THE AVERAGE DENSITY OF THE PATH OF PLANAR BROWNIAN MOTION

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Abstract: We show that the occupation measure μ on the path of a planar Brownian motion run for an arbitrary finite time interval has an average density of order three with respect to the gauge function $\varphi(t) = t^2 \cdot \log(1/t)$. More precisely, almost surely,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log|\log \varepsilon|} \int_{\varepsilon}^{1/e} \frac{\mu(B(x,t))}{\varphi(t)} \, \frac{dt}{|t \, \log t|} = 2 \quad \text{at } \mu\text{-almost every } x.$$

We also prove a refinement of this statement: Almost surely, at μ-almost every x,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log|\log \varepsilon|} \int_{\varepsilon}^{1/e} \delta_{\left\{\frac{\mu(B(x,t))}{\varphi(t)}\right\}} \, \frac{dt}{|t \, \log t|} = \int_0^{\infty} \delta_{\{a\}} \, a \, e^{-a} \, da \,,$$

in other words, the distribution of the φ–density function under the averaging measures of order three converges to a gamma distribution with parameter two.

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1 Introduction

In order to study the fine local properties of fractal sets and measures Bedford and Fisher (1992) introduced a range of average densities of different orders. Whereas the classical densities fail to exist for fractal measures, see for example Mattila (1995), the average densities of order two were shown to exist for a wide range of fractal measures, like for example deterministic and random self–similar sets, mixing repellers or random measures related to stable processes, see Bedford and Fisher (1992), Patzschke and M. Zähle (1993), Falconer (1992) and Falconer and Xiao (1995). Average densities were also used to characterize geometric regularity of sets, see Falconer and Springer (1994), Marstrand (1996), Mörters (1998), or symmetry properties of measures, see Mörters (1997).

For the class of random measures given by the occupation measures on the path of Brownian motions in different dimensions the density problem is understood in the following situations:

- For Brownian motion on the real line, with probability one, the classical (i.e. 'order zero') one–dimensional density of the occupation measure μ exists μ–almost everywhere. This is the classical local time of Paul Levy.
- For Brownian motion in dimension three and larger, with probability one, the two-dimensional average densities of order two of the occupation measure μ exist μ–almost everywhere. This was shown by Falconer and Xiao (1995).

In this paper we show that for the occupation measure of planar Brownian motion, with probability one, the average density of order three with respect to the gauge function $\varphi(t) = t^2 \cdot \log(1/t)$ exists and equals two at μ–almost every point. The average density of order two fails to exist.

This result on the average density of planar Brownian motion is remarkable in two ways. Firstly, it seems to be the first instance of average densities with a gauge function other than $\varphi(t) = t^\alpha$. Note that this is neither the case for the occupation measure of Brownian motion in dimensions $d \geq 3$ nor for the local time measure on the zero–set of a one–dimensional Brownian motion, although both cases have exact dimension gauges with multiply logarithmic correction factors. Secondly, this seems to be the first natural appearance of an average of third order in the context of average densities. However, a similar phenomenon was observed by Brosamler (1973) in his study of the long time behaviour of additive functionals of planar Brownian motion and recently limit theorems involving order three averages have been a subject of intensive study in probability theory, see for example Földes (1993) or Marcus and Rosen (1995).

Various refinements and generalizations of the average density approach of Bedford and Fisher were suggested, see for example Bandt (1992), Graf (1995) or Mörters (1997). In this paper we study the density distributions of the occupation measure of planar Brownian motion. More precisely, we show that, with probability one, the distribution of the φ–density function at the origin with respect to a random scale distributed according to the averaging measure of order three converges to the standard exponential distribution. This can be interpreted as a pathwise version of the two–dimensional Kallianpur–Robbins law. A similar result holds at almost every point of the path of a planar Brownian motion. Here the limiting distribution is the distribution of the sum of two standard exponentially distributed random variables, which is a gamma distribution with parameter two.

