Malardalen University
Mathematics and Physics Department

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Project in Analytical Finance I

Exotic Option II

Asian and Forward Options

Izabela Matusiak

Kwok-wai Choy

Maziar Saei

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1. **Asian Option**

1.1 Introduction

The Asian Option (also called Average Option) is the option whose payoff depends on the average value of the underlying asset over pre - specified period. They are used in many different markets, the most popular are foreign exchange and commodity contracts. The use of average makes the option more stable and less exposed to the manipulation in price by the underlying, which is especially important in illiquid markets such as commodities markets.

Pricing formula for the average option was first developed and used by two financiers David Spaughton and Mark Standish while they were working for Bankers Trust at the end of the eighties in Tokyo. That is why they called it *Asian*.

There are two basic forms of Asian Option:

Average price option is an option which at expiry pays the difference between the average value of the underlying during the life of the option (averaging period) and a fixed strike.

Average strike option is an option which at expiry pays the difference between the underlying market price and the strike which is an average of the underlying price over the specified averaging period.

In our work we will concentrate on the Average price option since it is much more common.

When talking about Asian Option we must specify:

- The averaging period
- The sampling frequency
- The averaging method:

Arithmetically average: $\frac{s_1 + s_2 + ... + s_n}{n}$

Geometric average $= \sqrt[n]{(s_1 * s_2 * ......\ s_n)}$

They can also be weighted with some weights w_i;

Weighted arithmetic average = $\frac{w_1 s_1 + w_2 s_2 + ... + w_n s_n}{w_1 + w_2 + ... + w_n}$

Weighted geometric average = $\sqrt[w_1 w_2 ..... w_n]{s_1^{w_1} s_2^{w_2} ....s_n^{w_n}}$

where $\boldsymbol{s_i}$ denotes the price of the underlying at time i and $\boldsymbol{w_i}$ denotes some weight

1.2 Binomial model

Let A(T) denotes the value of the average at date T, then the payoff of an average fixed strike call option at maturity is given by :

$$\text{AC(T)} = \max\left\{ A(T) - k, 0 \right\} = \left\{ \frac{1}{T}\int_0^T s_t d_t - k, 0 \right\}$$

The payoff for an average fixed strike put option at maturity will be therefore defined by :

$$\text{AP(T)} = \max\left\{ K - A(T), 0 \right\}$$

Let take a standard lattice:

T-2	T-1	T	probability
		S(T-2)U^2	P^2
	S(T-2)U		
S(T-2)		S(T-2)UD	$2P(1-P)$
	S(T-2)D		
		S(T-2) D^2	$(1-P)^2$

We use the lattice to calculate the average. The result is

T-2	T-1	T	probability	A(T)
		$S(T-2)U^2$	P^2	$\left[\left[S(T-2)+S(T-2)U+S(T-2)^2\right] /3\right]$
S(T-2)		$S(T-2)UD$	p(1-p)	$\left[\left[S(T-2)+S(T-2)U+S(T-2)UD\right] /3\right]$
		$S(T-2)UD$	p(1-p)	$\left[\left[S(T-2)+S(T-2)D+S(T-2)UD\right] /3\right]$
		$S(T-2)D^2$	$(1-p)^2$	$\left[\left[S(T-2)+S(T-2)D+S(T-2)D^2\right] /3\right]$

1.3 A mean value option

Here we construct an exotic European option with the payoff at time T_2 (maturity) equals:

$$X = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(u) du$$

where $T_2 < T_1$

It is known that

$$\begin{cases} dS(t) = r \cdot S(t) \cdot dt + \sigma \cdot S(t) \cdot dW \\ S(T) = s \end{cases}$$

After the integration we will obtain:

$$S(t) = s + r \cdot \int_0^t S(u) \cdot du + \sigma \int_0^t S(u) \cdot dW(u)$$

The price of an exotic European option is discounted expected value of future payoff that is given by:

$$\Pi[X|F] = e^{-r(T_2-t)} E_{t,s}^Q \left[\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(u) du \right] = \frac{e^{-r(T_2-t)}}{T_2 - T_1} \int_{T_1}^{T_2} E_{t,s}^Q [S(u)] du$$

