MÄLARDALENS HÖGSKOLA
Institution Mathematics and Physics
MT1410 Analytical Finance I, 5 points
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Black-Scholes Limitations

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Abstract

In this report we mainly study the limitations of the best-known continuous time model, the Black-Scholes model, and other analytical models which are the improvements of Black-Scholes formula.

We firstly discuss defects of Black-Scholes assumption (part 2). Later, we are going to give basic theory about jump-diffusion model and volatility models: Smile, stochastic and uncertain. (part 3 and 4).

The jump-diffusion process describes better the reality by both point of view, economic (microeconomic logic) and by the statistical time-series (explaining the skewness, fatter tails, and abnormal movements of prices).

Volatility smiles and surfaces show directly that volatility is not a simple constant. The classical way of dealing with random variables is to model them stochastically.We can do the same for volatility. There is also the uncertain volatility model which assume volatility to lie within a range of values.

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1. Introduction

Black-Scholes model was developed by Fischer Black and Myron Scholes in 1973. Myron Scholes and Robert C. Merton were awarded the Nobel Prize in Economics for their work in developing the Black-Scholes formula in 1997. Regrettably, Fisher Black died August 1995. The model describes the value of a European option on an asset with no cash flows and requires only five inputs: the asset price(S), the strike price(E), the time to maturity(T), the risk-free rate(R) of interest and the volitility(σ).

The model is used by everyone working in derivitives, whether they are salesman, trader or quants. In many ways, especially with regards to commercial success, the Black-Scholes model is remarkably robust. In words, we can not say that we have mastered option pricing theory unless we undersand the Black-Scholes formula. Nevertheless, there is room for improvement. Certainly, we can find models that better describe the underlying, such as volatility models and jump diffusion model.

Several models for volatility have been proposed in the option pricing literature. The simplest model assumes constant volatility. This was the approach taken by Black and Scholes (1973) and Merton (1973) in the work which laid the foundations for the modern analysis of options and still explain observed market prices for options. More complicated models assume volatility surfaces across underlying asset prices and time.

The classical way of dealing with random variables is to model them stochastically.We can do the same for volatility. *Stochastic volatility* is a generalization where the instantaneous volatility becomes a random variable $\sigma(t)$, which is then described by a stochastic process model.

The uncertain volatility model was independently developed by Lyons (1995) and Avellaneda et al. (1995). In this case, volatility is assumed to lie within a range of values. All that can be computed are the best case and the worst case prices, for a specified long or short position. By assuming the worst case, an investor can hedge his/her position and obtain a non-negative balance in the hedging portfolio, regardless of the actual volatility movement, provided that volatility remanins within the specified range.

Merton (1976) suggested that distributions with fatter tails than the lognormal might explain the tendency for deep-in-the-money, deep-out-of-the money, and short-maturity options to sell for more than their Black-Scholes value, and the tendency of near-the-money and longer-maturity options to sell for less. Merton priced options on jump-diffusion processes under the assumption of diversifiable jump risk and independent lognormal distributed jumps. Subsequent work by Jones (1984), Naik and Lee (1990), and Bates (1991) indicates that Merton's model with modified parameters is still relevant even under nondiversifiable jump risk. Others have proposed alternate option pricing models under fat-tailed shocks: McCulloch's (1987) stable Paretian model, Madan and Seneta's (1990) variance-gamma model, and Heston's (1993b) gamma process.[1]

[1] CENTRE OF BUSINESS ANALYSIS AND RESEARCH, (CoBAR), DIVISION OF BUSINESS & ENTERPRISE, UNIVERSITY OF SOUTH AUSTRALIA, WORKING PAPERS, ISSN 1443-2943: http://business.unisa.edu.au/cobar/workingpapers/cobar/2000-04.pdf

2. Black – Scholes defects[2]

2.1 Delta hedging is continuous

This is definitely not true because hedging must be done on a discrete time. Often the time between rehedges will depend on the level of transactions costs in the market for the underlying; the lower the costs, the more frequent the rehedgning.

2.1.1 What is delta hedging?

Delta hedging means holding one of the option and short a quantity Δ of the underlying. It is a way to reduce or even eliminate the risk by carefully choosing Δ.

Suppose Π is the value of a portfolio,

$$\Pi = V(S,t) - \Delta S$$

$$d\Pi = dV - \Delta dS$$

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta dS$$

The terms which with dS are random term and random terms are the risk in this portfolio. In order to eliminate risk, the random terms are better disappeared.

$$\left(\frac{\partial V}{\partial S} - \Delta\right) dS = 0 \Rightarrow \Delta = \frac{\partial V}{\partial S}$$

2.1.2 Black-Scholes assumption of continuous delta hedging

Delta can be expressed as a function of S and t, for example call option:

Call: $e^{-D(T-t)} N(d_1)$

and from Black-Scholes formula, we know

$$d1 = [\log(S/E) + (r - D - 1/2\,\sigma^2)(T-t)] / \sigma\sqrt{T-t}$$

Since variables S and t are ever-changing variables, the number of assets held (Δ) has to continuously change to maintain a delta neutral position.

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[2] Paul Wilmott on Quantitative Finance, Chapter 22 & Paul Wilmott Introduces Quantitative Finance, Chapters 8,10.

2.1.3 Delta hedging can not be done continuously in practice

Changing the number of assets held (Δ) requires the continual purchase or sale of the stock. This is called rehedging or rebanlancing the portfolio. The time between rehedges will depend on the level of transaction costs in the market. The lower the costs, the more frequent the rehedging.

In highly liquid market where it is relatively costless to buy and sell, the delta hedging may take place very frequently. But in less liquid markets, the hedging may take place less frequently since you may lose a lot on bid-offer spread. Moreover, you may not even be able to buy or sell in the quantities you want. Then, there is risk that has not been eliminated.

Therefore, we can say the at the Black-Scholes assumption of continuous delta hedging is too perfect. In practice, hedging must be done in discrete time.

2.2 There are no costs in delta hedging

In some markets the cost of delta hedging is insignificant while in other markets, it is expensive since there is a bid-offer spread on most underlyings. Therefore, we cannot rehedge as often as we can.

The difference between markets is due to the number of participants.

2.3 Volatility is a known constant (or a known deterministic function)

The Back-Scholes *formulae* require the volatility to be a known deterministic function of time while the Back-Scholes *equation* requires volatility to be a known function of time and the asset value. However, neither of this is true because volatility is very variable and unpredictable. Thus, volatility is not a constant nor a deterministic function of time and the underlying.

In order to observe or measure volatility, one must place a bound on its value which restricts it to lie within a given range.

We will give the details in the Part 3 of the report.

2.4 The underlying asset path is continuous

The market is discontinuous, meaning from time to time they 'jump', which is not incorporated in the lognormal asset price model.

It is said 'jump' because first, the sudden moves occur too frenquently and they are too large to be from a normaly distributed function and second because they are unhedgeable meaning the movements are too sudden for continuous hedging. Thus, a jump-diffusion model incorporates discontinuities into the price path, however risk elimination is no longer possible. (See part 4)

3. Volatility Models[3]

3.1. Volatility Smiles and Surfaces

One of the incorrect assumptions of the Black-Scholes world is that the volatility of the underlying is constant. If volatility is not a simple constant then perhaps it is a more complicated function of time and/or the underlying.

