Mälardalen University
MT 1410 Analytical Finance I
Teacher: Jan Röman, IMA
Seminar 2005-10-05

Poisson Processes and Jump Diffusion Model

Authors: Ying Ni, Cecilia Isaksson

Abstract

Besides the Wiener process (Brownian motion), there's an another stochastic process that is very useful in finance, the Poisson process. In a Jump diffusion model, the asset price has jumps superimposed upon the familiar geometric Brownian motion. The occurring of those jumps are modelled using a Poisson process. This paper introduces the definition and properties of Poisson process and thereafter the Jump diffusion process which consists two stochastic components, the "white noise" created by Wiener process and the jumps generated by Poisson processes.

Table of contents

1. Introduction

In option pricing theory, the geometric Brownian Motion is the most frequently used model for the price dynamic of an underlying asset. However, it is argued that this model fails to capture properly the unexpected price changes, called jumps. Price jumps are important because they do exist in the market. A more realistic model should therefore also take jumps into account. Price jumps are in general infrequent events and therefore the number of those jumps can be modelled by a Poisson processes.

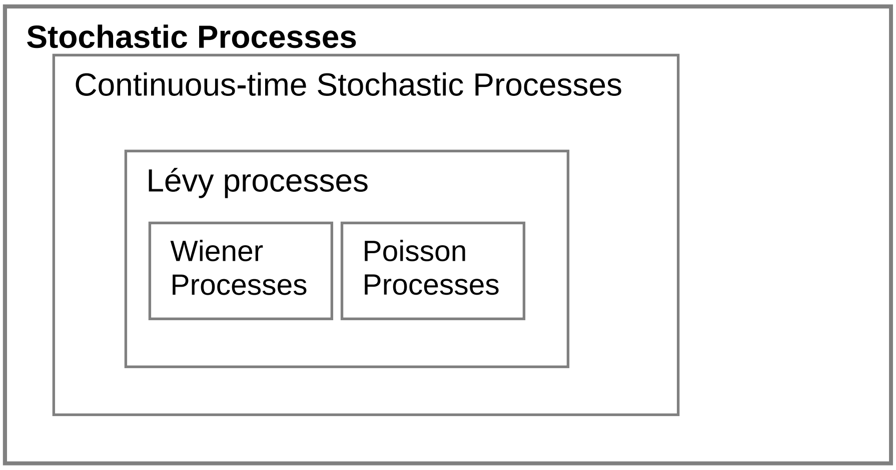
In the Jump diffusion model, the underlying asset price has jumps superimposed upon a geometric Brownian motion. The model therefore consists of a noise component generated by the Wiener process, and a jump component generated by the Poisson process.

2 The Poisson Process

2.1 Lévy Processes, Wiener Processes & Poisson Processes

Poisson processes (named after the French mathematician Siméon-Denis Poisson) are one of the most important classes of stochastic processes. A Poisson process is a stochastic process defined in terms of the occurrences of events in some space. Both the Wiener process and the Poisson process are two well-known examples of a more general family of stochastic processes called Lévy processes. In fact, many stochastic processes are Lévy processes. A Lévy process refers to any continuous-time stochastic process that has "stationary independent increments". These relations are illustrated in Figure 2.1

Figure 2.1 Lévy processes, Wiener processes & Poisson processes

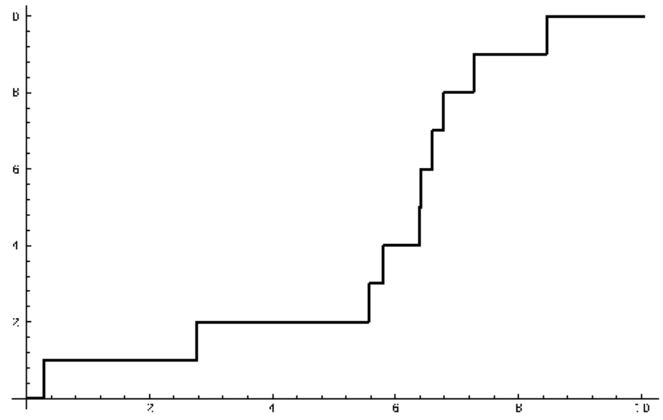


The awkward expression "stationary independent increments" may deserve some attention here. We know that a continuous-time stochastic process assigns a random variable X_t to each time point $t \geq 0$. The "increments" of the process represent the differences $X_s - X_t$ between its values at different times $t < s$. If the increments assigned to two disjoint (non-overlapping) time intervals are independent random variables, for example if $X_s - X_t$ and $X_u - X_v$ are independent, we then say the process has independent increments. For a process to also have "stationary increments", the probability distribution of any increment $X_s - X_t$ should only depend on the length of the time interval $s - t$; stationary increments for time intervals of equally length are therefore identically distributed.