The expected value of S(t) is equal to:

$$E[S(t)] = s + r \int_0^t E[S(u)] \cdot du + 0$$

Let $E[S(t)]$ = m and let take derivative of m

$$\begin{cases} \dot{m}(t) = t \cdot m(t) \\ m(0) = s \end{cases}$$

Then $m(t)$ = $E[S(t)]$ = se^{rt}

If we substitute it we'll receive the price of the option given by:

$$\Pi[X|F] = \frac{s \cdot e^{-r(T_2 - t)}}{T_2 - T_1} \int_{T_1}^{T_2} e^{r(u-t)} du = \frac{s/r}{T_2 - T_1} \cdot \left(1 - e^{-r(T_2 - T_1)}\right)$$

1.4 Pricing

- The price of Asian option is discounted expected value of the payoff at maturity. To calculate the expected value, the probability distribution for average should be known. Unfortunately, the sum of log-normally distributed variables is not log-normally distributed so there is no exact pricing formula for average option. However, there exists weak approximation by Turnbull and Wakeman in the book of Haug.

$$P_{call} \approx Se^{(b-r)T} N(d_1) - Ke^{-rT} N(d_2)$$

$$P_{put} \approx Ke^{-rT} N(-d_2) - Se^{(D-r)T} N(-d_1)$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(b + \frac{\sigma^2{}_A}{2}\right)T}{\sigma_A \sqrt{T}}$$

$$d_2 = d_1 - \sigma_A \sqrt{T}$$

$$b = \frac{\ln(M_1)}{T}$$

where M_1 is a first moment

$$M_1 = \frac{e^{(r-D)T} - e^{(r-D)\tau}}{(r-D)(T-\tau)}$$

And the adjusted volatility is given by

$$\sigma_A = \sqrt{\frac{\ln(M_2)}{T} - 2b}$$

Where M_2 is a second moment

$$M_2 = \frac{2e^{(2(r-D)+\sigma^2)T}S^2}{(r-D+\sigma^2)(2r-2q+\sigma^2)T^2} + \frac{2S^2}{(r-D)T^2}\left(\frac{1}{2(r-D)+\sigma^2} - \frac{e^{(r-D)T}}{r-D+\sigma^2}\right)$$

- The geometric average of log-normally distributed variables is also log-normally distributed so there exists exact formula to price geometric averaging options

$$P_{call} = Se^{(b-r)T}N(d_1) - Ke^{-rT}N(d_2)$$
$$P_{put} = Ke^{-rT}N(-d_2) - Se^{(b-r)T}N(-d_1)$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(b + \frac{\sigma^2{}_A}{2}\right)T}{\sigma_A\sqrt{T}}$$

$$d_2 = d_1 - \sigma_A\sqrt{T}$$

$$b = \frac{1}{2}\left(r - \frac{\sigma^2}{6}\right)$$

And the adjusted volatility equals:

$$\sigma_A = \sqrt{\frac{\ln(M_2)}{T} - 2b}$$

- To price any kind of Asian option we can also use **Monte Carlo simulation**. This method give us quite accurate value, what is more, it is possible to reduce the variance of estimates by applying the *control variate method*. Then we obtain estimate of price of similar derivative which analytical solutions is known.
  If the price of the geometric averaging Asian option is known, we can obtain the price of arithmetic one by applying the equation:

