Two-factor Hull-White Model

Analytical Finance 2

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1. Introduction

The interest rate models discussed in our lectures are widely used for pricing instruments. But there exists two limitations:

1. They involve only one factor (i.e., one source of uncertainty).
2. They do not give the user complete freedom in choosing the volatility structure.

In recent years there have been a number of attempts to extend the models introduced in our lectures so that they involve two or more factors. One of the examples is Two-factor Hull-White Model.

In our report we will first restate the One-factor Hull-White Model, and then introduce the Two-factor Hull-White Model which builds on that. Finally we will show the comparison by graphs.

2. The One-Factor Hull-White Model

Hull-Whit model is a generalization of the Vasicek model with time dependent parameters:

$$dr = (\theta(t) - a(t)r)dt + \sigma(t)dV(t)$$

where $\theta(t)$ is deterministic function of time. The parameters a and σ is calibrated against the volatility and $\theta(t)$ is calibrated against the bond prices, $\{p(0,T) : T \geq 0\}$ to observed curve $\{p^*(0,T) : T \geq 0\}$.

We recall the term structure equation:

$$\begin{cases} \dfrac{\partial F^T}{\partial t} + \{\mu(t,r) - \lambda(t,r)\sigma(t)\}\dfrac{\partial F^T}{\partial r} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 F^T}{\partial r^2} = r(t)F^T \\ F(r,T,T) = 1 \end{cases}$$

the drift is given by $\mu(t,r) - \lambda(t,r)\sigma(t)$. If we compare this drift terms we see that the parameters in the Hull-White model include the market price of risk and the volatility

$$\mu(t,r) - \lambda(t,r)\sigma(t) = \theta(t) - a(t)r(t)$$

This is why we say parameters a and σ is calibrated against the volatility.

The model has an affine term structure

$$p(t,T) = F(r(t),t,T) = e^{A(t,T) - B(t,T)r}$$

This model can be simplified if we let a be a constant. We get equations:

$$\begin{cases} \dfrac{\partial B}{\partial t} - aB = -1 \\ B(T,T) = 1 \end{cases}$$

Solve them, we obtain,

$$B(t,T) = \frac{1}{a}\{1 - e^{-a(T-t)}\}$$

Insert this in the equation of A, we get

$$\begin{cases} \dfrac{\partial A}{\partial t} - \theta(t)B(t,T) + \dfrac{1}{2}\sigma^2 B^2(t,T) = 0 \\ A(T,T) = 0 \end{cases}$$

$$A(t,T) = \int_t^T \left\{ \frac{\sigma^2}{2} B^2(s,T) - \theta(t)B(s,T) \right\} ds$$

Now we will calibrate the model to the observed initial yield curve. The initial forward rate is given by

$$f^*(0,T) = B_T(0,T)r(0) - A_T(0,T) = r(0)e^{-aT} + \int_t^T \theta(s)e^{-a(T-s)}ds - \frac{\sigma^2}{2a^2}(1-e^{-aT})^2$$

In order to solve this, we let

$$f^*(0,T) = x(T) + g(T)$$

where

$$\begin{cases} \dot{x}(t) = -ax(t) + \theta(t) \\ x(0) = r(0) \end{cases}$$

the solution is

$$x(t) = r(0)e^{-aT} + \int_t^T \theta(s)e^{-a(T-s)}ds$$

and

$$g(T) = \frac{\sigma^2}{2a^2}(1-e^{-aT})^2 = \frac{\sigma^2}{2a^2}B^2(0,T)$$

then we get

$$\begin{aligned} \theta = \dot{x}(T) + ax(T) &= f_T^*(0,T) - \dot{g}(T) + ax(T) \\ &= f_T^*(0,T) - \dot{g}(T) + a\left\{f^*(0,T) - g(T)\right\} \end{aligned}$$

With the function $\theta(T)$, the fixed values of a and σ and using martingale measure, the bond price is given by

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)}\exp\left\{B(t,T)f^*(0,t) - \frac{\sigma^2}{4a^2}B^2(t,T)(1-e^{-2at}) - B(t,T)r(t)\right\}$$

The spot rate volatility is the same as in the Vasicek model.

$$\sigma_{spot} = \frac{\sigma}{a(T-t)}(1-e^{-a(T-t)})$$

3. The Two-Factor Hull-White Model

As explained in Hull-White One factor model, the risk-neutral process for the short rate, *r*, is

$$df(r) = [\theta(t) + u - af(r)]dt + \sigma_1 dz_1$$

where *u* has an initial value of zero and follows the process

$$du = -budt + \sigma_2 dz_2$$

The parameter $\theta(t)$ is a deterministic function of time. The stochastic variable *u* is a component of the reversion level of *r* and itself reverts to a level of zero at rate *b*. The parameters *a*, *b*, σ_1, and σ_2 are constants and dz_1 and dz_2 are Wiener processes with instantaneous correlation ρ.