A Poisson process is similar to a Wiener process since they both belong to family of Lévy processes and therefore share the feature of "stationary independent increments". The main difference between the two processes lies in the probability distributions of increments $X_s - X_t$. That is, in the Wiener process, $X_s - X_t$ is normally distributed with expected value 0 and variance $s - t$; whereas in the Poisson process, $X_s - X_t$ has a Poisson distribution with expected value $\lambda(s - t)$. The parameter λ is a positive number and represents the "intensity" or "rate" of the process, i.e.the average number of occurrences per unit time.

An Example of Poisson process:
Nothing happens for a while, then there is a sudden change of state. (Willmot, 2001,p370)

Figure 2.2 A Poisson process with $\lambda = 1$



(McMullen,n.d., available online)

In notation, in a Poisson process the number of events in some subinterval $[t, s]$, $X_s - X_t$, is given by

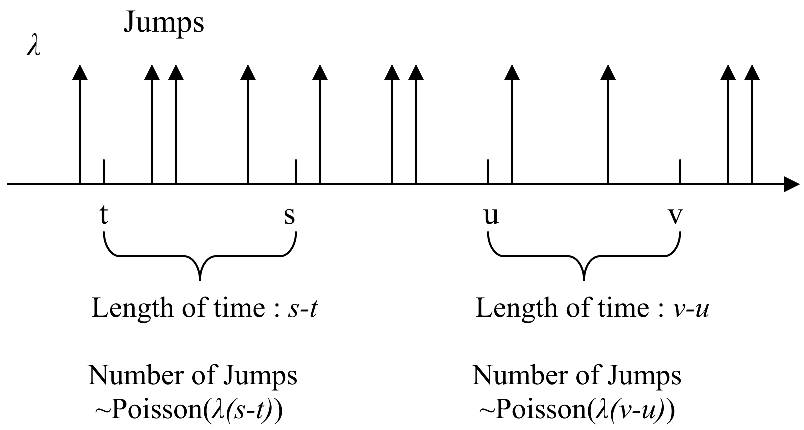
$$P[X_s - X_t = k] = \frac{e^{-\lambda(s-t)}(\lambda(s-t))^k}{k!} \qquad k = 0, 1, ... \qquad (2.1)$$

The occurrences of events in the Poisson process defined above can have different interpretations. For example, in queuing theory, the occurrences of events may be the arrivals of customers. In teletraffic theory "events" becomes calls or packets. Finally, in the world of finance we often talk about "jumps" (i.e. of stock prices) instead of "events", λ is therefore a measure of the frequency of jumps. Moreover, the number of jumps is often denote by N (instead of X above), with $N(t)$ referring to the number of jumps in the interval $(0, t)$. Because $N(0)$ is always defined to be equal to 0, the number of jumps N in a time interval $(0, t)$, $N(t)$, is easily derived from (2.1),

$$P[N(t) = k] = \frac{e^{-\lambda t}(\lambda t)^k}{k!} \qquad k = 0, \quad 1, ... \qquad (2.2)$$

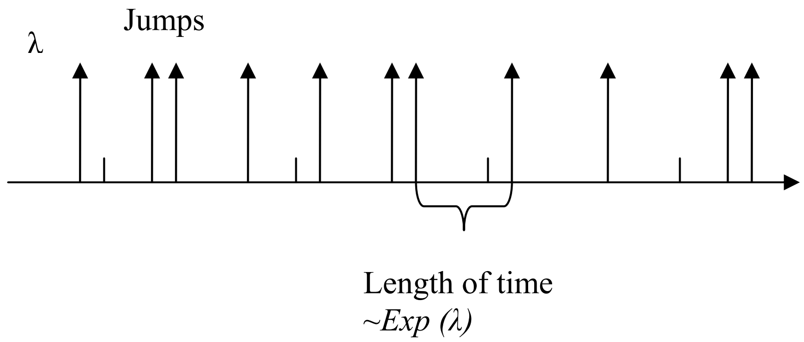
Figure 2.3 provides a schematic description of a Poisson process, where the arrows representing the occurrences of jumps.

