Department of Mathematics and Physics
MT 1410 Analytical Finance, 5 points
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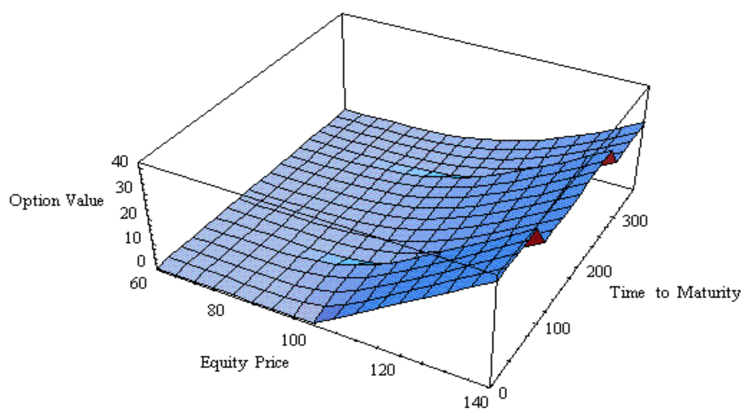
Black-Scholes Limitations



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Abstract

In this report we mainly study the limitation Black-Sholes model, as well other models which are improvements of Black-Scholes formulae.
First of all, we will consider the Black-Scholes defects and assumption of Black-Scholes model. We look at the delta hedging and later, we are going to give the basic theory about the volatility models such as smile, stochastic and theory about jump-diffusion.

The jump-diffusion process describes better the reality by both point of view, and by the statistical time series (explaining the skewness, fatter tails).

Volatility smiles and surface show directly that volatility is not a simple constant. The classical way of dealing with random variables is to model them stochastically. We can do the same for volatility to lie within a range of value.

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1. Introduction

In 1973, Fischer Black and Myron Scholes published "Value of Derivatives," the first successful model for pricing financial options. They introduced the Black-Scholes formula to aid in pricing stock options.

Over the ensuing 24 years, the financial community gradually accepted the Black-Scholes model and incorporated it into brokerage side of the financial business. In 1997, after years of real-world success, Scholes and Robert C. Merton of Harvard University were awarded Nobel Prize in economics "for a new method to determine the value of derivatives."

The Black and Scholes Option Pricing Model didn't appear overnight, in fact, Fisher Black started out working to create a valuation model for stock warrants. This work involved calculating a derivative to measure how the discount rate of a warrant varies with time and stock price. The result of this calculation held a striking resemblance to a well-known heat transfer equation. Soon after this discovery, Myron Scholes joined Black and the result of their work is a startlingly accurate option pricing model. Black and Scholes can't take all credit for their work; in fact their model is actually an improved version of a previous model developed by A. James Boness in his Ph.D. dissertation at the University of Chicago. Black and Scholes' improvements on the Boness model come in the form of a proof that the risk-free interest rate is the correct discount factor, and with the absence of assumptions regarding investor's risk preferences.

The Model:

$$C = SN(d_1) - Ke^{(-rt)}N(d_2)$$

C = Theoretical call premium

S = Current Stock price

t = time until option expiration

K = option striking price

r = risk - free interest rate

N = Cumulative standard normal distribution

e = exponential term (2.7183)

$$d_1 = \frac{\ln(S/K) + \left(r + {s^2}/{2}\right)t}{s\sqrt{t}}$$

$$d_2 = d_1 - s\sqrt{t}$$

s = standard deviation of stock returns

ln = natural logarithm

In order to understand the model itself, we divide it into two parts. The first part, SN(d1), derives the expected benefit from acquiring a stock outright. This is found by multiplying stock price [S] by the change in the call premium with respect to a change in the underlying stock price [N(d1)]. The second part of the model, Ke(-rt)N(d2), gives the present value of paying the exercise price on the expiration day. The fair market value of the call option is then calculated by taking the difference between these two parts.

2. Assumptions of the Black and Scholes Model:

1) The stock pays no dividends during the option's life

Most companies pay dividends to their share holders, so this might seem a serious limitation to the model considering the observation that higher dividend yields elicit lower call premiums. A common way of adjusting the model for this situation is to subtract the discounted value of a future dividend from the stock price.

2) European exercise terms are used

European exercise terms dictate that the option can only be exercised on the expiration date. American exercise term allow the option to be exercised at any time during the life of the option, making American options more valuable due to their greater flexibility. This limitation is not a major concern because very few calls are ever exercised before the last few days of their life. This is true because when you exercise a call early, you forfeit the remaining time value on the call and collect the intrinsic value. Towards the end of the life of a call, the remaining time value is very small, but the intrinsic value is the same.

3) Markets are efficient

This assumption suggests that people cannot consistently predict the direction of the market or an individual stock. The market operates continuously with share prices following a continuous Itô process. To understand what a continuous Itô process is, you must first know that a Markov process is "one where the observation in time period t depends only on the preceding observation." An Itô process is simply a Markov process in continuous time. If you were to draw a continuous process you would do so without picking the pen up from the piece of paper.

4) No commissions are charged

Usually market participants do have to pay a commission to buy or sell options. Even floor traders pay some kind of fee, but it is usually very small. The fees that Individual investor's pay is more substantial and can often distort the output of the model.

5) Interest rates remain constant and known

The Black and Scholes model uses the risk-free rate to represent this constant and known rate. In reality there is no such thing as the risk-free rate, but the discount rate on U.S. Government Treasury Bills with 30 days left until maturity is usually used to represent it. During periods of rapidly changing interest rates, these 30 day rates are often subject to change, thereby violating one of the assumptions of the model.

6) Returns are lognormal distributed

This assumption suggests, returns on the underlying stock are normally distributed, which is reasonable for most assets that offer options.