3.1.1 Implied volatility and volatility smiles

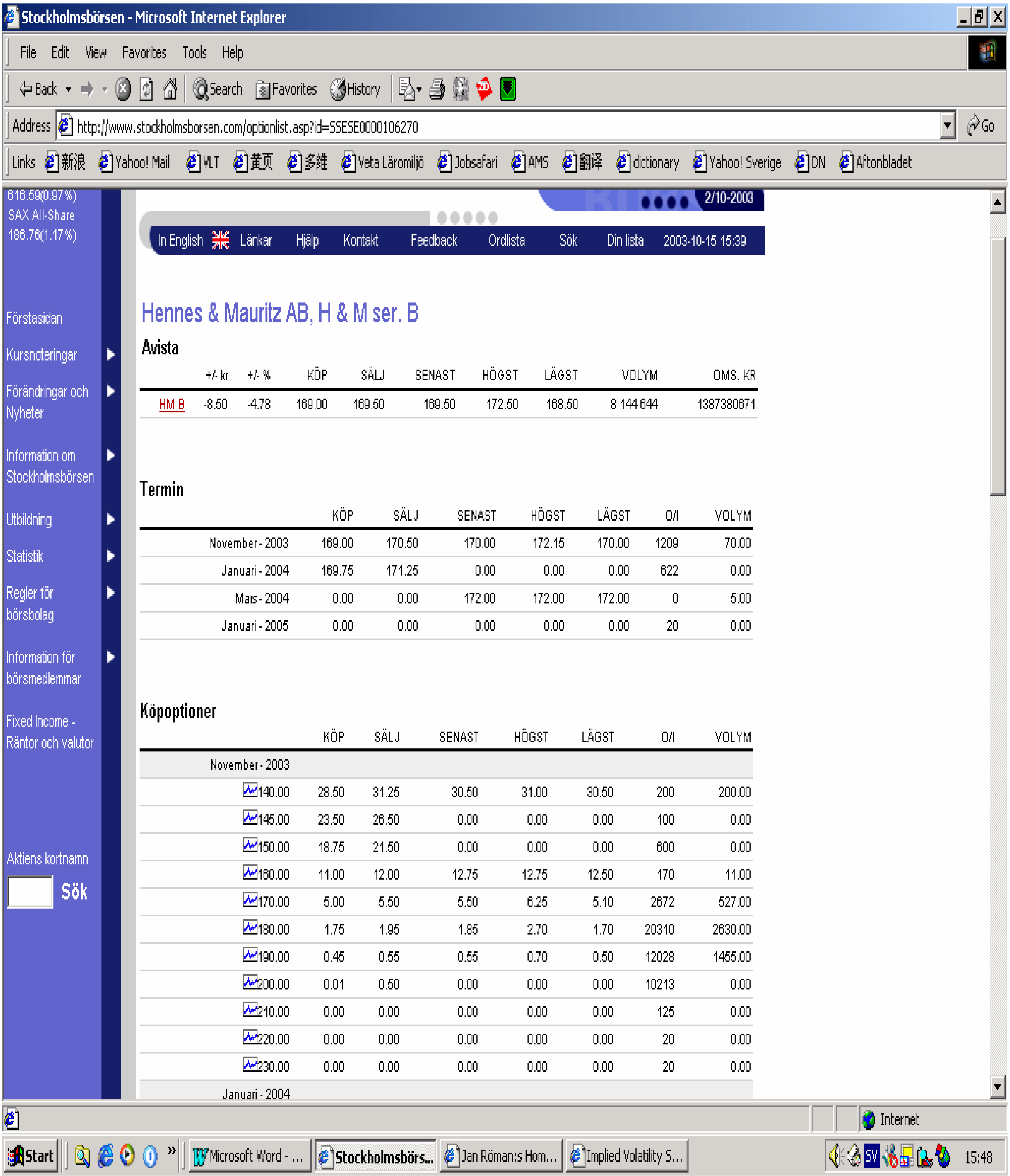
In the Black-Scholes formula, if we put in the expiry, the strike, the underlying and the interest rate together with the ***volatility***, we can get the price easily, since volatility is given. But in the real life, how do we know what volatility to put into the formulas? Normally, a trader can see on his screen that a certain call option with six months until expiry and a strike of 100 is trading at 6.51 with the underlying at 101.5 and a short-term interest rate of 8%. Can we use this information in some way? Yes, if we can see the price at which the option is trading, we can take the price and deduce the volatility. This is called the **implied volatility.** It is the volatility of the underlying which when substituted into the Black-Scholes formula gives a theoretical price equal to the market price.

The shape of this implied volatility versus strike curve is called the **smile**. In some markets it shows considerable asymmetry, a **skew**, and sometimes it is upside down in a **frown**.

In order to give reader an intuitive picture, we will show how the shape looks like by using the real data. Firstly we find the information of H&M call option from the Stockholm Exchange Market, then we can use this information to calculate the implied volatility and draw the graph. (see figure 1,2 and 3)

[3] Paul Wilmott on Quantitative Finance, Chapters 25, 26,27, 28.

The implied volatility and the smile implicate that the volatility is not constant, it varies with strike price.



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Avista

	+/- kr	+/- %	KÖP	SÄLJ	SENAST	HÖGST	LÄGST	VOLYM	OMS. KR
HM B	-8.50	-4.78	169.00	169.50	169.50	172.50	168.50	8 144 644	1387380671

Termin

	KÖP	SÄLJ	SENAST	HÖGST	LÄGST	OI	VOLYM
November - 2003	169.00	170.50	170.00	172.15	170.00	1209	70.00
Januari - 2004	169.75	171.25	0.00	0.00	0.00	622	0.00
Mars - 2004	0.00	0.00	172.00	172.00	172.00	0	5.00
Januari - 2005	0.00	0.00	0.00	0.00	0.00	20	0.00

Köpoptioner

	KÖP	SÄLJ	SENAST	HÖGST	LÄGST	OI	VOLYM
November - 2003							
140.00	28.50	31.25	30.50	31.00	30.50	200	200.00
145.00	23.50	26.50	0.00	0.00	0.00	100	0.00
150.00	18.75	21.50	0.00	0.00	0.00	600	0.00
160.00	11.00	12.00	12.75	12.75	12.50	170	11.00
170.00	5.00	5.50	5.50	6.25	5.10	2672	527.00
180.00	1.75	1.95	1.85	2.70	1.70	20310	2630.00
190.00	0.45	0.55	0.55	0.70	0.50	12028	1455.00
200.00	0.01	0.50	0.00	0.00	0.00	10213	0.00
210.00	0.00	0.00	0.00	0.00	0.00	125	0.00
220.00	0.00	0.00	0.00	0.00	0.00	20	0.00
230.00	0.00	0.00	0.00	0.00	0.00	20	0.00
Januari - 2004							

Figure 1: Option price of H&M from the Stockholm Stock Exchange.(www.stockholmsborsen.com)

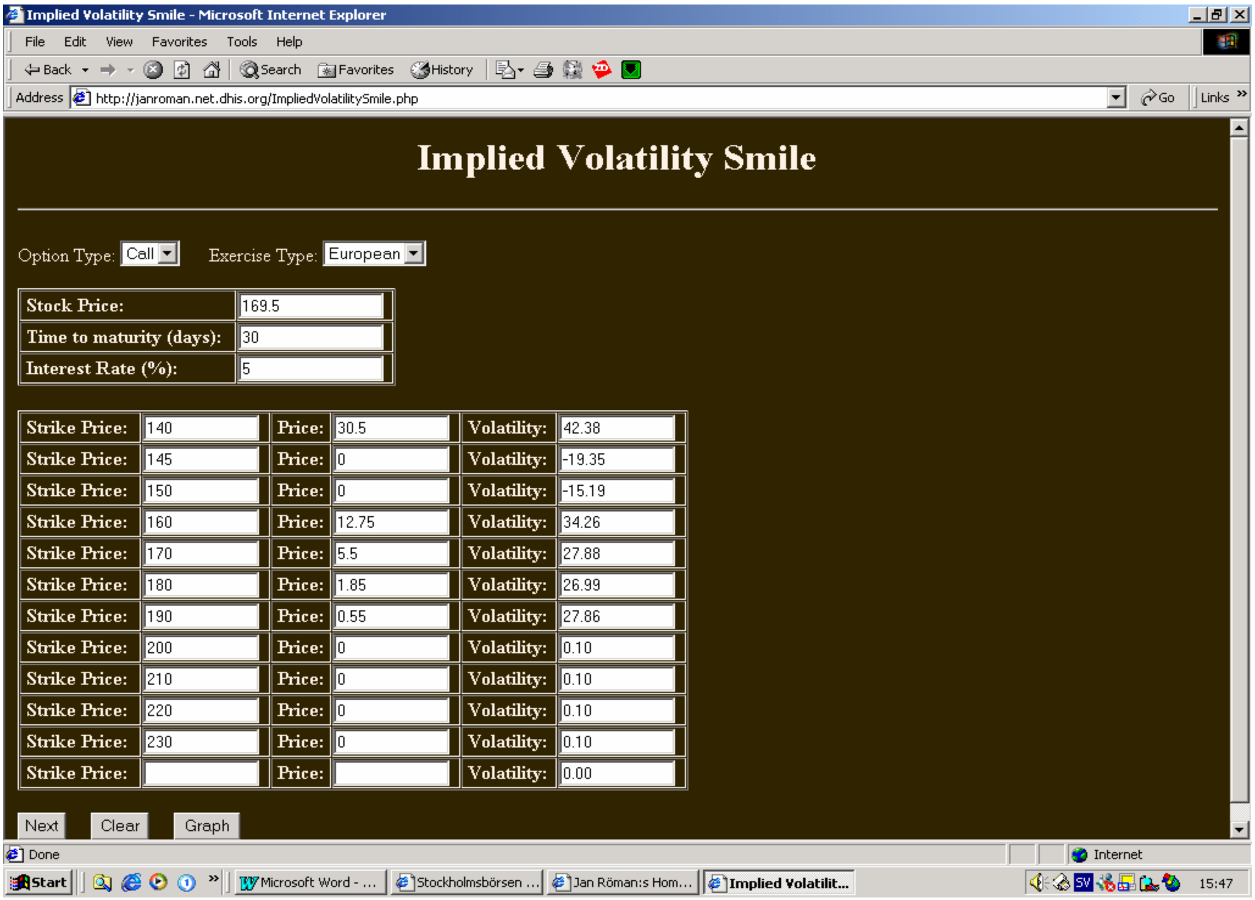

Figure 2: Calculation of Implied Volatility Smile.(http://janroman.net.dhis.org)