  $$V_A = V_A^* - V_B^* + V_B$$

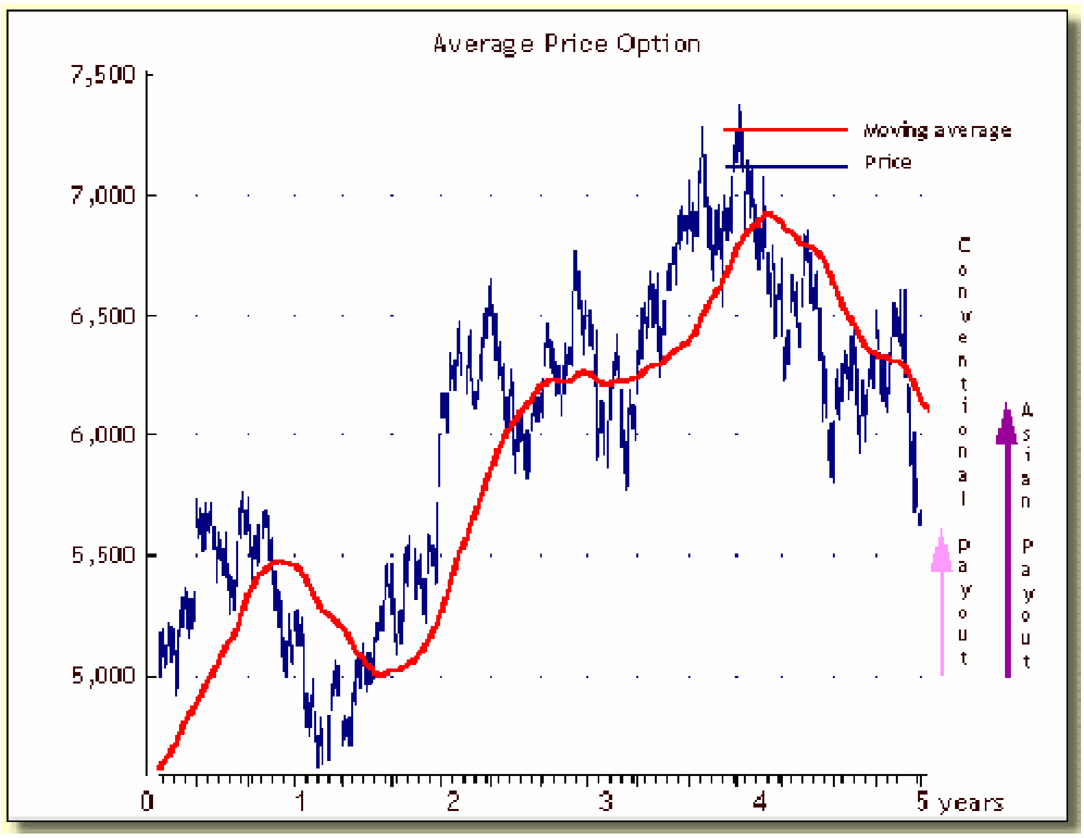
  where

  V'_A - estimated value of the arithmetic Asian through simulation
  V'_B - simulated value of the geometric Asian
  V_B - value of the geometric Asian

1.5 Conclusion

The advantage of Asian option is that any manipulation of underlying price or dramatic drop of the price close to the maturity date will be average up. From the other hand, any single market advance will have little effect on the average since it will be average down.
For that reason, the volatility of the option is lower as well as the price comparing to the standard option. Average price options are useful when the trader (hedger) is concerned only about the average price of a commodity which they regularly purchase.

Comparison of the payoff the Asian Option and the standard one.



2. **Forward options**

2.1 Introduction

Forward options are options with the strike set at some specified future date. They are also called: Forward start options or Delayed-start options. Although the actual strike is unknown, the contract specifies future parity ratio at present which is a relation between the strike and the underlying market at that time. For instance, the option may be set as at-the-money, or some percentage of in-the-money or out-of-the-money at the time it start.

Forward option can be either a call or put option. It can be regarded as a usual European call or put option start at some future time. The time to expiration is also specified at present.

Apart from forward options, there is another option called ratchet options which are related to forward options. It is a series of consecutive forward start options. The first is exercised at some future time. Then the second becomes active when the first expires, etc. Each option is set at-the-money when it becomes active. The effect of this progress is an option that periodically generates some profits.

An application of the forward options can be found in some companies where employees looking to receive an option, which are at-the-money on the day of grant.

2.2 Pricing

Any financial derivatives can basically be priced from the Black-Scholes PDE. Here we would like to review how to price typical European options through Black-Scholes PDE. Under general common assumptions, the pricing equation of the European options in a riskless world is as follow

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S^2} + rS(t)\frac{\partial V}{\partial S} - rV = 0 \quad (1)$$

where V is the price of the option
with some final condition such as $V(T,S) = G(S)$.

The solution of the above equation can be represented by the Feynman-Kac formula

$$V(S(t),t) = e^{-r(T-t)}\tilde{E}[G(S(T))|S(t)] \quad (2)$$

where the expectation operator is taken under the risk-neutral measure.

The forward start option is nothing but a European option that will start at some future time. Thus we can price it through equation (1) and (2).

Pricing procedure for the forward option

We assume the present time is t and the forward option will start at some future time T_1 and has maturity T.