3. Delta hedging

We described how certain derivatives could be hedged perfectly (that is, *replicated*) if an appropriate portfolio strategy is adopted. Suppose that *S*(*t*) is the price of a non-dividend-paying share at time *t*. Let *X* represent the payoff on a derivative at its maturity date *T* with *S*(*t*) as the underlying quantity, and let

$$V(t, s) = e^{-r(T-t)} E_Q[X | S(t) = s]$$

represent the arbitrage-free price at some earlier time *t* where *Q* is the risk-neutral pricing measure, *r* is the risk-free rate of interest, and $\mathcal{F}_t$ is the history of the process up to time t.[2] The derivative payoff can be replicated perfectly if at all times *t* we hold

$$\Delta(t) = \left.\frac{\partial V}{\partial s}\right|_{(t, S(t))} \quad \text{units of the share, } S(t)$$

with the remainder $V(t) - \Delta(t)S(t)$ in cash. This approach is referred to as *Delta hedging*.

Delta hedging works well if:

- We are using the correct model (for example, the Black-Scholes-Merton model) for *S*(*t*) with the correct values for *r* and σ;
- we are able to rebalance the portfolio continuously;
- There are no transaction costs.

In practice we are only able to rebalance at discrete time intervals. This is because continuous rebalancing is physically impractical and, more importantly, because it results in infinite transaction costs.

An example of the effect of discrete-time rebalancing is given in Figure 1 for a European call option. In the left-hand plots we have used the correct volatility for pricing and hedging. We can see that, approximately, the surplus (accumulated hedging error at *T*) is centered around zero. Also, we can see that the standard deviation of the surplus is approximately halved when we switch from 4-day rebalancing to 1-day rebalancing. This trend continues as we rebalance more and more frequently: that is, if Δt is the time between rebalances, then the standard deviation of the surplus at maturity is of the order of Δt.

The right-hand plots show the impact of underestimating the volatility (for both pricing and hedging). First, we can see that the average surplus is now negative, because we have undercharged. Second, the standard deviation of the errors is larger. Third, rebalancing more frequently does not reduce significantly the standard deviation because we are using the wrong Δt.

So we have identified two sources of hedging error:

- Due to discrete-time rebalancing;
- Due to errors in the estimated parameters.

We could add to this errors in the model used. For example, the" true" model might have jumps in prices rather than a Brownian motion type of diffusion, or price volatility might be stochastic rather than constant.

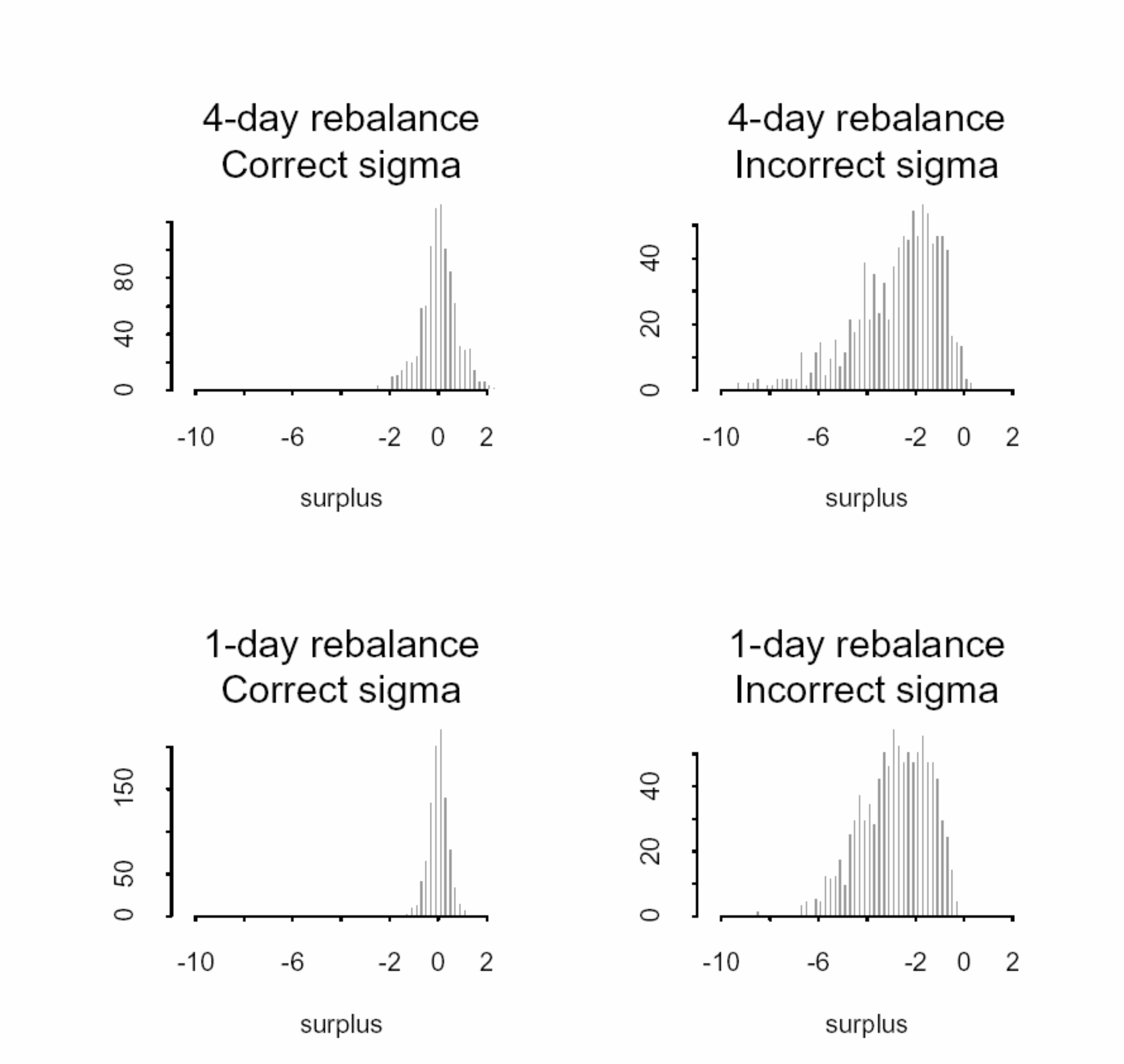


Figure 1.Hedging errors for a European call option.