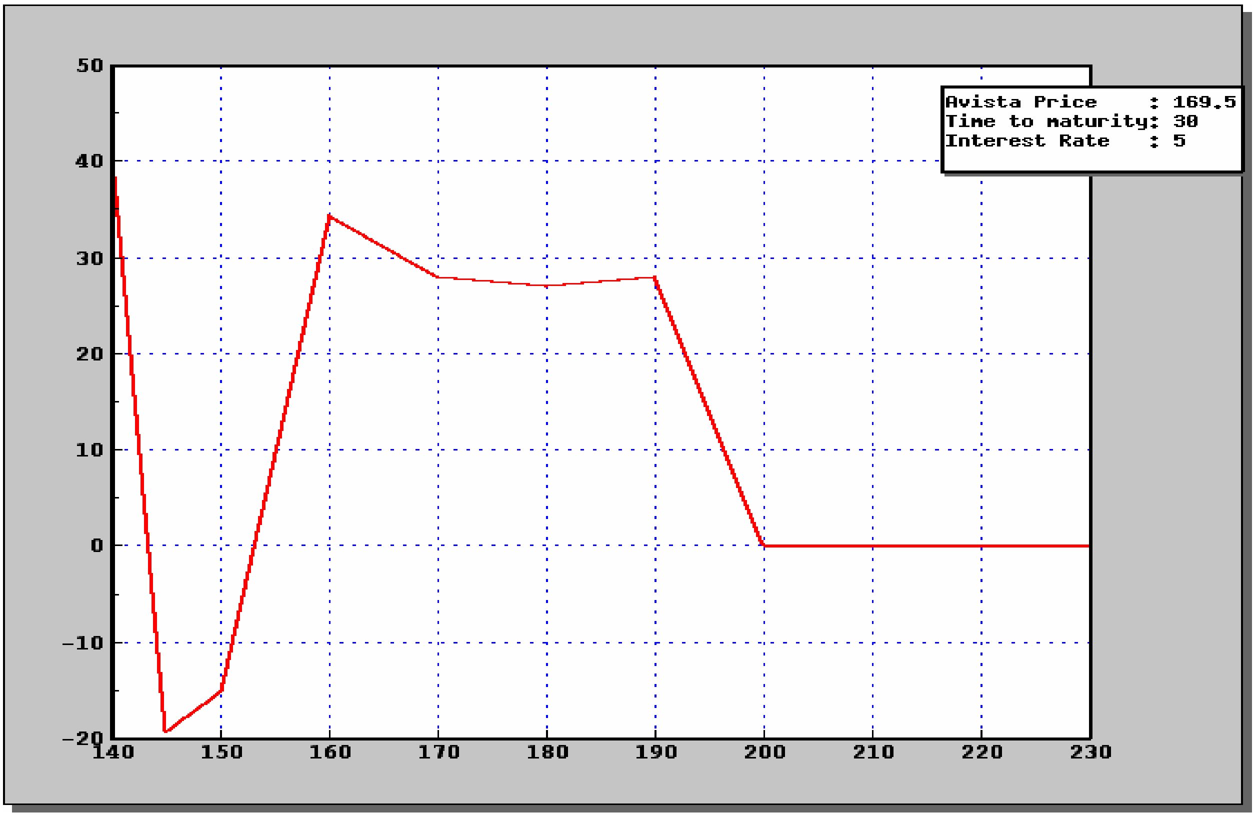

Figure 3: Implied volatility against strike price of H&M.(http://janroman.net.dhis.org).

3.1.2. Time-dependent volatility

In table 1 are the market prices of European call options with one, four and seven months until expiry. All have same strike price 105 and the underlying asset is currently 106.25. The short-term interest rate over this period is about 5.6%. If we calculate by substitution into the Black-Scholes call formula, these prices are consistent with volatility of 21.2%, 20.5%, and 19.4%for the one-, three- and seven-month options respectively. Clearly, if the volatility is constant for the whole seven months, the prices are cannot be correct.

Expiry	Strike price	Current price	Interest rate	Value of option	**Volatility**
1 month	105	106.25	5.6%	3.50	**21.2%**
3 month	105	106.25	5.6%	5.76	**20.5%**
7 month	105	106.25	5.6%	7.97	**19.4%**

Table 1. Market prices of European call oprions.

We better confirm this issue by using the Ericsson's history data. At figure 4, you can see how volatility appears to change with time.

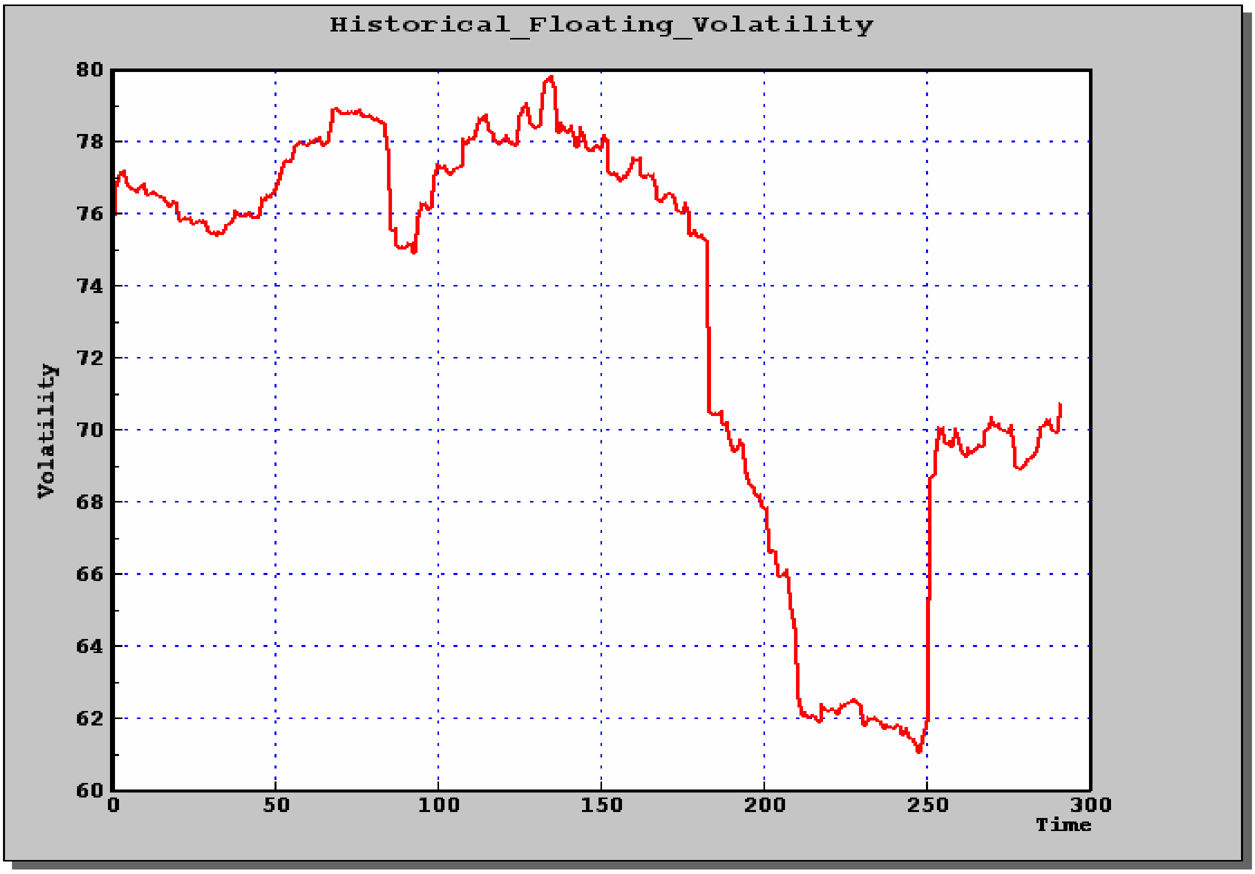

Figure 4: Volatiltiy against time to maturity of Ericsson**.** (http://janroman.net.dhis.org).