Figure 2.3 A schematic description of Poisson process



There's one important feature that is of particular relevance to finance, that is, the waiting time between the occurrences of each jump follows exponential distribution (~ *exp(λ)*), as described in figure 2.4.

Figure 2.4 Exponentially distributed the interarrival times



Therefore, denote τ as the first arrival time, namely the time between time 0 and the first jump, following the above argument τ is exponential distributed with pdf

$$f(\tau) = \lambda e^{-\lambda t} , \qquad (2.3)$$

which implies

$$P(\tau > t) = e^{-\lambda t} . \qquad (2.4)$$

2.2 The Markov Property of Poisson Processes[1]

In this section we pay attention to the Markov property of the Poisson process. Stated briefly, this is the '**Memorylessness**' property where the distribution of the Poisson process in the future is independent of the past. For e.g. at time 0 the probability of not observing a jump

[1] Section 2.2 is based on the *Jan Röman* , Lecture Notes in Analytical Finance One, p 125.

over a time horizon t is simply *exp(-λt)* by (2.4). Now assume that we return to the process after a time s and the process is still at 0 (i.e. no jump has yet occurred). The probability of not observing a jump for a further time t (i.e. no jump until time $t + s$) given that no jump has occurred until time *s* is:

$$P(\tau > t+s | \tau > s) = \frac{P(\tau > t+s)}{P(\tau > s)} = e^{-\lambda s} \qquad (2.5)$$

This probability is recognized by (2.4) simply as the time 0 probability of not observing a jump over a time horizon *s*. This illustrates the Markov property; the fact that the process has not jumped until time *s* (whatever *s* might be) does not dictate the probability of future jumps.

How is a Poisson process mathematically characterized? Quite simply, the value of a standard Poisson process after a time t has elapsed is simply:

$$N(t) = N(0) + \sum_{s<t} \left[N(s) - N(s^-) \right] \qquad (2.6)$$

In this expression, *N(0)* is simply the initial condition (set to zero in a standard Poisson process). The latter term is the mathematical expression for "the number of jumps in the time interval (*0,t*)". Since the process jumps finitely in infinitesimal time, the time s^- corresponds to an infinitesimal time step before time s, and where a jump is observed [$N(s) - N(s^-)$] is 1; otherwise it is 0. In its more useful form, the process can also be expressed as *dN(t)* which models the change in the Poisson process over a time step *dt*. Using the Markov property the value of *dN(t)* at any time *t* does not depend on the history of the Poisson process. Furthermore,

$$p(\tau < dt) = 1 - P(\tau < dt) = 1 - e^{-\lambda dt} = 1 - \left(1 - \lambda dt + \frac{1}{2}\lambda^2 (dt)^2 - \ldots\right) \sim \lambda dt\ , \qquad (2.7)$$

because dt is very small. Thus, *dN(t)* can be thought of as a random variable that increases by 1 over a time step *dt* with probability *λdt* and is zero with probability *1 -λdt*.

Expression (2.8) motivates the following simple definitions of Poisson Process.

> Poisson Process: A Process describing a situation where events happen at random. The probability of an event in time dt is *λ dt*, where *λ* is the intensity of the process.
> (Hull, 2003, p710)

or equivalently

> A Poisson process dq is defined by
>
> $$dq = \begin{cases} 0 & \text{with probability } 1 - \lambda dt \\ 1 & \text{with probability } \lambda dt \end{cases}$$
>
> *λ* is the intensity of the process.
>
> (Wilmott, 2001, p370)

Note that there're many different but equivalent definitions of Poisson process, the definition given above is probably the shortest but most abstract type, that is why we did it introduce it in the first place. This definition is actually equivalent to what is given in section 2.1 where Poisson process are defined as a Lévy process with Poisson distributed increments.