The plots show histograms of the hedging errors based on 1000 independent simulations. Comparison of the plots shows the effect of the time between rebalances and of incorrect estimation of the volatility. $S(0) = 100$, $\mu = 0{,}08$, $r = 0{,}06$, $T = 0{,}5$, $K = 100$, true $\sigma = 0{,}2$. In the left-hand plots the correct value for rσ (0.2) has been used for pricing and hedging. In the right-hand plots an incorrect value for σ (0.15) has been used.

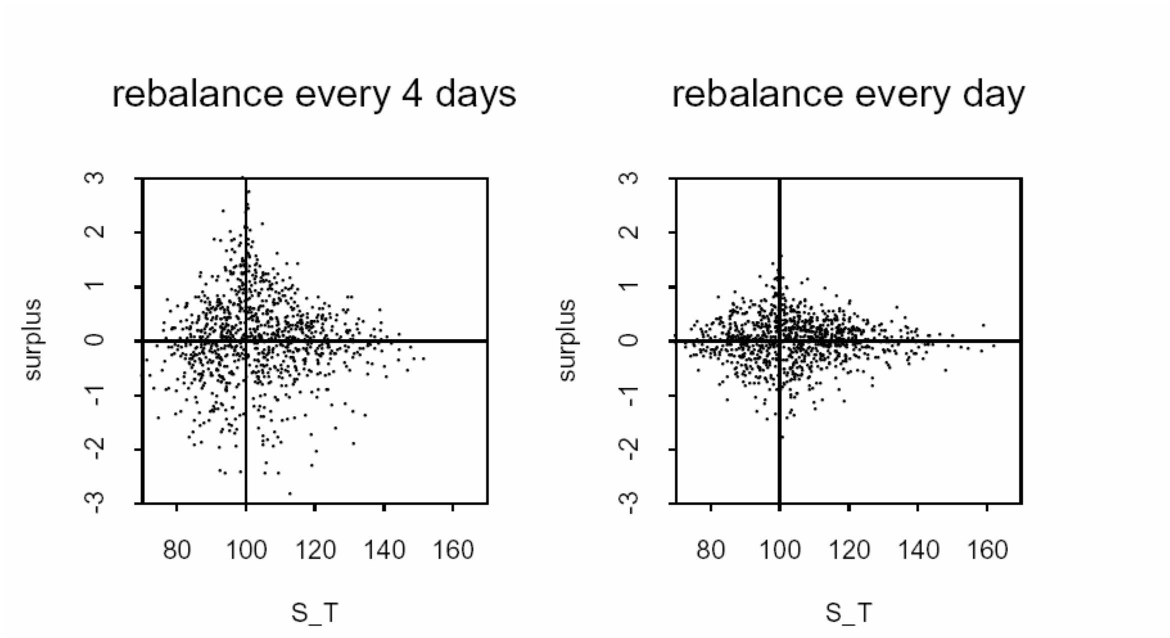


Figure 2: Hedging errors for a European call option based on 1000 simulations $S(0) = 100$, $\mu = 0{,}08$, $r = 0{,}06$, $T = 0{,}5$, $K = 100$, true $\sigma = 0{,}2$. We show how the size of hedging errors is related to the final share price, $S(T)$. The left-hand plot shows results for 4-day rebalancing and on the right 1-day rebalancing.

4. Black-Scholes equation

Under the effective market assumption Black and Scholes used the principle of the hedge position, that is, when the option price is properly determined then it should be possible to secure a return rate of a portfolio consisting of a short position in call options and a long position in its underlying asset. The Black-Scholes formula determines the option price under the assumption that no transaction costs are incurred, that funds can be borrowed or invested at a constant risk-free rate, that trading is continuous, and that the distribution of the stock's logarithmic return is normal.

The evolution of the asset price S at time t is assumed to follow the Geometric Brownian motion with the constant expected rate of return and the constant volatilityσ. The price, c, of a call option is a function of its underlying asset and time to mature; c = c(S; t). According to Ito's lemma:

$$dc = \left(\frac{\partial c}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 c}{\partial S^2}\right) dt + \frac{\partial c}{\partial S}dS.$$

Consider a portfolio which involves short selling of one unit of a European call option and long holding of Δ units of the underlying asset. The value of the portfolio Π is:

$\Pi = -c + \Delta S$

The in infinitesimal change in Π is

The last term in the right-hand side is the stochastic term which vanishes if

$$\begin{aligned} d\Pi &= -dc + \Delta dS \\ &= \left[-\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + \left(\Delta - \frac{\partial c}{\partial S} \right) \mu S \right] dt + \left(\Delta - \frac{\partial c}{\partial S} \right) \sigma S dZ. \quad (5.3) \end{aligned}$$

The last term in the right-hand side is the stochastic term which vanishes if $\Delta = \frac{\partial c}{\partial S}$.
Then the portfolio should earn the risk-free interest under the arbitrage free condition: $d\Pi = r\Pi dt$. Using this we obtain

$$\frac{\partial c}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + rS \frac{\partial c}{\partial S} - rc = 0.$$

Which is the Black-Scholes equation. It is important to note that the risk preferences, μ, of the investors do not affect the option price and the option pricing model involves five parameters: S, T, X, r and σ of which only σ is not an observable parameter.

5. Volatility Models

5.1. Volatility Smiles and Surfaces

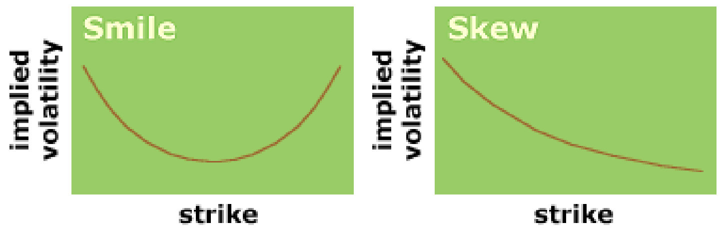
The one of the incorrect assumption of the Black-Scholes area is that the volatility of the underlying is constant. If volatility is not a simple constant then perhaps it is a more complicated function of time and/ or the underlying.