The paper is organized as follows: In Section 2 we recall the definitions concerning average densities, density distributions and Brownian occupation measure and state our theorems. In Section 3 we give the proof of the existence of the average densities of order three and disprove the existence of average densities of order two. In Section 4 we extend these results in order to prove the existence of the density distributions of order three.

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2 Average densities and density distributions

We recall the definition of the average densities, which is due to Bedford and Fisher (1992). Let μ be a locally finite Borel measure on $\mathbb{R}^d$ and suppose that $\varphi(t) = t^\alpha$ for some $0 \leq \alpha \leq d$ or some other gauge function. For every positive integer $n > 0$ define the *averaging measures of order n* as the family (A_ε^n) given by

$$\int \delta_{\{t\}} \, dA_\varepsilon^n(t) = \frac{1}{a_n(1/\varepsilon)} \int_{1/b_n}^{1/\varepsilon} \delta_{\{1/t\}} \, K_n(t) \, dt = \frac{1}{a_n(1/\varepsilon)} \int_\varepsilon^{b_n} \delta_{\{t\}} \, K_n(1/t) \, \frac{dt}{t^2} \quad \text{for } 0 < \varepsilon < b_n,$$

with $b_n = [\exp^{(n)}(-\infty)]^{-1}$, $a_n(x) = \log^{(n-1)}(x)$ and $K_n(x) = \frac{d}{dx}(a_n(x))$, where $\exp^{(n)}$, $\log^{(n)}$ denotes the n-th iterate of exp, respectively log. The averaging measures of order n are the Hardy–Riesz log averages composed with the mapping $t \mapsto 1/t$. Further material on these averaging procedures can be found in Fisher (1987) or Bedford and Fisher (1992).

Denote by $B(x,t)$ the closed ball of radius t around $x \in \mathbb{R}^d$. The *average density of order n with respect to φ of μ at x* is the limit

$$\lim_{\varepsilon \downarrow 0} \int \frac{\mu(B(x,t))}{\varphi(t)} \, dA_\varepsilon(t) \, ,$$

if it exists. Explicitly written, the average density of order two is

$$\lim_{\varepsilon \downarrow 0} \frac{1}{|\log \varepsilon|} \int_\varepsilon^1 \frac{\mu(B(x,t))}{\varphi(t)} \, \frac{dt}{t}$$

and the average density of order three is

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log|\log \varepsilon|} \int_\varepsilon^{1/e} \frac{\mu(B(x,t))}{\varphi(t)} \, \frac{dt}{|t \, \log t|} \, .$$

The basic properties of average densities are listed in Bedford and Fisher (1992).

As a refinement of the notion of average density one can study the limit of the distributions of the density function $\mu(B(x,t))/\varphi(t)$ with respect to the averaging measures of order n as $\varepsilon \downarrow 0$. More explicitly, for a fixed measure μ, gauge function φ and $x \in \mathbb{R}^d$, define a probability distribution on the real line as

$$\int \delta_{\left\{ \frac{\mu(B(x,t))}{\varphi(t)} \right\}} \, dA_\varepsilon(t) \, ,$$

where $\delta_{\{a\}}$ stands for the point mass in a. The *φ–density distribution of order n of μ at x* is the limit in the sense of weak convergence, as $\varepsilon \downarrow 0$, of these distributions, if it exists. Explicitly written, the φ–density distribution of order three at x is defined as the measure P on the real line such that, for all continuous, bounded functions $F : \mathbb{R} \to \mathbb{R}$,

$$\int F(a) \, dP(a) = \lim_{\varepsilon \downarrow 0} \frac{1}{\log|\log \varepsilon|} \int_\varepsilon^{1/e} F\Big(\tfrac{\mu(B(x,t))}{\varphi(t)}\Big) \, \frac{dt}{|t \, \log t|} \, .$$

