# *Two-factor Hull-White Model*

Analytical Finance 2

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# The One-Factor Hull-White Model

$$dr = (\theta(t) - a(t)r)dt + \sigma(t)dV(t)$$

$$\begin{cases} \dfrac{\partial F^T}{\partial t} + \{\mu(t,r) - \lambda(t,r)\sigma(t)\}\dfrac{\partial F^T}{\partial r} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 F^T}{\partial r^2} = r(t)F^T \\ F(r,T,T) = 1 \end{cases}$$

$$\mu(t,r) - \lambda(t,r)\sigma(t) = \theta(t) - a(t)r(t)$$

$$p(t,T) = F(r(t),t,T) = e^{A(t,T) - B(t,T)r}$$

# The One-Factor Hull-White Model

$$\begin{cases} \dfrac{\partial B}{\partial t} - aB = -1 \\ B(T,T) = 1 \end{cases} \qquad B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}$$

$$\begin{cases} \dfrac{\partial A}{\partial t} - \theta(t)B(t,T) + \dfrac{1}{2}\sigma^2 B^2(t,T) = 0 \\ A(T,T) = 0 \end{cases}$$

$$A(t,T) = \int_t^T \left\{\frac{\sigma^2}{2} B^2(s,T) - \theta(t)B(s,T)\right\} ds$$

# The One-Factor Hull-White Model

$$f^*(0,T) = B_T(0,T)r(0) - A_T(0,T) = r(0)e^{-aT} + \int_t^T \theta(s)e^{-a(T-s)}ds - \frac{\sigma^2}{2a^2}(1-e^{-aT})^2$$

$$f^*(0,T) = x(T) + g(T)$$

$$\begin{cases} \dot{x}(t) = -ax(t) + \theta(t) \\ x(0) = r(0) \end{cases}$$

$$x(t) = r(0)e^{-aT} + \int_t^T \theta(s)e^{-a(T-s)}ds \quad g(T) = \frac{\sigma^2}{2a^2}(1-e^{-aT})^2 = \frac{\sigma^2}{2a^2}B^2(0,T)$$

# The One-Factor Hull-White Model

$$\theta = \dot{x}(T) + ax(T) = f_T^*(0,T) - \dot{g}(T) + ax(T)$$
$$= f_T^*(0,T) - \dot{g}(T) + a\left\{f^*(0,T) - g(T)\right\}$$

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)} \exp\left\{B(t,T) f^*(0,t) - \frac{\sigma^2}{4a^2} B^2(t,T)(1 - e^{-2at}) - B(t,T) r(t)\right\}$$

$$\sigma_{spot} = \frac{\sigma}{a(T-t)}(1 - e^{-a(T-t)})$$

# The Two-Factor Hull-White Model

$$df(r) = [\theta(t) + u - af(r)]dt + \sigma_1 dz_1$$

$$du = -budt + \sigma_2 dz_2$$

$$P(t,T) = A(t,T)\exp[-B(t,T)r - C(t,T)u]$$

# The Two-Factor Hull-White Model

$$\ln A(t,T) = \ln \frac{P(0,T)}{P(0,t)} + B(t,T)F(0,t) - \eta$$

$$\eta = \frac{\sigma_1^2}{4a}(1-e^{-2at})B(t,T)^2 - \rho\sigma_1\sigma_2[B(0,t)C(0,t)B(t,T) + \gamma_4 - \gamma_2]$$
$$-\frac{1}{2}\sigma_2^2[C(0,t)^2 B(t,T) + \gamma_6 - \gamma_5]$$

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$$

$$C(t,T) = \frac{1}{a(a-b)}e^{-a(T-t)} - \frac{1}{b(a-b)}e^{-b(T-t)} + \frac{1}{ab}$$

# The Two-Factor Hull-White Model

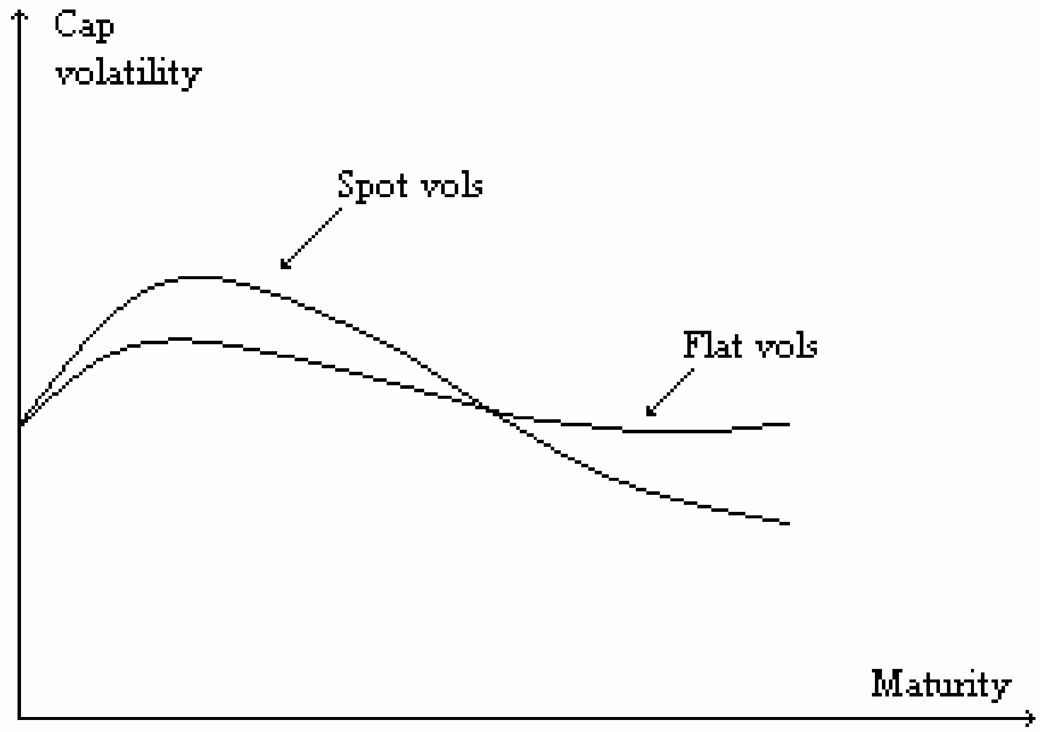
$$c = LP(0,s)N(h) - KP(0,T)N(h - \sigma_p)$$