2.3 Applications of Poisson Processes in Finance

Example

Price jumps occurs for a given stock follow a Poisson process with rate $\lambda = 5$ jumps per year.

(a) What is the probability that 3 jumps will occur in the next half-year?
(b) What is the probability that 10 or more jumps will occur in the next two years?
(c) What is the probability that it will be more 1/6 years before the occurrence of the next jump?

Solution:

(a) $\lambda = 5$ per year, and $\Delta = 0.5$ year, so $\lambda\Delta = 2.5$ and X is distributed as Poisson(2.5), where

X = the number of jumps in the next half-year.

$P(X=3) = e^{-2.5}\, 2.5^3/3! = 0.2138.$

(b) $\lambda = 5$ per year, and $\Delta = 2$ years, so $\lambda\Delta = 10$.

X = the number of jumps in next 2 years ~ Poisson(10)
$P(X \geq 10) = 1 - P(X < 10) = 1 - F(9) = 1 - 0.4579 = 0.5421.$

(c) $\lambda = 5$ per year, so the waiting time T (in years) between two consecutive jumps is distributed as *Exp(5)*.

P(Wait more than 1/6 years) = $P(T > 1/6)$
$= 1 - P(\boldsymbol{T} \leq 1/6) = 1-[1-e^{-5/6}] = e^{-5/6} = 0.4346.$

3. Jump Diffusion Model

3.1 The model[2]

If X is a stochastic diffusion process that jumps, then it is called jump diffusion:

$$dX_t = A_t(t,X)dt + B_t(t,X)dW + C_t(t,X)dN \tag{3.1}$$

The first two terms are the drift and noise, where dW is a standard Wiener process. They have been used to model stock prices in finance. The last term introduces the possibility of a jump occurring. *'dN'* constitutes a standard Poisson process; over a time interval dt a jump of size 1 can be observed with probability dt. The scaling by $C_t(x,t)$ allows the jump size to vary. This jump component has zero mean during a finite interval h:

$$E[\Delta N_t] = 0 \tag{3.2}$$

We need to make this assumption, since the term is part of the unpredictable innovation term. Any predicable part of the jumps may be included in the drift component A_t. We assume the following structure for the jump: Between jumps, N remain constant. We assume that there are k possible types of jumps, with sizes $\{A_i, i = 1,....,k\}$. The jumps occur at a rate λ that may depend on the latest observed X_t. Ones the jump occurs, the jump type is selected randomly and independently. The probability that the jump of size A_i will occur is given by p_i.

$$\Delta N_t = \Delta J_t - \left[\lambda_t h\left(\sum_{i=1}^{k} A_i p_i\right)\right] \tag{3.3}$$

where J_t is a process that represents the sum of all jumps up to time t. The term $\sum_{i=1}^{k} A_i p_1$ is the expected size of a jump, whereas $\lambda_t h$ represents the probability that the jump will occur. This is subtracted from ΔJ_t to make ΔN_t unpredictable.
Under these conditions , the drift coefficient A_t is representing the sum of two separate drifts, one belonging to the Wiener continuous component, the other to the pure jumps in X_t,

$$A_t = \alpha_t + \lambda_t\left(\sum_{i=1}^{k} A_i p_i\right) \tag{3.4}$$

Where α_t is a drift coefficient of the continuous movements in X_t.

These models are becoming increasingly important in modelling stocks as they result in distributions with 'fatter tails' than the standard Ito processes. They are also being used to model energy and power prices where the jump behaviour is very often observed.

The key for mathematical finance is to now derive the SDE for a function $F(X)$. The key is to consider the process X as the sum of 2 processes:

$$dX^c = A(t,X)dt + B(t,X)dW \tag{3.5}$$

[2] Section 3.1 is based on the *Jan Röman* , Lecture Notes in Analytical Finance One, p 127. and an extension from Salih N. Neftci, An Introduction to the Mathematics of Financial Derivatives, p 248.

and a pure jump process:
$$dY = C(t, X)dN \tag{3.6}$$

Then, to consider the Taylor series expansion of $F(x)$ by first considering the contribution from the continuous process and then the jump process:

$$dF = \frac{dF}{dX}dX^c + \frac{1}{2}\frac{d^2F}{dX^2}(dX^c)^2 + \left[F(X + C(t, X)) - F(X)\right]dN \tag{3.7}$$

The last term arises from the jump component. $[x + C(x,t)]$ denotes the value of the process x just after a jump. The majority of the times the last term is zero because $dN=0$. Only in those cases when a jump occurs the last term is non-zero and the jump in x is also observed in the function F.