5.2. Implied volatility and volatility smiles

A standard way to estimate volatility for a given underliyng is to use the price of an option on that underlier. Suppose a call option on the underlying is actively trade, so the option's price is readily obtainable. Then, by applying a suitable option pricing formula—in a sense backwards—we calculate the annual volatility that would have to be input into the option pricing formula to obtain that price for the option. In this manner, we obtain the volatility implied by the option price—what is called the **implied volatility** for the underlying.

Most derivatives markets exhibit persistent patterns of volatilities varying by strike. In some markets, those patterns form a smile. In others, such as equity index options markets, it is more of a skewed curve. This has motivated the name volatility skew. In practice, either the term "volatility smile" or "volatility skew" (or simply skew) may be used to refer to the general phenomena of volatilities varying by strike. Indeed, you may even hear of "volatility smirks" or "volatility sneers", but such names are often as much whimsical as they are descriptive of any particular volatility pattern.

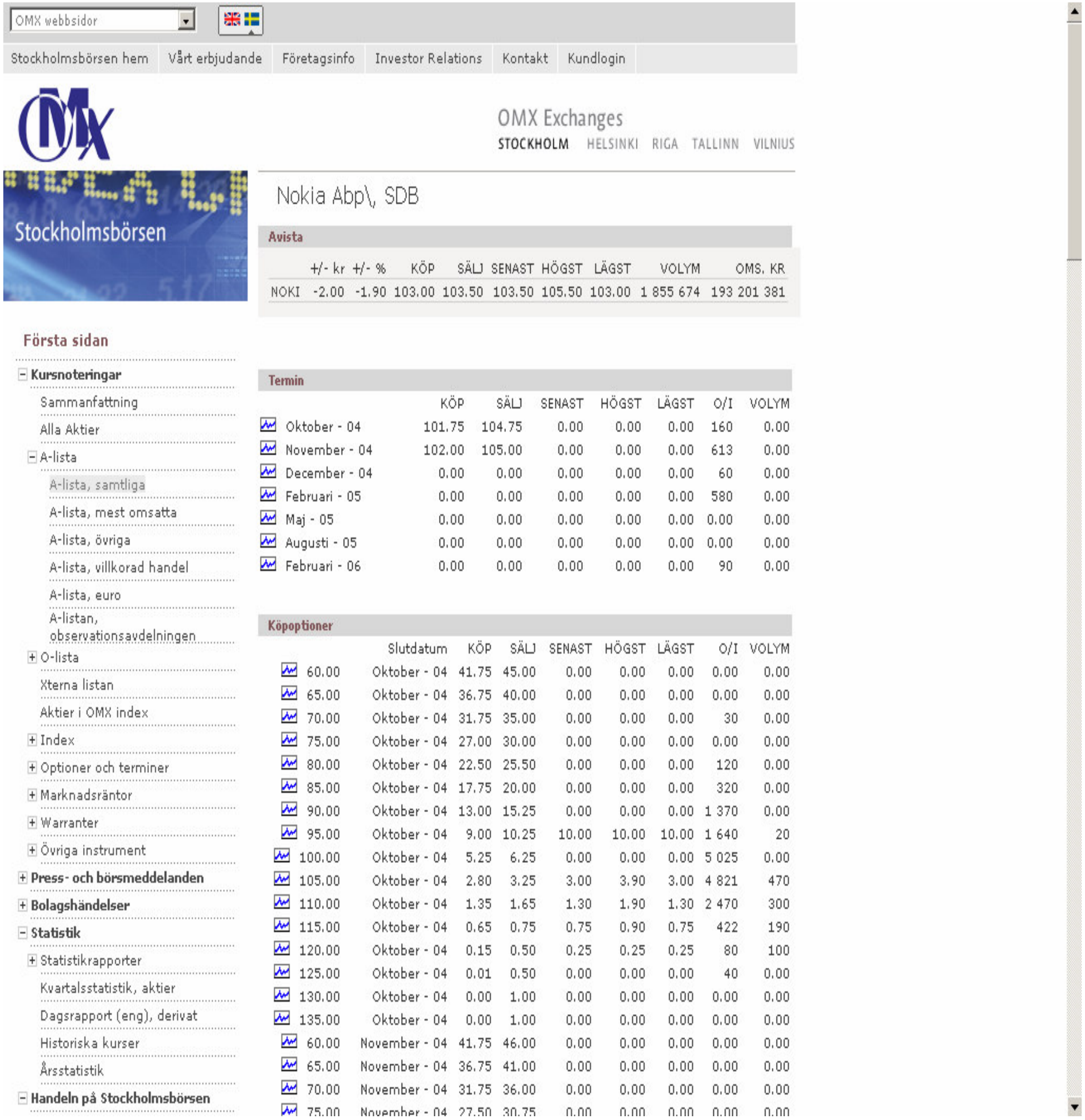
Smile and Skew



Volatility smile (left) is compared to volatility skew (right).

There are various explanations for why volatilities exhibit skew. Different explanations may apply in different markets. In most cases, multiple explanations may play a role. Some explanations relate to the idealized assumptions of the Black-Scholes approach to valuing options.

To consider an example, we will show how the shape looks like by using the real data. Firstly, we take the information of, for example, Nokia call option from the Stockholm Exchange Market, then we can use this information to calculate the implied volatility and see the graph.