Now look at the particular case of planar Brownian motion. Throughout this paper we assume that $(B_t)_{t \in \mathbb{R}}$ is a Brownian motion in the plane with $B_0 = 0$. For any finite interval $[t_0, t_1]$ we define the *occupation measure* or *occupation time* of the Brownian motion to be the random measure $\mu[t_0, t_1]$ given by

$$\mu[t_0,t_1](A) = \mathcal{L}\Big(t \in [t_0,t_1] \, : \, B_t \in A\Big) \, ,$$

where $\mathcal{L}$ denotes Lebesgue measure. It was shown by Ray (1962) and Taylor (1964) that $\mu[t_0, t_1]$ coincides with a constant multiple of the Hausdorff measure on the graph $\{B_t : t \in [t_0, t_1]\}$ for the gauge function $t^2 \log(1/t) \log\log\log(1/t)$. The main result of this paper is the existence of the average densities of order three for the occupation measures at almost every point.

Theorem 2.1 *Suppose $\mu = \mu[0,T]$ is the occupation measure of a planar Brownian motion run for a finite time interval of arbitrary length $T > 0$ and define the gauge function $\varphi(t) = t^2 \log(1/t)$. Then the following statements hold with probability one:*

a) The average density of order three with respect to φ exists at μ-almost every x and we have

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log|\log \varepsilon|} \int_\varepsilon^{1/e} \frac{\mu(B(x,t))}{\varphi(t)} \, \frac{dt}{|t \, \log t|} = 2 \, .$$

b) At μ-almost every x the average density of order two with respect to φ fails to exist.

Our second theorem determines the density distributions for the occupation measure of planar Brownian motion.

Theorem 2.2 *Suppose $\mu = \mu[0,T]$ is the occupation measure of a planar Brownian motion run for a finite time interval of arbitrary length $T > 0$ and define the gauge function $\varphi(t) = t^2 \log(1/t)$. Then the following statements hold with probability one:*

a) The φ–density distribution of order three of μ at the origin exists and is equal to a standard exponential distribution. More explicitly,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log|\log \varepsilon|} \int_{\varepsilon}^{1/e} \delta_{\left\{ \frac{\mu(B(0,t))}{\varphi(t)} \right\}} \frac{dt}{|t \log t|} = \int_0^\infty \delta_{\{a\}}\, e^{-a}\, da\,.$$

b) The φ–density distribution of order three of μ exists at μ–almost every x and equals a gamma distribution with parameter two, which is the distribution of the sum of two independent standard exponentially distributed random variables. More explicitly,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log|\log \varepsilon|} \int_{\varepsilon}^{1/e} \delta_{\left\{ \frac{\mu(B(x,t))}{\varphi(t)} \right\}} \frac{dt}{|t \log t|} = \int_0^\infty \delta_{\{a\}}\, a e^{-a}\, da\,.$$

Remarks:

- The first statement can be interpreted as a pathwise version of the two-dimensional Kallianpur–Robbins law, which states in one of many equivalent formulations, that for the distribution W of planar Brownian motion

$$\lim_{t \downarrow 0} \int \delta_{\left\{ \frac{\mu(B(0,t))}{\varphi(t)} \right\}}\, dW = \int_0^\infty \delta_{\{a\}} e^{-a}\, da\,.$$

- Using the methods of Falconer and Xiao (1995) it can be shown that for Brownian motion in dimensions exceeding two and $\varphi(t) = t^2$, the φ–density distributions of order two of the occupation measure μ agrees almost surely μ-almost everywhere with the distribution of $\mu(B(0,1))$, which was calculated explicitly by Ciesielski and Taylor (1962).

- A further characteristic, which appears to be worth studying, is the tangent measure distribution of the occupation measures of a planar Brownian motion. For the definition and results about tangent measure distributions see for example Bandt (1990), Graf (1995) or Mörters and Preiss (1997).

3 The order–three density of planar Brownian motion

In this section we prove Theorems 2.1 and 2.1. Let us first see, why we can restrict our attention to the behaviour of the occupation measure at the origin. An elegant approach to this problem is the idea of Palm distributions, which also appears in many other branches of probability like queuing theory or point processes. For a general reference see Kallenberg (1983), for an account from the point of view of fractal geometry see U. Zähle (1988) or Patzschke and M. Zähle (1992, 1993). The following definition is based on a well–known characterization theorem of Mecke.