This model provides a richer pattern of term structure movements and a richer pattern of volatility structures than the one-factor model. For example, when $f(r) = r$, $a = 1$, $b = 0.1$, $\sigma_1 = 0.01$, $\sigma_2 = 0.0165$, and $\rho = 0.6$ the model exhibits, at all times, a "humped" volatility structure similar to that observed in the market.

When $f(r) = r$ the model is analytically tractable. The price at time *t* of a zero-coupon bond that provides a payoff of $1 at time *T* is

$$P(t,T) = A(t,T)\exp[-B(t,T)r - C(t,T)u]$$

where

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$$

$$C(t,T) = \frac{1}{a(a-b)}e^{-a(T-t)} - \frac{1}{b(a-b)}e^{-b(T-t)} + \frac{1}{ab}$$

and $A(t,T)$ is as given in the following.

The prices, *c* and *p*, at time zero of European call and put options on a zero-coupon bond are given by

$$c = LP(0,s)N(h) - KP(0,T)N(h - \sigma_p)$$

$$p = KP(0,T)N(-h - \sigma_p) - LP(0,s)N(-h)$$

where *T* is the maturity of the option, *s* is the maturity of the bond, *K* is the strike price, *L* is the bond's principal

$$h = \frac{1}{\sigma_p} Ln \frac{LP(0,s)}{P(0,T)K} + \frac{\sigma_p}{2}$$

and σ_p is as given in the following.

The $A(t,T)$ ***,*** σ_p ***, and*** $\theta(t)$ ***Functions in the Two-factor Hull-White Model***

In this part, we provide some of the analytic results for the two-factor Hull-White model when $f(r) = r$.

The $A(t,T)$ function is

$$InA(t,T) = In\frac{P(0,T)}{P(0,t)} + B(t,T)F(0,t) - \eta$$

where

$$\eta = \frac{\sigma_1^{\ 2}}{4a}(1-e^{-2at})B(t,T)^2 - \rho\sigma_1\sigma_2[B(0,t)C(0,t)B(t,T) + \gamma_4 - \gamma_2]$$
$$-\frac{1}{2}\sigma_2^{\ 2}[C(0,t)^2 B(t,T) + \gamma_6 - \gamma_5]$$

$$\gamma_1 = \frac{e^{-(a+b)T}(e^{(a+b)t}-1)}{(a+b)(a-b)} - \frac{e^{-2aT}(e^{2at}-1)}{2a(a-b)}$$

$$\gamma_2 = \frac{1}{ab}(\gamma_1 + C(t,T) - C(0,T) + \frac{1}{2}B(t,T)^2 - \frac{1}{2}B(0,T)^2 + \frac{t}{a} - \frac{e^{-a(T-t)} - e^{-aT}}{a^2})$$

$$\gamma_3 = -\frac{e^{-(a+b)t}-1}{(a-b)(a+b)} + \frac{e^{-2at}-1}{2a(a-b)}$$

$$\gamma_4 = \frac{1}{ab}(\gamma_3 - C(0,t) - \frac{1}{2}B(0,t)^2 + \frac{t}{a} + \frac{e^{-at}-1}{a^2})$$

$$\gamma_5 = \frac{1}{b}[\frac{1}{2}C(t,T)^2 - \frac{1}{2}C(0,T)^2 + \gamma_2]$$

$$\gamma_6 = \frac{1}{b}[\gamma_4 - \frac{1}{2}C(0,t)^2]$$

where $B(t,T)$ and $C(t,T)$ functions are as we mentioned before and $F(t,T)$ is the instantaneous forward rate at time *t* for maturity *T*.