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  - Sammanfattning
  - Alla Aktier
  - A-lista
    - A-lista, samtliga
    - A-lista, mest omsatta
    - A-lista, övriga
    - A-lista, villkorad handel
    - A-lista, euro
    - A-listan, observationsavdelningen
  - O-lista
  - Xterna listan
  - Aktier i OMX index
  - Index
  - Optioner och terminer
  - Marknadsräntor
  - Warranter
  - Övriga instrument
- Press- och börsmeddelanden
- Bolagshändelser
- Statistik
  - Statistikrapporter
  - Kvartalsstatistik, aktier
  - Dagsrapport (eng), derivat
  - Historiska kurser
  - Årsstatistik
- Handeln på Stockholmsbörsen

Nokia Abp\, SDB

Avista

	+/- kr	+/- %	KÖP	SÄLJ	SENAST	HÖGST	LÄGST	VOLYM	OMS. KR
NOKI	-2.00	-1.90	103.00	103.50	103.50	105.50	103.00	1 855 674	193 201 381

Termin

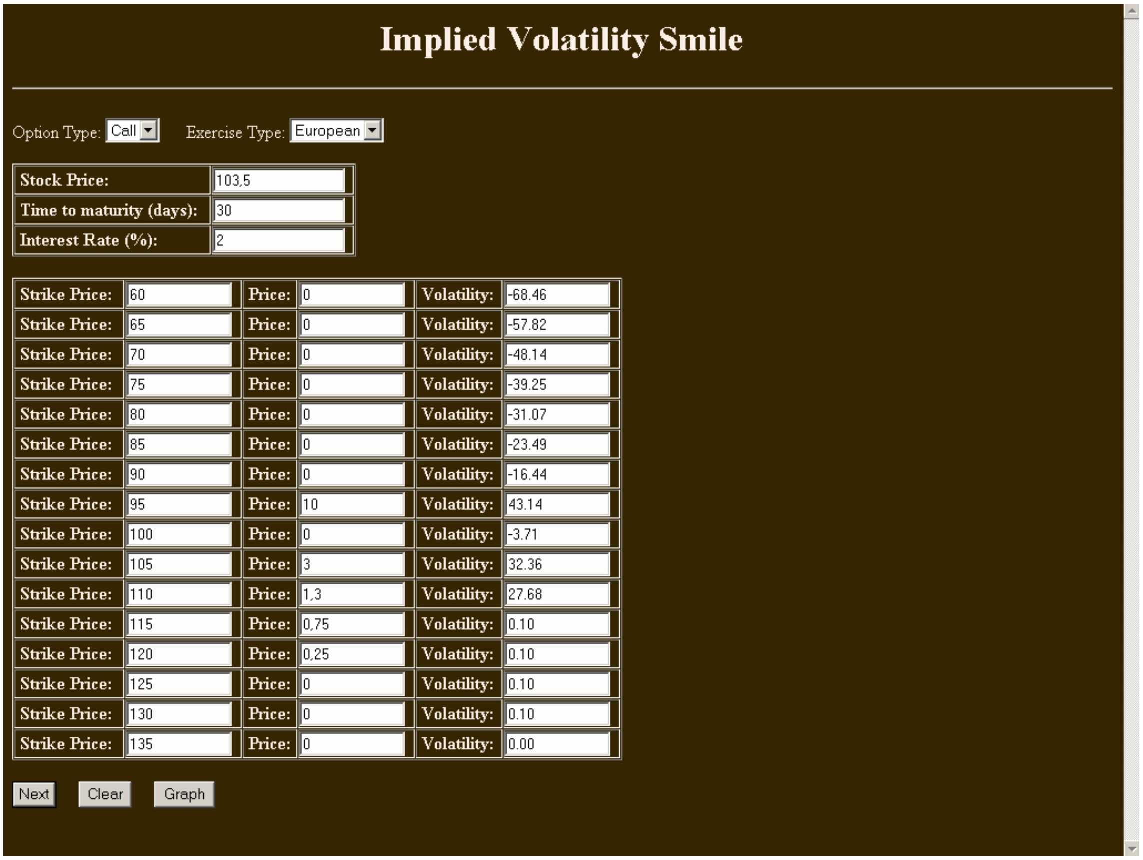
	KÖP	SÄLJ	SENAST	HÖGST	LÄGST	O/I	VOLYM
Oktober - 04	101.75	104.75	0.00	0.00	0.00	160	0.00
November - 04	102.00	105.00	0.00	0.00	0.00	613	0.00
December - 04	0.00	0.00	0.00	0.00	0.00	60	0.00
Februari - 05	0.00	0.00	0.00	0.00	0.00	580	0.00
Maj - 05	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Augusti - 05	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Februari - 06	0.00	0.00	0.00	0.00	0.00	90	0.00

Köpoptioner

	Slutdatum	KÖP	SÄLJ	SENAST	HÖGST	LÄGST	O/I	VOLYM
60.00	Oktober - 04	41.75	45.00	0.00	0.00	0.00	0.00	0.00
65.00	Oktober - 04	36.75	40.00	0.00	0.00	0.00	0.00	0.00
70.00	Oktober - 04	31.75	35.00	0.00	0.00	0.00	30	0.00
75.00	Oktober - 04	27.00	30.00	0.00	0.00	0.00	0.00	0.00
80.00	Oktober - 04	22.50	25.50	0.00	0.00	0.00	120	0.00
85.00	Oktober - 04	17.75	20.00	0.00	0.00	0.00	320	0.00
90.00	Oktober - 04	13.00	15.25	0.00	0.00	0.00	1 370	0.00
95.00	Oktober - 04	9.00	10.25	10.00	10.00	10.00	1 640	20
100.00	Oktober - 04	5.25	6.25	0.00	0.00	0.00	5 025	0.00
105.00	Oktober - 04	2.80	3.25	3.00	3.90	3.00	4 821	470
110.00	Oktober - 04	1.35	1.65	1.30	1.90	1.30	2 470	300
115.00	Oktober - 04	0.65	0.75	0.75	0.90	0.75	422	190
120.00	Oktober - 04	0.15	0.50	0.25	0.25	0.25	80	100
125.00	Oktober - 04	0.01	0.50	0.00	0.00	0.00	40	0.00
130.00	Oktober - 04	0.00	1.00	0.00	0.00	0.00	0.00	0.00
135.00	Oktober - 04	0.00	1.00	0.00	0.00	0.00	0.00	0.00
60.00	November - 04	41.75	46.00	0.00	0.00	0.00	0.00	0.00
65.00	November - 04	36.75	41.00	0.00	0.00	0.00	0.00	0.00
70.00	November - 04	31.75	36.00	0.00	0.00	0.00	0.00	0.00
75.00	November - 04	27.50	30.75	0.00	0.00	0.00	0.00	0.00

Fig.1. Option price of Nokia from the Stockholm Stock Exchange (www.stockholmborsen.com).