Definition

Denote by $\mathcal{M}(\mathbb{R}^d)$ the Polish space of all locally finite Borel measures on $\mathbb{R}^d$ equipped with the vague topology. The distribution P of a random measure is a **Palm distribution** if, for every measurable $G : \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d \to [0, \infty]$, the following formula holds

$$\int\int G(\nu, u)\, d\nu(u)\, dP(\nu) = \int\int G(T^u\nu, -u)\, d\nu(u)\, dP(\nu)\,, \tag{1}$$

where the measure $T^u\nu$ is given by $T^u\nu(A) = \nu(u + A)$.

Proposition 3.1 *Let $T > 0$ be fixed and pick a random number $X \in [0, T]$ uniformly and independently of the Brownian motion. Then the distribution of the random measure $\mu[X - T, X]$ is a Palm distribution.*

Proof: We first show that the random measure $\mathcal{L}[X - T, X]$ defined to be Lebesgue measure on the interval $[X - T, X]$ is Palm distributed. For any $G : \mathcal{M}(\mathbb{R}^2) \times \mathbb{R}^2 \to [0, \infty]$ we have

$$\begin{aligned} &\int_0^T \int_{\tau-T}^{\tau} G\Big(T^u\mathcal{L}[\tau - T, \tau], -u\Big)\, du\, d\tau \\ &= \int_0^T \int_{\tau-T}^{\tau} G\Big(\mathcal{L}[\tau - T - u, \tau - u], -u\Big)\, du\, d\tau \\ &= \int_0^T \int_0^T G\Big(\mathcal{L}[v - T, v], v - \tau\Big)\, dv\, d\tau \\ &= \int_0^T \int_{v-T}^{v} G\Big(\mathcal{L}[v - T, v], u\Big)\, du\, dv\,. \end{aligned} \tag{2}$$

The occupation measure $\mu[X - T, X]$ is now the image measure of $\mathcal{L}[X - T, X]$ under the independent random map $\sigma_0[B]$, where $\sigma_u[B] : t \to B_{t+u} - B_u$. Denote the distribution of Brownian motion by W and observe that, as (B_t) has stationary increments,

$$W \circ (\sigma_0[\,\cdot\,])^{-1} = W \circ (\sigma_u[\,\cdot\,])^{-1} \text{ for all } u \in \mathbb{R}. \tag{3}$$

Also note that

$$T^u\nu \circ \sigma_u[B]^{-1} = T^{\sigma_0[B](u)}[\nu \circ \sigma_0[B]^{-1}] \text{ and } \sigma_u[B](-u) = -\sigma_0[B](u). \tag{4}$$

Denote the distribution of $\mu[X - T, X]$ by P, and use (2), (3), (4) and Fubini's theorem to get

$$\begin{aligned} &\int\int G(\mu, w)\, d\mu(w)\, dP(\nu) \\ &= \int\int_0^T \int_{\tau-T}^{\tau} G\Big(\mathcal{L}[\tau - T, \tau] \circ \sigma_0[B]^{-1}, \sigma_0[B](u)\Big)\, du\, d\tau dW(B) \\ &\overset{(2)}{=} \int\int_0^T \int_{\tau-T}^{\tau} G\Big(T^u\mathcal{L}[\tau - T, \tau] \circ \sigma_0[B]^{-1}, \sigma_0[B](-u)\Big)\, du\, d\tau dW(B) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \int_{\tau-T}^{\tau} \int G\Big(T^u \mathcal{L}[\tau-T,\tau] \circ \sigma_0[B]^{-1}, \sigma_0[B](-u)\Big)\, dW(B)\, du\, d\tau \\
&\overset{(3)}{=} \int_0^T \int_{\tau-T}^{\tau} \int G\Big(T^u \mathcal{L}[\tau-T,\tau] \circ \sigma_u[B]^{-1}, \sigma_u[B](-u)\Big) dW(B)\ du\, d\tau \\
&\overset{(4)}{=} \int \int_0^T \int_{\tau-T}^{\tau} G\Big(T^{\sigma_0[B](u)}\Big[\mathcal{L}[\tau-T,\tau] \circ \sigma_0[B]^{-1}\Big], -\sigma_0[B](u)\Big)\, du\, d\tau\, dW(B) \\
&= \int \int G\Big(T^w \mu, -w\Big)\, d\mu(w)\, dP(\nu)\,.
\end{aligned}
$$