3.1.3. Volatility Surfaces

As the above discussion, we have already known that volatility is not only time-dependent but also price-dependent. In fact, we may as well go all the way and assume that volatility is a function of both the asset and the time.

The volatility surfaces can show us how implied volatility against both time and strike in a three dimensional plot. Now, the volatility surfaces are commonly used for pricing and hedging exotic contracts.

In this part, we just want to give a rough picture about how volatility surface looks like and it's dynamic behaviour.

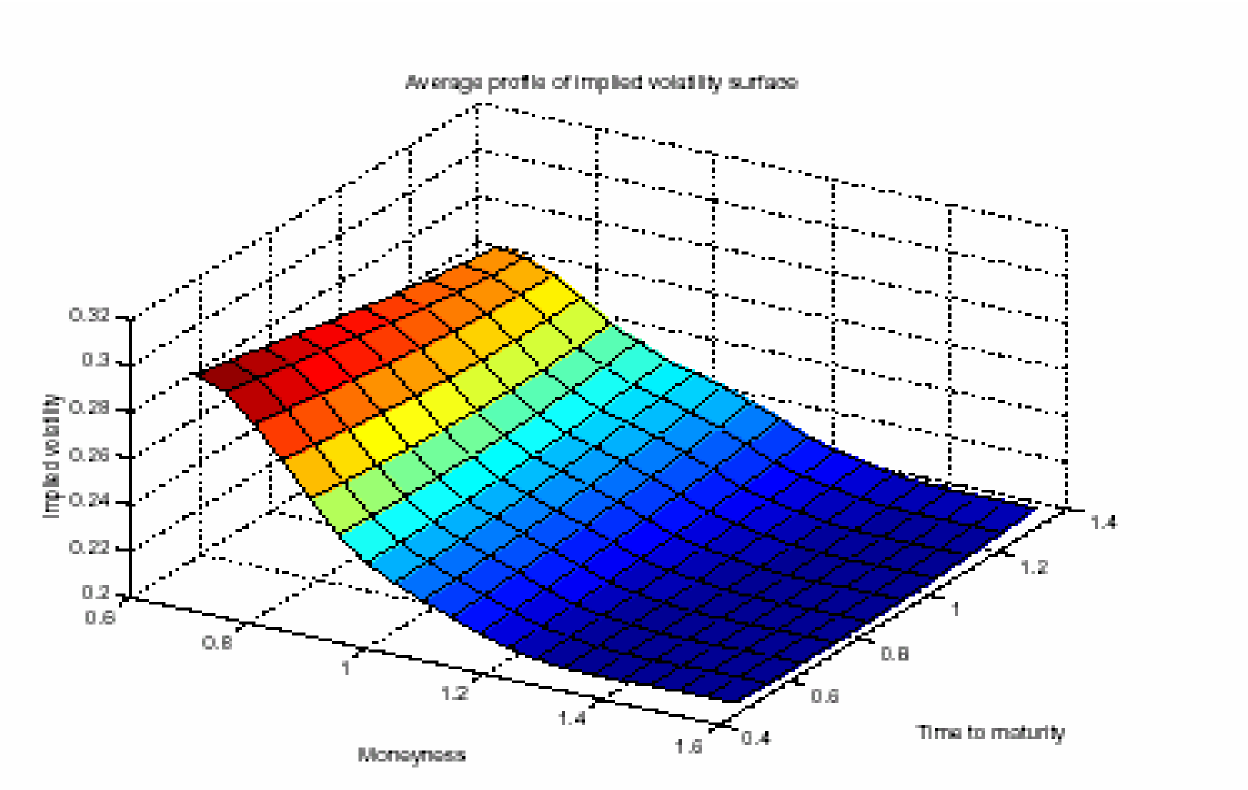

Figure 5: shows the average profile of the implied volatility of DAX options as a function of time to maturity and strike, 1999-2001. It is a non-flat surface. ("Stochastic Models of Implied Volatility Surfaces"---http://papers.ssrn.com).

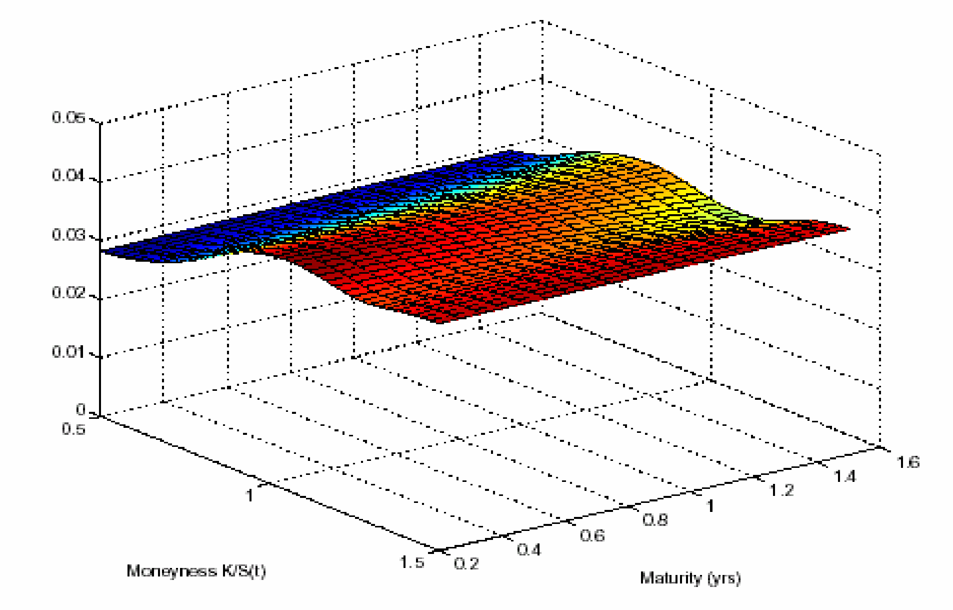

Figure 6: the daily implied volatility variations for SP 500 Index options. It reflects an overall shift in the level of all implied volatilities. ("Stochastic Models of Implied Volatility Surfaces”---http://papers.ssrn.com).

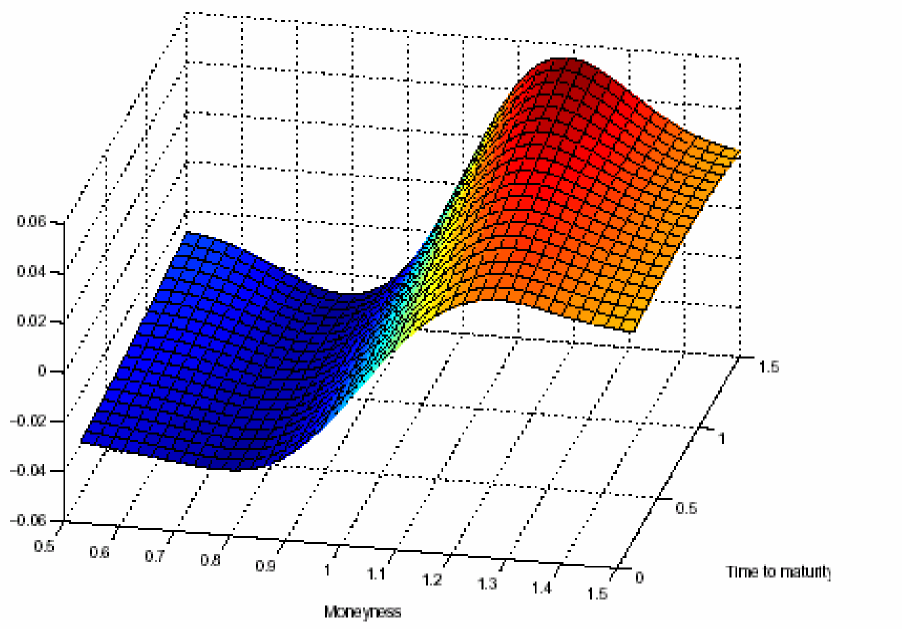

Figure 7: the daily implied volatility variations for SP 500 Index options. It reflects opposite movements in (out of the money) call and put implied volatilities. ("Stochastic Models of Implied Volatility Surfaces”---http://papers.ssrn.com).