3.2 Practical Problems[3]

The advantage of jump-diffusion process is that describes better the reality by both economic (microeconomic logic) and by the statistical time-series (explaining the skewness, fatter tails, abnormal movements of ex. oil prices) point of view. But there are some problems with jump-diffusion processes: it's impossible to build a riskless portfolios and it is difficult to estimate the parameters.

The first problem when considering jumps in the option valuation is that is impossible to build a perfect hedge. So, in general is not possible to build a riskless portfolio as in Black-Scholes-Merton contingent claims approach.

The alternatives are:

a. assume that the jump-risk is non-systematic (uncorrelated with the market portfolio) and so returning the risk-free interest rate (Merton, 1976);
b. look for the minimum variance of the portfolio for hedging and valuation purposes;
c. specify an utility function for the investor (single agent optimality or a detailed equilibrium description);
d. assume that the firm is risk-neutral (some people argues that the individual investor is risk-averse but the firm is risk-neutral); or
e. use the dynamic programming with an exogenous risk-adjusted discount rate, or with a "market estimated" discount rate as proxy.

The second problem is to estimate the parameters. There are several parameters to estimate, and in general is hard to estimate the law (and the parameters) for the jump-size distribution (mainly because we are interested in large but rare jumps, so there is a lack of data to estimate the jump-size parameters).
The two general approaches to estimate parameters in options models are:
(1) statistical approach, from historical data; and
(2) implied approach from the market data, that is, from the options prices (as is performed with implied volatility, to get implied jump-arrival).

[3] Source has been taken from Dias, Marco A.G.'s financial web page.

For the statistical approach there are several methods, from the Classical School using Kalman Filter; to the Bayesian School using Markov Chain Monte Carlo.
Researchers working with jump-diffusions processes has been using several approaches to joint estimate of parameters in jump-diffusion processes. Others authors take a more practical approach and separated the parameters estimative from these two independent processes. By taking out of the sample the jumps, you get a time-series to estimate the diffusion parameters (like volatility, and in case of mean-diffusion, the long-run mean and the diffusion speed). The out sample data, the jumps, is counted to determine the jump frequency λ. In this approach, is necessary to define jump in the data. The mean size of the jump and the dispersion of this size could be inferred with the data, but as the number of jump samples is too small, a practical feeling about the jump-size is also acceptable.

4. Conclusion

A Poisson process is a contiuous-time stochastic process and are one of the most important classes of stochastic processes. A Poisson process is similar to a Wiener process since they both belong to family of Lévy processes and therefore share the feature of "stationary independent increments".

In finance we often talk about jumps instead of events, where λ is a measure of the frequency of jumps. A Poisson process is a pure jump process: a process that changes instantaneosly form one value to another at random times.

The diffusion models are becoming increasingly important in modelling stocks as they result in distributions with 'fatter tails' than the standard Ito processes. They are also being used to model energy and power prices where the jump behaviour is very often observed. But there are some problems with jump-diffusion processes. It is impossible to build a riskless portfolio and it’s difficult to estimate the parameters.

5. References

Dias, Marco A.G., Pontifícia Universidade Católica do Rio de Janeiro (PUC-RJ)), Industrial Engineering Department, viewed 2005-10-05 <http://sphere.rdc.puc-rio.br/marco.ind/rev-jump.html>

Hull,.J.C., Options, Futures, and Other Derivatives, 5ed, Pearson Education Inc,. Upper Saddel River, N. J. 2003

McMullen, C.T., *Poisson Process*, Harvard University, viewed 2005-10-05 <http://abel.math.harvard.edu/~ctm/home/text/class/harvard/191/97/html/index.html>

Röman, J., *Lecture Notes in Analytical Finance One*, working paper, Mälardalens University

Salih N. Neftci, An Introduction to the Mathematics of Financial Derivatives, 2ed, Elsevier Science (USA), 2000

Wilmot, P., *Introduces Quantitative Finance*, Wiley, Chichester, 2001