Thus the Palm formula (1) is proved. ■

Corollary 3.2 *The average density of the random measure $\mu[X-T,X]$ exists at the origin almost surely if and only if the average density (of the same order) of $\mu=\mu[0,T]$ exists μ-almost everywhere almost surely. Moreover, if the former is almost surely equal to a constant c, then so are the latter.*

Proof: This follows from (1) by using the function

$$
G(\nu,u) = \begin{cases} 1 & \text{if the average density of } \nu \text{ at } u \text{ fails to exist,} \\ 0 & \text{otherwise,} \end{cases}
$$

and observing that $G(T^u\nu,-u)=0$ if and only if the average density of ν at the origin exists. ■

We now study the occupation measure of Brownian motion at the origin. The principal problem is that we cannot use the scaling invariance of Brownian motion directly, as in Falconer and Xiao (1995), because the rescaling procedure changes our time scale. We overcome this problem by means of a method of Ray (1962), which is based on a subdivision technique of Knight (1962). For this purpose fix a number $B>0$ and let τ be the stopping time given by

$$
\tau = \inf\{t>0 \,:\, |B_t| \geq B\}\,.
$$

Fix the occupation measure $\mu=\mu[0,\tau]$ of the stopped Brownian motion. For the moment we also fix a number $b>0$ and define $a_n = Be^{-bn}$. By N_n we denote the number of returns from a_{n-1} to a_n before the Brownian motion crosses B. Ray (1962) proved the following lemma.

Lemma 3.3 *The sequence (N_n) of random variables is a null-recurrent Markov chain with initial value $N_1=0$, individual distributions given by*

$$
P(N_n=k) = \frac{1}{n}\Big(1-\frac{1}{n}\Big)^k, \tag{5}
$$

and a stationary transition function given by

$$
\mathbb{E}\Big\{\sigma^{N_m}|N_n=k\Big\} = \frac{\Big[\sigma+(1-\sigma)(m-n)\Big]^k}{\Big[1+(1-\sigma)(m-n)\Big]^{k+1}} \text{ for } m \geq n. \tag{6}
$$

Ray (1962) also showed a connection between the occupation measure of balls around the origin and the Markov chain (N_n). For the convenience of the reader we include the proof here. This is in order to demonstrate how the scaling invariance of Brownian motion is used and also to correct a minor error in Ray's statement.

Lemma 3.4 *For all $0 < r \le 1$, almost surely, as $n \to \infty$,*

$$\mu\Big(B(0, ra_n)\Big) = br^2 \cdot a_n^2 \cdot N_n + o(na_n^2)\,.$$

Proof: Define stopping times

$$\begin{aligned} t_0(\omega) &= 0\,, \\ t_{2k+1}(\omega) &= \inf\{t > t_{2k}(\omega) \,:\, |B_t(\omega)| > a_{n-1}\}\,, \\ t_{2k}(\omega) &= \inf\{t > t_{2k-1}(\omega) \,:\, |B_t(\omega)| < a_n\}\,, \end{aligned}$$

For any $0 < r \le 1$ let

$$T_k = T_k(\omega) = \int_{t_{2k}(\omega)}^{t_{2k+1}(\omega)} \mathbf{1}_{\{|B_s(\omega)| \le ra_n\}}\, ds\,.$$