We can see, that implied volatility and smile implicate that the volatility is not constant, it varies with strike price.



Implied Volatility Smile

Option Type: Call Exercise Type: European

Stock Price:	103,5
Time to maturity (days):	30
Interest Rate (%):	2

Strike Price:	60	Price:	0	Volatility:	-68.46
Strike Price:	65	Price:	0	Volatility:	-57.82
Strike Price:	70	Price:	0	Volatility:	-48.14
Strike Price:	75	Price:	0	Volatility:	-39.25
Strike Price:	80	Price:	0	Volatility:	-31.07
Strike Price:	85	Price:	0	Volatility:	-23.49
Strike Price:	90	Price:	0	Volatility:	-16.44
Strike Price:	95	Price:	10	Volatility:	43.14
Strike Price:	100	Price:	0	Volatility:	-3.71
Strike Price:	105	Price:	3	Volatility:	32.36
Strike Price:	110	Price:	1,3	Volatility:	27.68
Strike Price:	115	Price:	0,75	Volatility:	0.10
Strike Price:	120	Price:	0,25	Volatility:	0.10
Strike Price:	125	Price:	0	Volatility:	0.10
Strike Price:	130	Price:	0	Volatility:	0.10
Strike Price:	135	Price:	0	Volatility:	0.00

Next Clear Graph

Fig. 2. Calculation of Implied Volatility Smile (http://janroman.net.dhis.org)

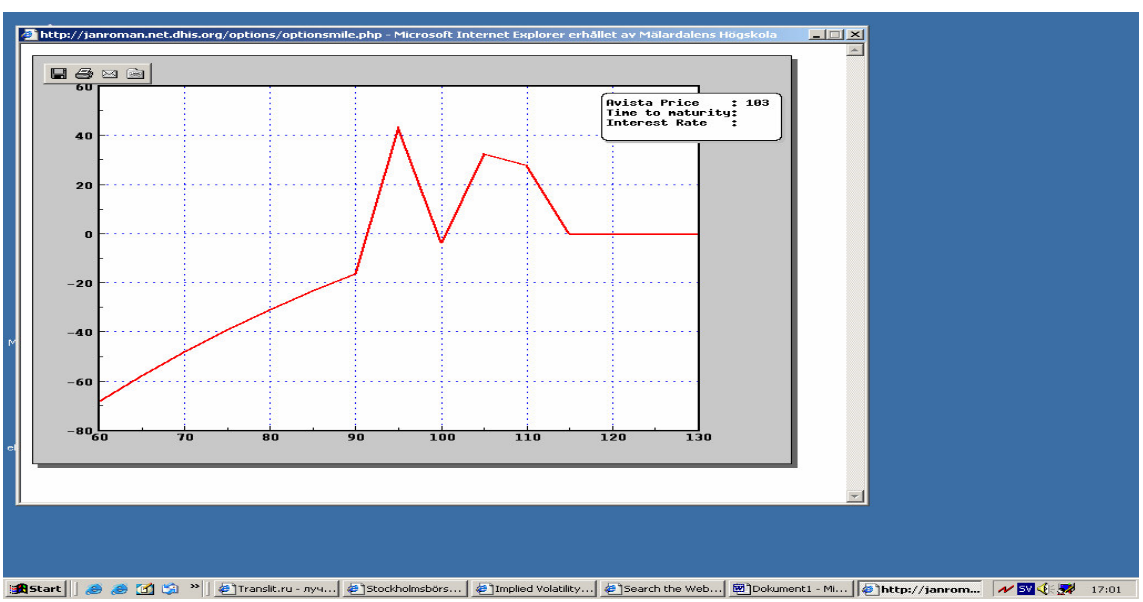


Fig.3. Implied volatility against strike price of Nokia (http://janroman.net.dhis.org)

5.3. Time-depended volatility

We will consider of European call with one, four and seven months until expiry. All have same strike price and the underlying asset price. Clearly, we can see that if the volatility is constant for the whole seven months, the prises are cannot be correct.
And we show this issue by using the Nokia's history data. At figure below, you can see how volatility appears to change with time.

