volatility,

$S(t)$ is the asset's price at time *t*,

$W(t)$ is a Wiener process.

3.3 The Jump Diffusion Model

- The jump diffusion model incorporates some of the known characteristics of stock prices that are not consistent with the simple diffusion model.
- The jump diffusion model builds on the simple diffusion model. Rather than having all variability reflected in a normally distributed "surprise," it has a second source of variability in asset returns. This is the effect of a random number of "jumps," either upward or downward, in stock returns.

$$dS(t) = \alpha \cdot S(t)dt + \sigma \cdot S(t)dW(t) + J(t) \cdot dN(t)$$

α is the drift rate,

σ is the volatility,

$S(t)$ is the asset's price at time t,

$W(t)$ is a Wiener process,

$J(t)$ is a random jump size at time t,

$N(t)$ is a standard Poisson process.