Observe that

$$\mu(B(0, ra_n)) = \sum_{k=0}^{N_n} T_k\,. \tag{7}$$

As T_k depends only on the process $|B_t|$, the T_k are independent. By the scaling invariance of Brownian motion all T_k have the same distribution as $a_n^2 T(x, b)$ for some x on the unit circle, where

$$T(x, b) = T(x, b; \omega) = \int_0^{\rho} \mathbf{1}_{\{|X_t(\omega)| \le ra_n\}}\, dt$$

for a Brownian motion X_t started at x and stopped at the time ρ when it first crosses the boundary of $B(0, e^b)$. It is well known, see for example Stroock (1993), that

$$\mathbb{E}T(x, b) = \int_{B(0,r)} G(x, y)\, dy = \frac{1}{\pi} \int_{B(0,r)} \log\Big(\frac{|e^b x - e^{-b} y|}{|x - y|}\Big)\, dy\ ,$$

where G is the Green function for the Laplace equation with boundary value zero on the circle of radius e^b. To evaluate the integral, differentiate with respect to b under the integral sign, observe that by symmetry we may assume $x = 1$, and use polar coordinates and the Poisson integral formula to get

$$\begin{aligned} \frac{\partial}{\partial b}\mathbb{E}T(x, b) &= \frac{1}{\pi} \int_{B(0,r)} \frac{|e^b x|^2 - |e^{-b} y|^2}{|e^{-b} y - e^b x|^2}\, dy \;=\; \frac{1}{\pi} \int_{B(0,r)} \frac{1 - |e^{-2b} y|^2}{|e^{-2b} y - 1|^2}\, dy \\ &= \frac{e^{4b}}{\pi} \int_0^{re^{-2b}} \int_0^{2\pi} \frac{1 - \varrho^2}{\varrho^2 - 2\varrho \cdot \cos\varphi + 1}\, d\varphi\, \varrho\, d\varrho \;=\; r^2\,. \end{aligned}$$

Hence $\mathbb{E}T(x, b) = br^2$ for all $|x| = 1$ and thus $\mathbb{E}\{T_k\} = br^2 \cdot a_n^2$. Since the T_k are independent and T has finite fourth moment, we get, for some constant $C > 0$,

$$\mathbb{E}\Big\{\Big(\sum_{k=1}^{N} (T_k - br^2 \cdot a_n^2)\Big)^4\Big\}$$

$$
\begin{aligned}
&= N \cdot \mathbb{E}\Big\{(T_k - br^2 \cdot a_n^2)^4\Big\} + 6\binom{N}{2}\Big(\mathbb{E}\Big\{(T_k - br^2 \cdot a_n^2)^2\Big\}\Big)^2 \\
&< C \cdot N^2 a_n^8 .
\end{aligned}
$$

By Markov's inequality, for every $\varepsilon > 0$,

$$
P\Big\{\Big|\sum_{k=1}^{N}\Big(T_k - br^2 \cdot a_n^2\Big)\Big| > \varepsilon \cdot n a_n^2\Big\} < \frac{1}{\varepsilon^4 n^4 a_n^8}\,\mathbb{E}\Big\{\Big(\sum_{k=1}^{N}\Big(T_k - br^2 \cdot a_n^2\Big)\Big)^4\Big\} < C \cdot \frac{N^2}{n^4 \varepsilon^4} .
$$

As the T_k are independent of N_n, using (7),

$$
\begin{aligned}
&P\Big(\Big|\mu\Big(B(0, r a_n)\Big) - T_0 - br^2 \cdot a_n^2 N_n\Big| > \varepsilon \cdot n a_n^2\Big) \\
&= \mathbb{E}\Big\{P\Big(\Big|\sum_{k=1}^{N_n}\Big(T_k - br^2 \cdot a_n^2\Big)\Big| > \varepsilon \cdot n a_n^2\Big)\Big| N_n\Big\} \\
&< C \cdot \frac{\mathbb{E}\{N_n^2\}}{n^4 \varepsilon^4} \le \frac{2C}{n^2 \varepsilon^4} .
\end{aligned}
$$