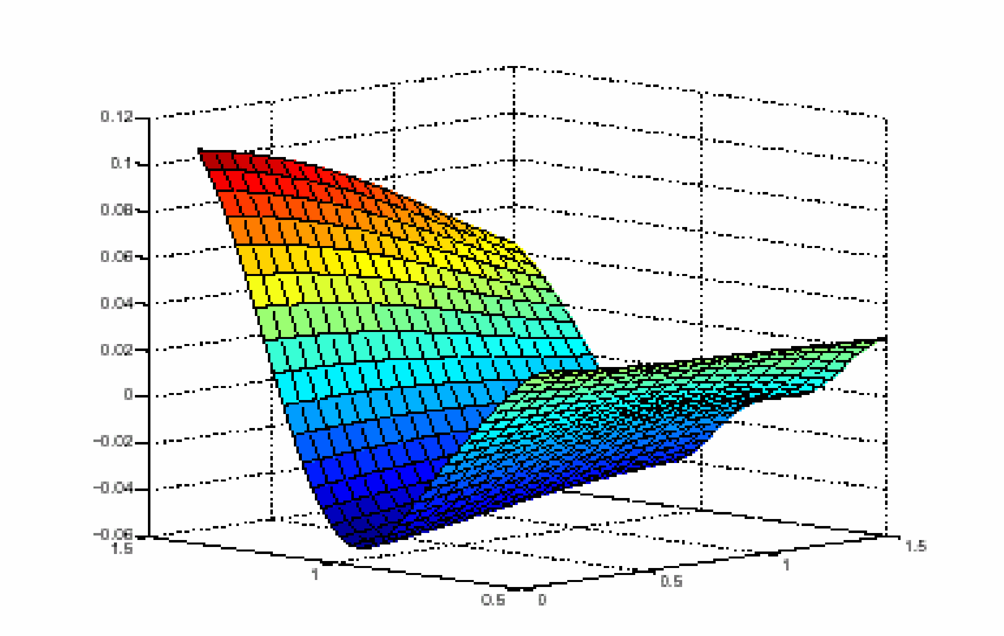

Figure 8: the daily implied volatility variations for SP 500 Index options. It reflects changes in the convexity of the surface. ("Stochastic Models of Implied Volatility Surfaces”---http://papers.ssrn.com).

3.2. Stochastic Volatility

In this part, we focus on modelling volatility as a stochastic variable, how to price contracts when volatility is stochastic, what is market price of volatility risk, and some named stochastic volatility models.

3.2.1. Modelling volatility as a stochastic volatility

We continue to assume that S satisfies

$$dS = \mu S dt + \sigma S dX_1 ,$$

and we further assume that volatility satisfies

$$d\sigma = p(S, \sigma, t)\, dt + q(S, \sigma, t)\, dX_2$$

The tow increments dX_1 and dX_2 have a correlation of ρ. The choice of functions $p(S, \sigma, t)$ and $q(S, \sigma, t)$ is important to the evolution of the volatility, and thus to the pricing of derivatives. (The choice of these functions will not be discussed in this report)

The value of an option with stochastic volatility is a function of three variables, $V(S, \sigma, t)$.

3.2.2. Price contracts when volatility is stochastic

When volatility is stochastic we are faced with the problem of having a source of randomness that cannot be easily hedged away. Because we have two sources of randomness we must hedge our option with two other contracts, one being the underlying asset as usual, but now we also need another option to hedge the *volatility risk*. We therefore must set up a portfolio containing one option, with value denoted by $V(S, \sigma, t)$, a quantity $-\Delta$ of the asset and a quantity $-\Delta_1$ of another option with value $V_1(S, \sigma, t)$.

Thus we have

$$\Pi = V - \Delta S - \Delta_1 V_1$$

By using Itô's lemma on function of S, σ and t, the change in this portfolio in a time dt is given by

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2}\right) dt - \Delta_1 \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V_1}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial \sigma^2}\right) dt$$

$$+ \left(\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta\right) dS + \left(\frac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial \sigma}\right) d\sigma$$

To eliminate all randomness from the portfolio we must choose

$\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta = 0$ to eliminate dS terms, and $\frac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial \sigma} = 0$ to eliminate dσ terms.

This leaves us with

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2}\right) dt - \Delta_1 \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V_1}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial \sigma^2}\right) dt$$

$$= r\,\Pi\, dt = r\,(V - \Delta S - \Delta_1 V_1)\, dt$$

where we used arbitrage arguments to set the return on the portfolio equal to the risk-free rate.

This is one equation with two unknowns, V and V_1. So we collect all V terms on the left-hand side and all V_1 terms on the right-hand side, we find that

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} + rS \frac{\partial V}{\partial S} - rV\right) / \frac{\partial V}{\partial \sigma} =$$

$$\left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V_1}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial \sigma^2} + rS \frac{\partial V_1}{\partial S} - rV_1\right) / \frac{\partial V_1}{\partial \sigma}$$

Since the two options will typically have different payoffs, strikes or expires, the only way for this to be possible equal is for both sides to be independent of the contract type. Both sides can only be function of the *independent* variables,. Thus we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} + rS \frac{\partial V}{\partial S} - rV = -(p - \lambda q) \frac{\partial V}{\partial \sigma}$$

for some function λ(S, σ, t). We usually write this equation as

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma qS\frac{\partial^2 V}{\partial S\partial\sigma}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2}+rS\frac{\partial V}{\partial S}+(p-\lambda q)\frac{\partial V}{\partial\sigma}-rV=0$$

It is the pricing equation. If we can solve it, then we will find the value of the option and the hedge rations. Note that we find two hedge ratios, $\frac{\partial V}{\partial S}$ and $\frac{\partial V}{\partial \sigma}$, since we have two sources of randomness that we must hedge away. The quantity **p- λq** is called the **risk-neutral drift rate** of the volatility.

3.2.3. The market price of volatility risk

The function λ(S, σ, t) is called the **market price of volatility risk**. What does this mean?

Suppose we hold one of the options with value V, and satisfying the above pricing equation, if delta hedged with the underlying asset only, we have

$$\Pi = V - \Delta S$$

The change in this portfolio value is

$$d\Pi=\left(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma qS\frac{\partial^2 V}{\partial S\partial\sigma}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2}\right)dt+\left(\frac{\partial V}{\partial S}-\Delta\right)dS+\frac{\partial V}{\partial\sigma}d\sigma$$

Because we are delta hedging the coefficient of dS is zero. We find that

$$d\Pi-r\Pi dt=\left(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma qS\frac{\partial^2 V}{\partial S\partial\sigma}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2}\right)dt+\frac{\partial V}{\partial\sigma}d\sigma-r\Pi dt$$

$$=\left[\left(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma qS\frac{\partial^2 V}{\partial S\partial\sigma}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2}\right)-r(V-\Delta S)\right]dt+\frac{\partial V}{\partial\sigma}d\sigma$$

$$=\left(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma qS\frac{\partial^2 V}{\partial S\partial\sigma}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2}+rS\frac{\partial V}{\partial S}-rV\right)dt+\frac{\partial V}{\partial\sigma}d\sigma$$

$$= [- (p - \lambda q)\frac{\partial V}{\partial\sigma}]\, dt + \frac{\partial V}{\partial\sigma}\, d\sigma$$

$$= -p\frac{\partial V}{\partial \sigma}dt + \lambda q\frac{\partial V}{\partial \sigma}dt + \frac{\partial V}{\partial \sigma}(pdt + qdX_2)$$

$$= \lambda q\frac{\partial V}{\partial \sigma}dt + q\frac{\partial V}{\partial \sigma}dX_2$$

$$= q\frac{\partial V}{\partial \sigma}(\lambda dt + dX_2)$$

Now, we can observe that for every unit of volatility risk, represented by dX_2, there are λ units of extra return, represented by dt. Hence the name" market price of risk".

3.2.4. Named stochastic volatility models

Hull & White (1987)

One of the stochastic volatility models considered by Hull & White was

$$d(\sigma^2) = a(b - \sigma^2)\,dt + c\sigma^2\,dX_2$$

Heston (1993)

In Heston's model

$$d\sigma = -\gamma\sigma\,dt + \delta\,dX_2,$$

with arbitrary correlation between the underlying and its volatility.