The volatility function, σ_p , is

$$\sigma_p{}^2 = \int_0^t \{\sigma_1{}^2[B(\tau,T)-B(\tau,t)]^2 + \sigma_2{}^2[C(\tau,T)-C(\tau,t)]^2 + 2\rho\sigma_1\sigma_2[B(\tau,T)-B(\tau,t)][C(\tau,T)-C(\tau,t)]\}d\tau$$

This shows that $\sigma_p{}^2$ has three components. Define

$$U = \frac{1}{a(a-b)}(e^{-aT} - e^{-at})$$

$$V = \frac{1}{b(a-b)}(e^{-bT} - e^{-bt})$$

The first component of $\sigma_p{}^2$ is

$$\frac{\sigma_1{}^2}{2a}B(t,T)^2(1-e^{-2at})$$

The second is

$$\sigma_2{}^2(\frac{U^2}{2a}(e^{2at}-1)+\frac{V^2}{2b}(e^{2bt}-1)-2\frac{UV}{a+b}(e^{(a+b)t}-1))$$

The third is

$$\frac{2\rho\sigma_1\sigma_2}{a}(e^{-at}-e^{-aT})(\frac{U}{2a}(e^{2at}-1)-\frac{V}{a+b}(e^{(a+b)t}-1))$$

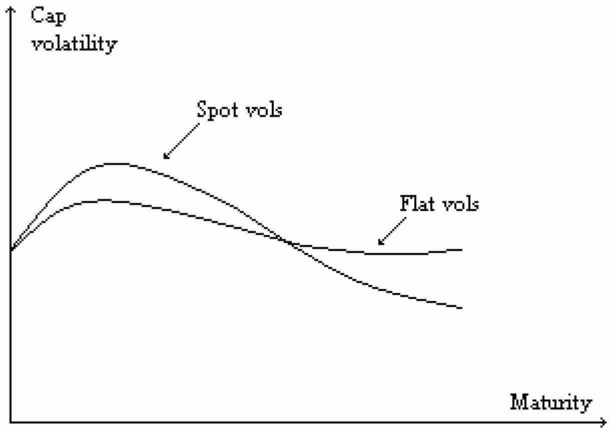
Finally, the $\theta(t)$ function is

$$\theta(t) = F_t(0,t) + aF(0,t) + \phi_t(0,t) + a\phi(0,t)$$

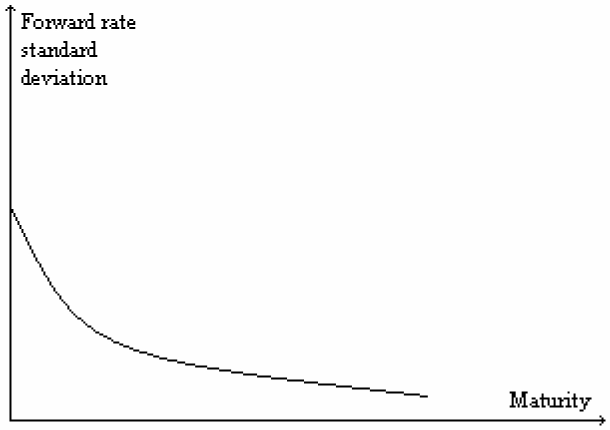
where the subscript denotes a partial derivative and

$$\phi(t,T) = \frac{1}{2}\sigma_1{}^2B(t,T)^2 + \frac{1}{2}\sigma_2{}^2C(t,T)^2 + \rho\sigma_1\sigma_2B(t,T)C(t,T)$$

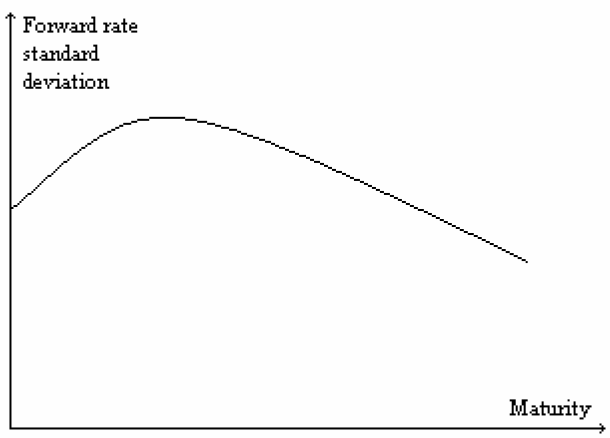
4. Graphs



The " humped" volatility structure observed in the market



Volatility structure in one factor Hull-White model



Volatility structure in two factor Hull-White model

5. Conclusion

The two-factor model provides a richer pattern of term structure movements and a richer pattern of volatility structures than the one-factor model.

This model also exhibits, at all times, a “humped” volatility structure similar to that observed in the market. The correlation structure implied by the model is also plausible.

There exist other two-factor models like the Yield-factor Model published by Duffie and Kan which we can study later.

6. Reference

Options, Futures, and Other Derivatives, John C. Hull, Fifth Ed.

Lecture notes in Analytical Finance 2, Jan R. M. Röman, 2005.