The following backward procedure can be used to calculate the price of the forward option:

(1) Calculate the final payoff of the option at time T.
(2) Calculate the value of the payoff at time T_1; this is given by equation (2) with $t = T_1$.
(3) Check the conditions and calculate the value of the option at $t < T_1$ again by equation (2) to get the solution

$$V(S(t),t) = e^{-r(T_1-t)}\tilde{E}[V(S(T_1),T_1)|S(t)] \quad (3)$$

Rubinstein formula

On the other hand, a forward option can also be priced using the Rubinstein formula. To simplify the notation, we assumed the present time is zero. The forward option will start at a future time t and expire at time T. The strike is set at a constant α time the asset price, other notation will be the same as the pricing formula for the European options. b is the cost of carry rate:

$$P_{call} = Se^{(b-r)t}[e^{(b-r)(T-t)}N(d_1) - \alpha e^{-r(T-t)}N(d_2)]$$
$$P_{put} = Se^{(b-r)t}[\alpha e^{-r(T-t)}N(-d_2) - e^{(b-r)(T-t)}N(-d_1)] \quad (4)$$
$$d_1 = \frac{\ln(\frac{1}{\alpha}) + (b + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, d_2 = d_1 - \sigma\sqrt{T-t}$$

Pricing of a cliquet option

The pricing of a cliquet option can be done once we know the price of each forward options.
It consists of several forward options where the strike price for the next exercise date is proportional to the asset price as of the previous exercise date.
For example, a one-year ratchet call option with half-year payments will normally have two payments which is equal to the difference between the asset price and the strike price at the previous exercise date. Usually, the strike price of the first option equals the asset price of today. Price of a ratchet option is the sum of the prices of the forward starting options. We can use Black-Scholes formula to price the cliquet option.

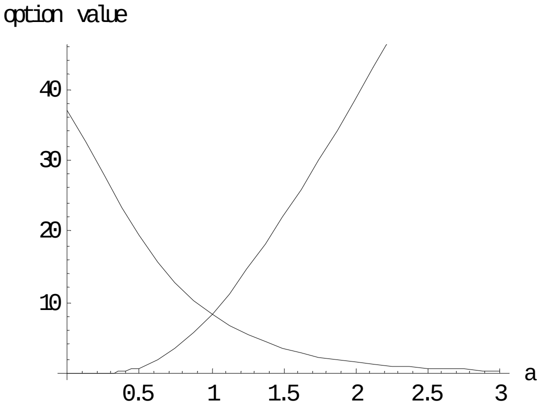
$$P_{call} = \sum_{i=1}^{n} Se^{(b-r)t_i}[e^{(b-r)(T_i-t_i)}N(d_1) - \alpha e^{-r(T_i-t_i)}N(d_2)] \quad (5)$$

where n is the number of settlements, t_i is the time the forward option start, and T_i is the maturity time of the forward option. Equation (5) is the sum of each forward call option.
Similarly, the price of a ratchet put option is a sum of forward put options.

Moreover, out-of-the-money, in-the-money cliquets can also be handled in the same partial differential equation pricing framework.

Price Curves

Here we use the Rubinstein formula, equation 4, to calculate the change of the value of the put and call option with respect with the change of α. We put in the asset price 50, risk-free rate 10%, volatility 40%, future time at year 1, and maturity at year 3. We assume the cost of carry rate is zero.



From the graph, we can find that as α becomes larger, the price of the put option will increase, on the other hand, the price of the call option will decrease. When α becomes smaller, the situation will be opposite.

This can be explained as follow. As α is the ratio between the strike price and the asset price. The larger the α, the larger the strike price, which in turns result in a smaller value of the call option, but a higher value of the put option.

2.3 Conclusion

Here we have provided an approach to price the forward options under the standard Black-Scholes model**.** Although we can price the forward option in the above methods, hedging this type of option still remains a difficult and an open area. The reason is that we used to assume the volatility of the underlying asset is constant over the time period. This is, however, not the case in the real situation because it is well known that the volatility of the asset is also a stochastic variable. So the main risk with forward options is the wrong assumption of the constant volatility, thus this may result in a significant risk in pricing these options with the above pricing method.

Thus hedging this type of option remains difficult in some sense if we do not know how the volatility changes over some future time period.

3. References

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