GARCH

Generalized autoregressive conditional heteroskedasticity, or GARCH for short, is a model for an asset and its associated volatility. As the time step tends to zero, the simplest GARCH model becomes the same as the stochastic volatility model

$$d(\sigma^2) = \phi(\theta - \sigma^2)\,dt + v\sigma^2\,dX_2$$

3.3. Uncertain Volatility

(The Model of Avellaneda, Levy&Parás and Lyons 1995)

3.3.1. Motivation

There are two traditional ways of measuring volatility: implied and historical. Whichever way is used, the result cannot be the future value of volatility; either it is the market's estimate of the future or an estimate of values in the past. The correct value of volatility to be used in an option calculation cannot be known until the option has expired. Therefore, how to value option when volatility is uncertain is a problem. Normally, we can be more certain about the correctness of a range of values than a single value: we will be happier to say that the volatility of a stock lies within the range 20-30% over the next six months than to say that the average volatility over this period will be 24%. The range we choose may be the range of past historical volatility, or implied volatilities, or include both of these. Then, the range for volatility leads to ranges for the option's value. Working in this area was started by Avellaneda, Levy, Paràs and Lyons with Uncertain Volatility Model in 1995.

The Uncertain volatility Model is an extension of the Black-Scholes framework that incorporates uncertainty in the volatility of the underlying asset in the pricing and hedging of derivative securities. In uncertain volatility model, no statistical distribution for the stochastic volatility is specified; rather, a worst-case and best-case are considered. By assuming the worst case, an investor can hedge his/her position and obtain a non-negative balance in the hedging portfolio, regardless of the actual volatility movement, provided that volatility remains within the specified range.

3.3.2. The basics of the uncertain volatility model

Let us suppose that the volatility lies within the band

$$\sigma^- < \sigma < \sigma^+$$

Same with the Black-Scholes hedging and no-arbitrage arguments, we firstly construct a portfolio of one option, with value V(S,t), and hedge it with $-\Delta$ of the underlying asset. The value of this portfolio is thus

$$\Pi = V - \Delta S$$

We still have

$$dS = \mu S dt + \sigma S dX$$

So the change in the value of this portfolio is

$$d\,\Pi = dV - \Delta dS$$

$$\Rightarrow \quad d\Pi = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt - \Delta dS$$

$$\Rightarrow \quad d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \left(\frac{\partial V}{\partial S} - \Delta\right)dS$$

since the choice of $\Delta = \frac{\partial V}{\partial S}$ eliminates the risk:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt$$

We will assume that the volatility over the next time step is such that our portfolio increases by the least amount. If we have a long position in a call option, we assume that the volatility is at the lower bound σ^-; for a short call we assume that the volatility is high. This amounts to considering the *minimum* return on the portfolio, where the minimum is taken over all possible values of the volatility within the given range. The return on this worst-case portfolio is then set equal to the risk-free rate:

$$\min_{\sigma^- < \sigma < \sigma^+} (d\,\Pi) = r\,\Pi\,dt$$

$$\Rightarrow \quad \min_{\sigma^- < \sigma < \sigma^+} \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt = r\,(V - S\frac{\partial V}{\partial S})\,dt$$

Now observe that the volatility term in the above is multiplied by the option's gamma. σ will be considered as function of gamma. Therefore the value of σ will give this its minimum value depends on the sign of the gamma. When the gamma is positive, we muse choose σ to be the lowest value σ^- and when it is negative we choose σ to be its highest value σ^+ for keeping the amounts to be minimum return.

Then we find that the worst-case value V^- satisfies:

$$\frac{\partial V-}{\partial t} + \frac{1}{2}\sigma(\Gamma)^2 S^2 \frac{\partial^2 V^-}{\partial S^2} + rS\frac{\partial V^-}{\partial S} - rV^- = 0$$

Where

$$\Gamma = \frac{\partial^2 V^-}{\partial S^2}$$

and

$$\sigma(\Gamma) = \begin{cases} \sigma^+ & \text{if } \Gamma < 0 \\ \sigma^- & \text{if } \Gamma > 0 \end{cases}$$

we can find the best option value V^+, by solving

$$\frac{\partial V+}{\partial t} + \frac{1}{2}\sigma(\Gamma)^2\, S^2 \frac{\partial^2 V^+}{\partial S^2} + rS\frac{\partial V^+}{\partial S} - rV^+ = 0$$

where

$$\Gamma = \frac{\partial^2 V^+}{\partial S^2}$$

but this time

$$\sigma(\Gamma) = \begin{cases} \sigma^+ & \text{if } \Gamma > 0 \\ \sigma^- & \text{if } \Gamma < 0 \end{cases}$$

In the real life, we won't find much use for the problem for the best case in practice since it would be financially suicidal to assume the best outcome. We go from the worst-case equation to the best. In other words, the problem for the worst price for the long and short position in a particular contract is mathematically equivalent to valuing a long position only, but in worst and best cases.

4. Jump diffusion[4]

Equities, currencies or interest rates do not follow the lognormal random walk. One evidence is the sudden moves of these financial quantities that can lead to unexpected fall or crash. These sudden moves occur more frequently than the return with Normal-distribution follow with a reasonable volatility. These moves looks discontinuous which indicates that the asset has *jumped*.

The difference between the real distribution of the assets and the Normal distribution is that the peak of the real distribution is higher than the Normal distribution, which means that that there is a likelihood of a small move that we would expect. In addition, the real distribution has flatter tails meaning there is a greater chance of large rise or fall than the Normal Distribution.

Before, when calculating random walks, continuous **Brownian motion,** which represents high level of normal activity, is based on normal distribution increment. In other words, we add the return from one day to the next a normal distribution random variable with a variance proportional to timestep to the calculation of the asset price. Now, we must also add the **jump-diffusion model** for an asset. For this extra part we use the **Poisson process,** which represents rare and extreme events. The size of the jump is constant but the probability of a jump increases with duration.

A Poisson process *dq* is defined by

$$dq = \begin{cases} 0 \text{ with probability } 1 - \lambda\, dt \\ 1 \text{ with probability } \lambda\, dt \end{cases},$$

where $\lambda\, dt$ is the probability of a jump in q in the timestep *dt*.

The parameter λ is called the **intensity** of the Poisson process. Thus, the price of an asset can be written as

$$dS = \mu S dt + \sigma S dX + (J - 1) S dq$$

There are two sources of risk, the diffusion, dX and the jump, dq and if there is a jump (dq = 1) then S immediately goes to the value JS under the assumption that there is no correlation between Brownian motion and Poisson Process .

[4] Paul Wilmott on Quantitative Finance, Chapter 29.

The random walk in log S is

$$d(\log S) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\, dX + (\log J)dq.$$

This is just a jump-diffusion version of Itô.

The price of a European non-path-dependent option can be written as

$$\sum_{n=1}^{\infty} \frac{1}{n!} e^{-\lambda'(T-t)} \left(\lambda'(T-t)^n\right) V_{BS}\left(S,t;\sigma_n, r_n\right).$$

In the above

$$\lambda' = \lambda(1+k),\ \ \sigma_n^2 = \sigma^2 + \frac{n\sigma^2}{T-t} \text{ and } r_n = r - \lambda k + \frac{n\log(1+k)}{T-t}$$

And V_{BS} is The Black- Scholes formula for the option the absence of jumps and k = E[J-1] . The formula of the price of a European non-path-dependent option represents the sum of individual Black-Scholes values, each are weighted according to the probability that there will have been *n* jumps before expiry.

	A	B	C	D	E	F
1	**Asset**	100		**Timestep**	**Asset with jump**	**Asset without jump**
2	**Drift**	20%		0	100	100
3	**Volatility**	20%		0,01	104,4508263	99,59308023
4	**Intensity**	0,1		0,02	106,2856503	100,7776901
5	**Jump**	20%		0,03	106,167958	99,39788434
6	**Timestep**	0,01		0,04	104,3289272	100,270379
7				0,05	102,4205944	102,624792
8	**Internal Rand**			0,06	105,0266332	103,0638148
9				0,07	105,4203776	101,4122392
10	0,5[illegible]			[illegible]	[illegible]	101,2206065
11				[illegible]	[illegible]	103,7212043
12				[illegible]	[illegible]	102,4171387
13				0,11	[illegible]	99,38041856
14				0,12	106,6722611	101,6961032
15				0,13	104,9304384	104,7652287
16				0,14	105,7728998	104,6698733
17				0,15	110,5924629	104,8933937
18				0,16	111,6736932	101,8811282
19				0,17	110,4327999	99,76465471
20				0,18	109,3887442	103,3093014
21				0,19	115,0449382	103,4514778
22				0,2	119,977663	106,6010859
23				0,21	121,9138842	104,0417555
24				0,22	123,8506101	101,2291064
25				0,23	125,7672543	100,4486804
26				0,24	127,9418484	102,139647
27				0,25	129,3114102	98,54407917
28				0,26	129,092342	97,93419652
29				0,27	126,5334529	98,07295726
30				0,28	121,804261	99,03905209
31				0,29	125,9365183	96,47479537
32				0,3	125,1386177	98,82987429
33				0,31	124,2836448	101,8708321
34				0,32	121,9778771	101,5885635
35				0,33	119,7847908	102,7628026

= E5*(1+$B$2*$B$6+$B$3*SQRT($B$6)*(RAND() + RAND() + RAND() + RAND() +RAND() +RAND() + RAND() +RAND() +RAND() +RAND() +RAND() +RAND() -6) - B$5*IF(RAND()<$B$4*$B$6, 1, 0)

Figure 9: Spreadsheet simulation of a jump-diffusion process and a non-jump-diffusion process. In this example the stock jumps by 20% at random times given by a Poisson process.