$$p = KP(0,T)N(-h - \sigma_p) - LP(0,s)N(-h)$$

$$h = \frac{1}{\sigma_p} Ln \frac{LP(0,s)}{P(0,T)K} + \frac{\sigma_p}{2}$$

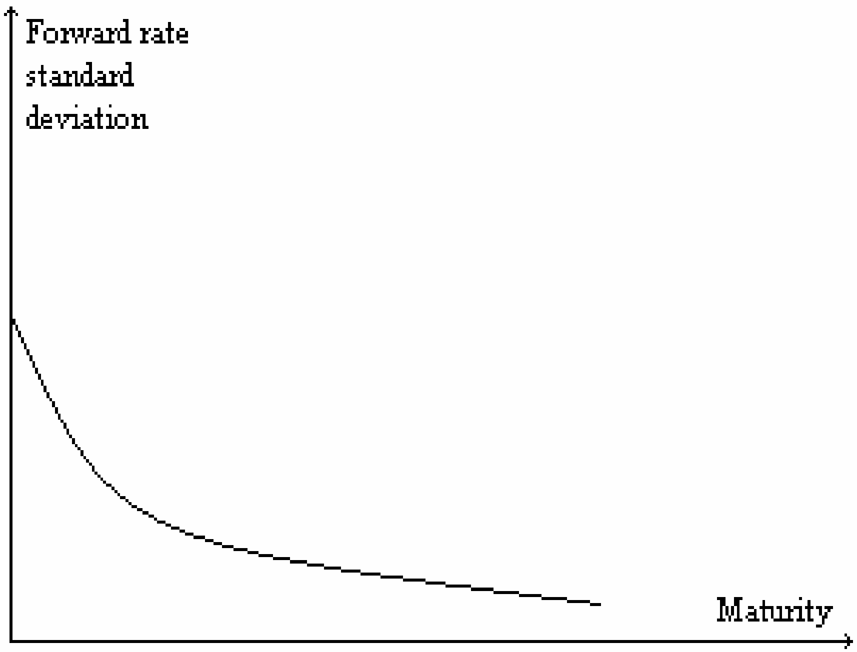
$$\sigma_p{}^2 = \int_0^t \{\sigma_1{}^2[B(\tau,T) - B(\tau,t)]^2 + \sigma_2{}^2[C(\tau,T) - C(\tau,t)]^2 + 2\rho\sigma_1\sigma_2[B(\tau,T) - B(\tau,t)][C(\tau,T) - C(\tau,t)]\}d\tau$$

# Graphs

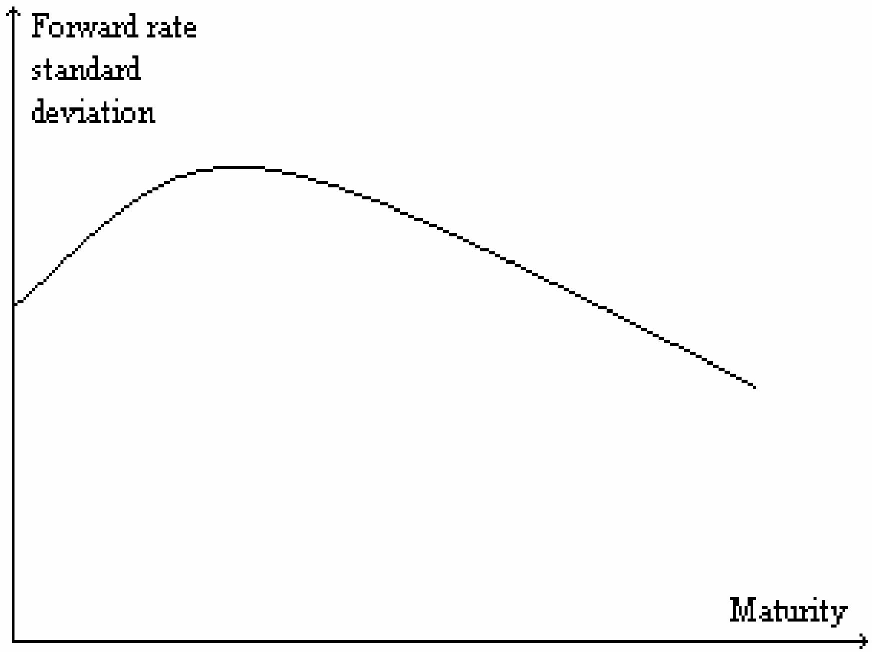



The " humped" volatility structure observed in the market

# *Graphs*




Volatility structure in one factor Hull-White model




Volatility structure in two factor Hull-White model