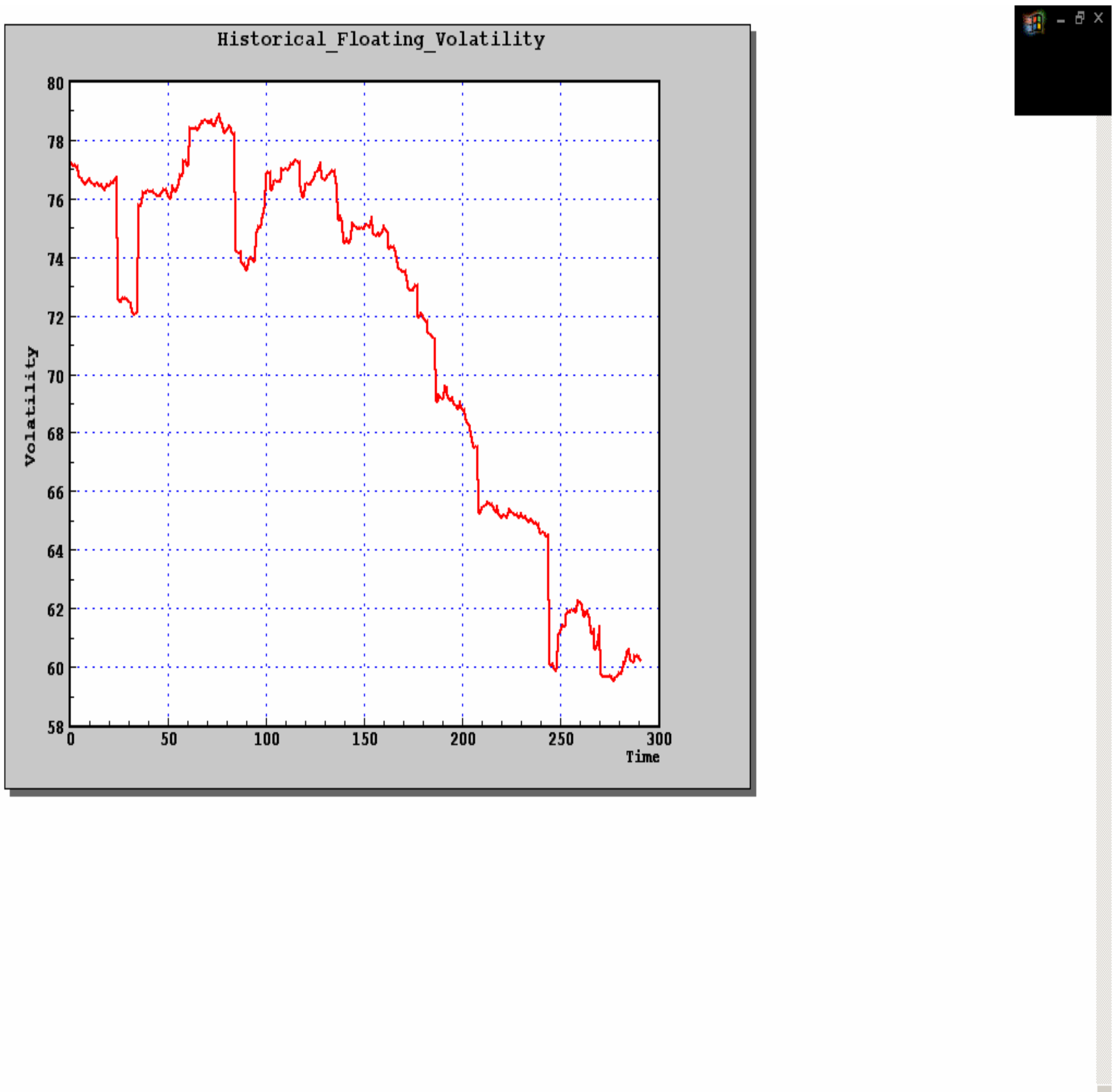
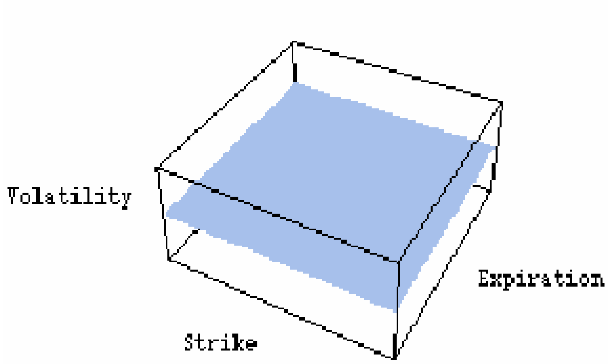


Fig.4. Volatility against time to maturity of Nokia (http://janroman.net.dhis.org)

5.4. Volatility Surfaces

According to classic theory, the Black- Scholes implied volatility of an option should be independent of its strike and expiration. Plotted as a surface, it should be flat, as shown at

The volatility surface according to Black-Scholes

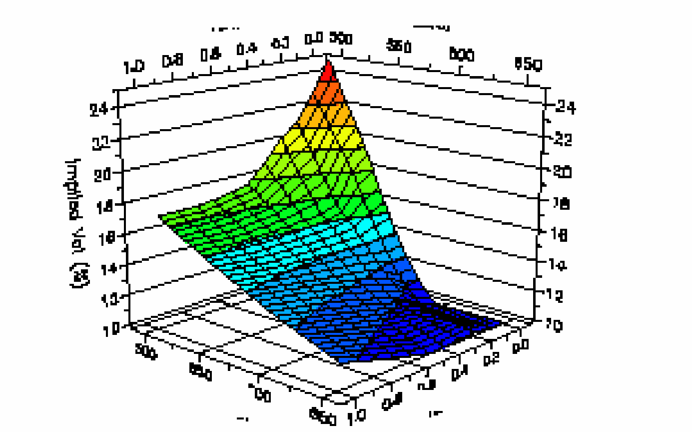


right.

Prior to the stock market crash of October 1987, the volatility surface of index options was indeed fairly flat. Since the crash, the volatility surface of index options has become skewed. Referred to as the volatility smile, the surface changes over time.

Its level at any instant is a varying function of strike and expiration, as shown at left.

The volatility surface according to S&P options markets



The smile phenomenon has spread to stock options, interest-rate options, currency options, and almost ever other volatility market. Since the Black-Scholes model cannot account for the smile, trading desks have begun to use more complex models to value and hedge their options.

After 15 years, there is still no overwhelming consensus as to the correct model. Each market has its own favorite (or two). This talk covers a short history of attempts to model the smile. Despite initial optimism about finding a model to replace Black-Scholes, we are still in many ways searching in the dark.

6. A Jump-Diffusion Model

Brownian motion and normal distribution have been widely used in the Black–Scholes option-pricing framework to model the return of assets. However, two puzzles emerge from many empirical investigations: the leptokurtic feature that the return distribution of assets may have a higher peak and two (asymmetric) heavier tails than those of the normal distribution, and an empirical phenomenon called “volatility smile” in option markets. To incorporate both of them and to strike a balance between reality and tractability, this paper proposes, for the purpose of option pricing, a double exponential jump-diffusion model.
In particular, the model is simple enough to produce analytical solutions for a variety of option-pricing problems, including call and put options, interest rate derivatives.

Despite the success of the Black–Scholes model based on Brownian motion and normal distribution, two empirical phenomena have received much attention
recently:

(1) The return distribution is skewed to the left, and has a higher peak and two heavier tails than those of the normal distribution.
(2) The volatility smile. More precisely, if the Black–Scholes model is correct, then the implied volatility should be constant.