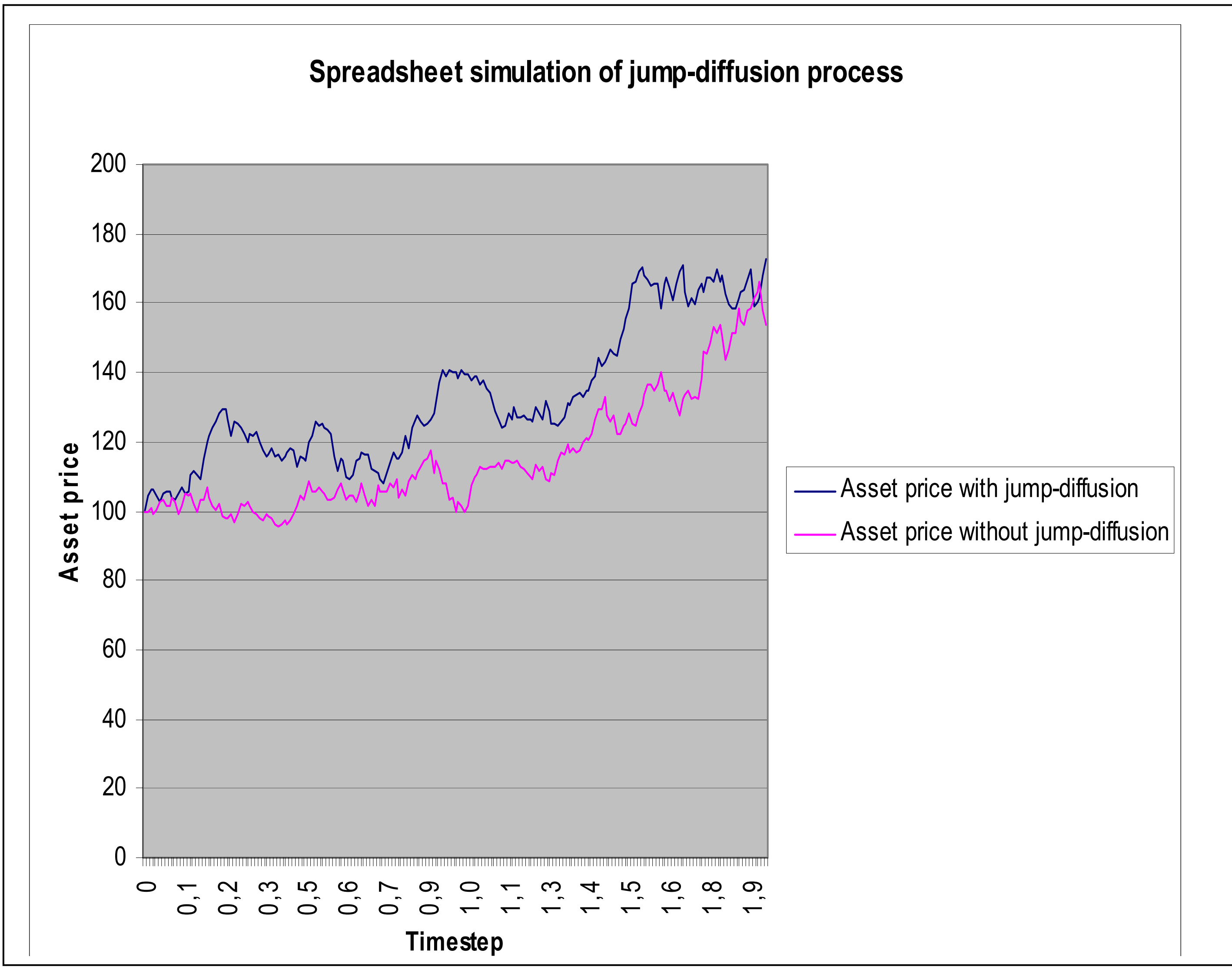

Figure 10. Graph of an asset price with and without jump.

4.1. Hedging when there are jumps

Holding a portfolio of one long option position and a short position in some quantity Δ of the underlying:

$$\Pi = V(S, t) - \Delta S$$

The change in the value of this portfolio from time t to $t+dt$ is due partly to the change in the underlying and partly to the change in the underlying:

$$d\Pi = dV - \Delta dS$$

Notice that Δ has not changed during the timestep because there are no changes in S. From Itô we have

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt$$

Thus the portfolio changes by

$$d\prod = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \left(\frac{\partial V}{\partial S} - \Delta\right)dS + (V(JS,t) - V(S,t) - \Delta(J-1)S)dq.$$

This is also a jump-diffusion version of Itô.

If there is no jump at time t so that $dq = 0$ then we could eliminate the risk by chosen $\Delta = \partial V/\partial S$. If there is a jump and $dq = 1$ then the portfolio changes in value by an amount which cannot be hedged away. In that case perhaps we choose Δ to minimize the variance of $d\Pi$. This presents us with a dilemma. We don't know weather to hedge the small diffusive changes in the underlying which are always present or the large moves which happen rarely.

4.1.1. Hedging the diffusion

If we chose $\Delta = \partial V/\partial S$ then we are reducing the random term, the risk in our portfolio, to zero. Thus, we are following a Black-Scholes type of strategy, hedging the diffusive movements. The changes in the portfolio value is then

$$d\prod = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + (V(JS,t) - V(S,t) - (J-1)S\frac{\partial V}{\partial S}dq.$$

The portfolio now evolves in a deterministic and nondeterministic (cannot be determine in advance) jump in its value. According to Merton (1976) the jump component of the asset price process is uncorrelated with the market as a whole, and then the risk in the discontinuity should not be priced into the option. Diversifiable risk should not be rewarded. Therefore, we set the expectation of this expression and set equal to the risk-free return (no arbitrage) from the portfolio.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + \lambda E[(V(JS,t) - V(S,t)] - \lambda\frac{\partial V}{\partial S}SE[J-1] = 0.$$

4.1.2. Hedging the jumps

Another possible way is to hedge both the diffusion and the jumps as much as we can.

The change in the value of the portfolio with an arbitrary Δ is, to leading order,

$$d\prod = \left(\frac{\partial V}{\partial S} - \Delta\right)dS + (-\Delta(J-1)S + V(JS,t) - V(S,t))dq + \ldots.$$

The variance in this changes, which is a measure of the risk in the portfolio is

$$\text{var}[d\prod] = \left(\frac{\partial V}{\partial S} - \Delta\right)^2 \sigma^2 S^2 dt + \lambda E[(-\Delta(J-1)S + V(JS,t) - V(S,t))^2 dt + \ldots.$$

This is minimized by the choice

$$\Delta = \frac{\lambda E[(J-1)(V(JS,t) - V(S,t))] + \sigma^2 S \dfrac{\partial V}{\partial S}}{\lambda SE[(J-1)^2] - \sigma^2 S}$$

(This formula is obtained by differentiation with respect to Δ and set the resulting expression equal to zero).

So, a value of an option as pure discounted real expectation under this best-hedge strategy then we find that

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial S}\left(\mu - \frac{\sigma^2}{d}\big(\mu + \lambda k - r\big)\right) - rV + \lambda E\left[(V(JS,t) - V(S,t))\left(1 - \frac{J-1}{d}\big(\mu + \lambda k - r\big)\right)\right] = 0.$$

where, $d = \lambda E[(J-1)^2] + \sigma^2$

when $\lambda = 0$ this collapses to the Black- Scholes equation. And if there is no diffusion, $\sigma = 0$, then we have

$$\Delta = \frac{E[(J-1)(V(JS,t) - V(S,t))]}{SE[(J-1)^2]}$$

and

$$\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} - rV + \lambda E\left[(V(JS,t) - V(S,t))\left(1 - \frac{J-1}{d}\left(\mu + \lambda k - r\right)\right)\right] = 0.$$

4.2. Jump Volatility

As we model volatility as a jump process, one can see that volatility is constant for a while, and then it randomly jumps to another value. Thus, volatility is in one of two states σ^- or $\sigma^+ > \sigma^-$. The jump from lower to higher value will be modelled by Poisson process with intensity λ^+ and intensity λ^- going the other way.

If we hedge the random movement in S with the underlying, then take real expectations, and see the return on the portfolio equal to the risk-free rate, we arrive at

$$\frac{\partial V^+}{\partial t} + \frac{1}{2}\sigma^{+2} S^2 \frac{\partial^2 V^+}{\partial S^2} + r\frac{\partial V^+}{\partial S} - rV^+ + \lambda^-(V^- - V^+) = 0.$$

for the value V^+ of the option when the volatility is σ^+. Similarly, we find that

$$\frac{\partial V^-}{\partial t} + \frac{1}{2}\sigma^{-2} S^2 \frac{\partial^2 V^-}{\partial S^2} + r\frac{\partial V^-}{\partial S} - rV^- + \lambda^+(V^+ - V^-) = 0.$$

for the value V^- when volatility is σ^-.

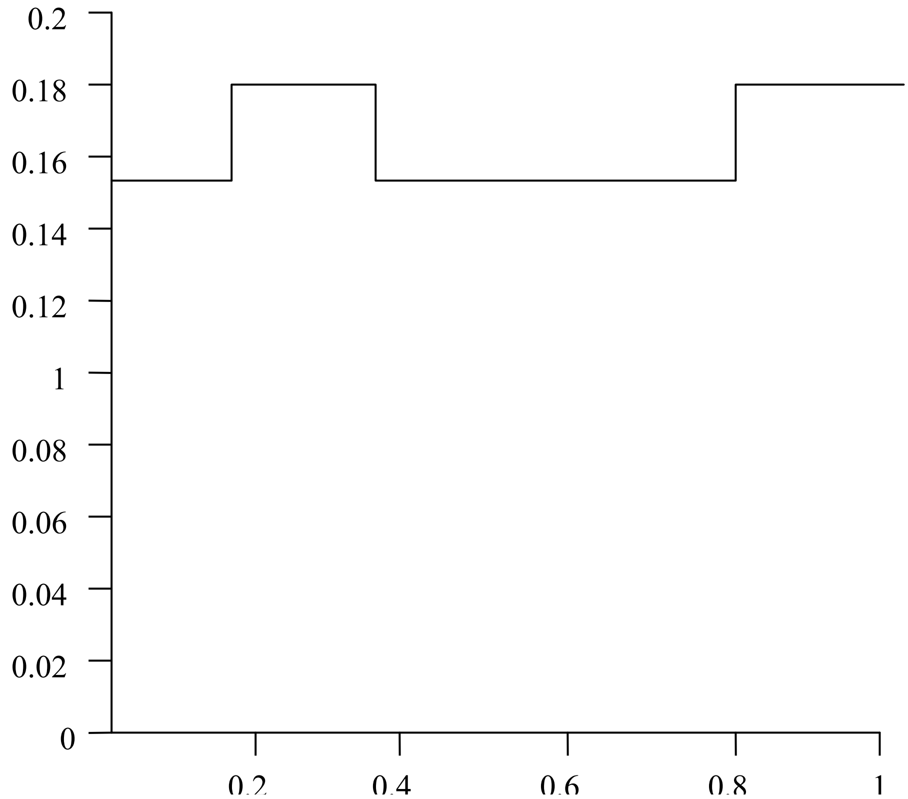

Figure 11. Jump volatility. (Paul Wilmott on Quantitative Finance, Chapter 29)

4.3. Jump Volatility with deterministic decay

A jump volatility that resembles the real behaviour of volatility also contains exponential decay of the volatility after the jump.

$$\sigma(\tau) = \sigma^- + (\sigma^+ - \sigma^-)\, e^{-v\tau}$$

where τ is the time since the last sudden jump in the volatility and v is a decay parameter. At any time, governed by a Poisson process with intensity λ, the volatility can jump from its present level to σ^+.

The value of an option is given by V(S, t, τ),

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma(\tau)^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + \lambda(V(S,t,0) - V(S,t,\tau) = 0.$$

This value means that we have delta hedge with the underlying to eliminate the risk due to movement of the asset but we have taken real expectations with respect to the volatility jump.

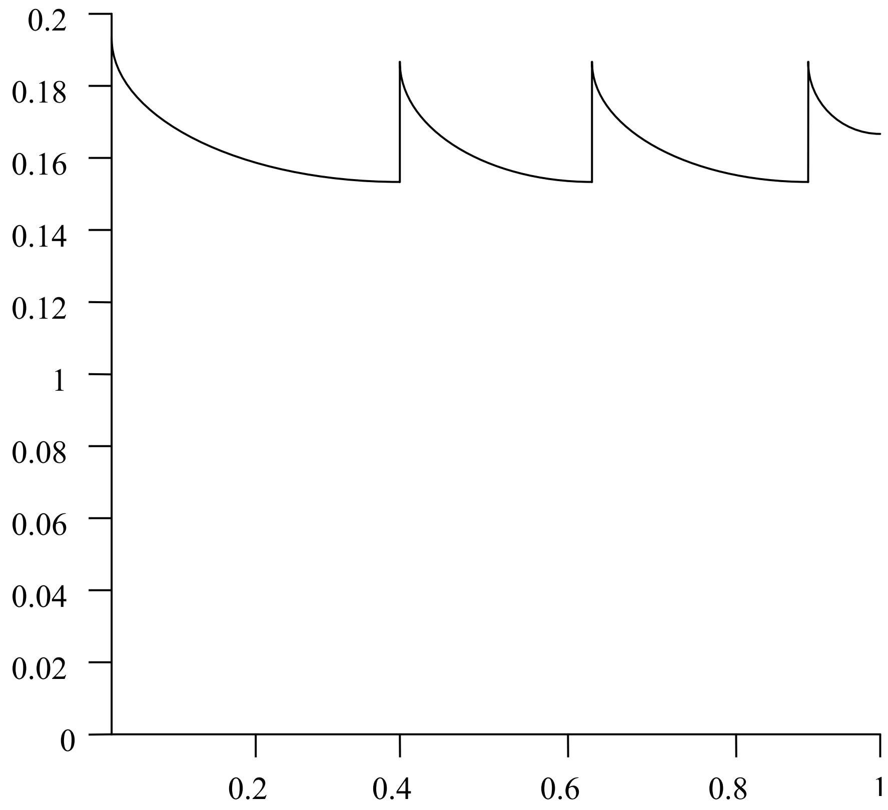

Figure 4. Jump volatility with exponential decay. (Paul Wilmott on Quantitative Finance, Chapter 29)

5. Conclusion

Black-Scholes model is the foundation for the modern analysis of options. However, there are many faults with it's assumptions. Therefore, some models was created to improve on Black-Scholes from a technical and mathematical point of view, such as volatility smile and surfaces, stochastic volatility, uncertain volatility and jump-diffusion.

Volatility smiles and surfaces show directly that volatility is not a simple constant. It is a more complicated function of time and function of assets ,as well as function of both. Because of the profound importance of volatility in the pricing of options, and because volatility is hard to estimate, observe or predict, the classical way is to model it stochastically.

In uncertain volatility model , volatility is assumed to lie within a range of values, and this range of volatility lead to range for the option price: best case and worst case. The best case is not usually used in practice since it would be financially suicidal to assume the best outcome. Moreover, the problem for the worst price for long and short positions in a particular contract is mathematically equivalent to valuing a long position only. In other words, if you calculate the worst case for long position, it means the best case for short position.

The advantage of jump-reversion process is that describes better the reality by both point of view, economic (microeconomic logic) and by the statistical time-series (explaining the skewness, fatter tails, and abnormal movements of prices). But there is a cost. The two problems with jump-diffusion processes are: the impossibility to build a riskless portfolio; and the difficulty with parameters estimative.

The first important problem when considering jumps in the option valuation is that is impossible to build a perfect hedge. So, in general is not possible to build a riskless portfolio as in Black-Scholes-Merton contingent claims approach, unless (1) assume that the jump-risk is non-systematic (uncorrelated with the market portfolio) and so returning the risk-free interest rate or (2) look for the minimum variance of the portfolio for hedging and valuation purposes.

The second problem is the job of the parameters estimative: There are several parameters to estimate, and in general is hard to estimate the law (and the parameters) for the jump-size distribution (mainly because we are interested in large but rare jumps, so there is a lack of data to estimate the jump-size parameters).

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