In reality, it is widely recognized that the implied volatility curve resembles a “smile,” meaning it is a convex curve of the strike price.

The model is very simple. The logarithm of the asset price is assumed to follow a Brownian motion plus a compound Poisson process with jump sizes double exponentially distributed. Because of its simplicity, the parameters in the model can be easily interpreted, and the analytical solutions for option pricing can be obtained. The explicit calculation is made possible partly because of the memoryless property of the double exponential distribution.

Merton has suggested model where the asset price has jumps superimposed upon a geometric Brownian motion:

μ: Expected return from asset net of the dividend yield, q

λ: Average number of jumps per year

k: Average jump size measured as a percentage of the asset price

The percentage jump size is assumed to be drawn from a probability distribution in the model. The probability of a jump in time δt is $\lambda\ \delta t$. The average growth rate in the asset price from the jumps is therefore λk. The process for the asset price is

$dS/S=(\mu-\lambda k)dt+\sigma dz+dp$

where dz is a Wiener process,

dp is the Poisson process generating the jumps

σ is the volatility of the geometric Brownian motion.

σ

The processes dz and dp are assumed to be independent. The key assumption made by Merton is that the jump component of the asset’s return represents nonsystematic risk (i.e., risk not priced in the economy). This means that a Black-Scholes type of portfolio, which eliminates the uncertainty arising from the geometric Brownian motion (but not the jump), must earn the riskless rate.

7. Stochastic Volatility Models

When the instantaneous volatility is known function of time, the risk-neutral process followed by the stock price is
$dS = (r-q)Sdt+\sigma(t)Sdz$

The Black-Scholes formulas are the correct providing the variance rate is set equal to the average variance rate during the life of the option

Advantage and Limitations of the Black-Scholes model. The variance rate is the square of the volatility. Suppose that during a one-year period the volatility of the stock will be 20% during the first six month and 30% during the second six month. The average variance rate is

$0.5*(0.20)^2+0.5*(0.30)^2 = 0.065$

It is correct to use Black-Scholes with a variance rate of 0.065. This corresponds to a volatility of $\sqrt{0.065} = 0.255$, or 25.5 %.
This equation assumes that the instantaneous volatility of an asset is perfectly predictable. In practice volatility varies stochastically. This has led some researches to develop more complex models where there are two stochastic variables: the stock price and its volatility.
Hull and White consider the following stochastic volatility model for the risk-neutral behavior of price:

$dS/S = (r-q)dt+\sqrt{V}\, dzs$
$dV = a(VL-V)dt+\xi V^{\wedge}\, dzv$

where a,VL, ξ, and α are constant, and dzv and dzs are Wiener process. The variable V in this model is the asset's variance rate. The variance rate has a drift that pulls it back to a level VL at rate a. Hull and White compare the price given by this model with the price given by the Black-Scholes is to put equal to the expected average variance rate during the life of the option.
Hull and White show that, when volatility is stochastic but uncorrelated with the asset price the price of a European option is the Black-Scholes price integrated over the probability distribution of the average variance rate during the life of the option. Thus a European call price is
$\int c(V'')g(V'')dV''$

where V" is the average value of the variance rate, c is the Black-Scholes price expressed as a function of V", and g is the probability density function of V" in a risk-neutral world. This result can be used to show that Black-Scholes overprice options that are at the money or close to the money, and under prices options that are deep in or deep out of the money. The model is consistent with the pattern of implied volatilities observed for currency options.

The case where the asset price and volatility are correlated is more complicated Option price can be obtained using Monte Carlo simulation. In the particular case where the $\alpha = 0.5$, Hull and White provide a series expansion and Heston gives an analytic result. The pattern of implied volatility is negatively correlated with the asset price is similar to that observed to equities.

8. Conclusion

Advantage: The main advantage of the Black-Scholes model is speed -- it lets you calculate a very large number of option prices in a very short time.

Limitation:

The Black-Scholes model has one major limitation: it cannot be used to accurately price options with an American-style exercise as it only calculates the option price at one point in time -- at expiration. It does not consider the steps along the way where there could be the possibility of early exercise of an American option.
As all exchange traded equity options have American-style exercise (i.e. they can be exercised at any time as opposed to European options which can only be exercised at expiration) this is a significant limitation.

The exception to this is an American call on a non-dividend paying asset. In this case the call is always worth the same as its European equivalent as there is never any advantage in exercising early.
Various adjustments are sometimes made to the Black-Scholes price to enable it to approximate American option prices (e.g. the Fischer Black Pseudo-American method) but these only work well within certain limits and they don't really work well for puts.

Volatility smiles and surfaces can show directly that volatility is not a simple constant. It is more complicated function of time and function of assets, as well as function of both. Volatility is hard to estimate, observe or predict, the classical method is to model it stochastically.

The advantage of jump reversion process is that describes better the reality by both point of view, economic and statistical time-series. We have two problems with jump-diffusion processes such as the impossibility to build a risk-less portfolio and difficulties with parameters estimate.

The serious criticism is the difficulties with hedging. Due to the jump part, the market is incomplete, and the conventional riskless hedging arguments are not applicable here. However, it should be pointed out that the riskless hedging is really a special property of continuous-time Brownian motion, and it does not hold for most of the alternative models. Even within the Brownian motion framework, the riskless hedging is impossible if one wants to do it in discrete time.

10. Literature and references:

- Peter Hoadley's Option Strategy Analysis Tools
- http://www.riskglossary.com/
- Jan Röman /http://janroman.net.dhis.org/
- **Black-Scholes-Option Pricing Models**
  http://bradley.bradley.edu/~arr/bsm/pg01.html
- Stockholm Stock Exchange http://stockholmborsen.com
- Paul Willmot. Introduce to Quantitative Finance, 2001
- http://papers.ssrn.com
- Emanuel Derman, Laughter in the Dark - The Problem of the Volatility Smile