By the Borel–Cantelli lemma we get, almost surely, as $n \to \infty$,

$$
\mu\Big(B(0, r a_n)\Big) = T_0 + br^2 \cdot a_n^2 \cdot N_n + o(n a_n^2) .
$$

Furthermore T_0 is bounded by the first exit time from the ball $B(0, a_{n-1})$. As the second moment of this stopping time is bounded by a constant multiple of a_n^4, Markov's inequality and the Borel–Cantelli lemma yield again that $T_0 = o(n a_n^2)$. ■

Ray (1962) used this statement to show that

$$
\limsup_{t \to 0} \frac{\mu(B(0,t))}{t^2 \log(1/t) \log\log\log(1/t)} = 1 \text{ almost surely.}
$$

Denote $X_n = N_n/n$. We study the behaviour of the average density of μ at the origin by means of an analysis of the averages of X_n. This analysis is based on the calculation of the moments of X_n, which can be carried out using Lemma 3.3.

Lemma 3.5 *The following (in-)equalities hold for the moments of (X_i).*

a) $\mathbb{E}X_i = \dfrac{i-1}{i}$ *for all* $i > 0$,

b) $\mathbb{E}\{X_i X_j\} = \dfrac{j-1}{ij}(i+j-1)$ *and* $\mathrm{Cov}\{X_i, X_j\} = \dfrac{j-1}{i}$ *for* $i \ge j > 0$,

c) $\mathbb{E}\{X_i X_j X_k\} \le 3$ *for all* $i, j, k \ge 0$.

Proof: From (5) we get

$$
\mathbb{E}X_i = \sum_{k=0}^{\infty} \frac{k}{i} \cdot P\Big(X_i = \frac{k}{i}\Big) = \sum_{k=0}^{\infty} \frac{k}{i^2} \cdot \Big(1 - \frac{1}{i}\Big)^k = \frac{i-1}{i} , \tag{8}
$$

as $\sum_{k=0}^{\infty} kq^k = q/(1-q)^2$ for $q<1$. Now let $i \geq j > 0$ and differentiate (6) to get

$$\mathbb{E}\{N_i|N_j=k\} = \frac{d}{d\sigma}\Big[\mathbb{E}\{\sigma^{N_i}|N_j=k\}\Big]_{\sigma=1} = k+i-j\,. \tag{9}$$

We thus get

$$\mathbb{E}\{X_iX_j\} = \mathbb{E}\Big\{\mathbb{E}\{X_iX_j|N_j\}\Big\} = \sum_{k=0}^{\infty} \frac{k}{j}\cdot\frac{k+i-j}{i}\cdot\Big(1-\frac{1}{j}\Big)^k\frac{1}{j}\,.$$

As $\sum_{k=0}^{\infty} k^2q^k = 2q^2/(1-q)^3 + q/(1-q)^2$ for $q<1$, we obtain

$$\mathbb{E}\{X_iX_j\} = \frac{j-1}{ij}\cdot(i+j-1)\,.$$

and this yields

$$\mathrm{Cov}\{X_i,X_j\} = \mathbb{E}\{X_iX_j\} - \mathbb{E}X_i\mathbb{E}X_j = \frac{j-1}{i}\,. \tag{10}$$

In order to study the third moment we use that, by Lemma 3.3, for $k \leq j \leq i$,

$$\begin{aligned}\mathbb{E}\Big\{X_iX_jX_k\Big\} &= \mathbb{E}\Big\{\mathbb{E}\{X_iX_jX_k|N_j,N_k\}\Big\}\\ &= \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{lm}{jk}\cdot\frac{l+i-j}{i}\cdot\frac{1}{k}\Big(1-\frac{1}{k}\Big)^m\cdot P(N_j=l|N_k=m)\,.\end{aligned}$$

As before we have

$$\sum_{l=0}^{\infty} l\,P\Big(N_j=l\Big|N_k=m\Big) = \frac{d}{d\sigma}\Big[\mathbb{E}\{\sigma^{N_j}|N_k=m\}\Big]_{\sigma=1} = m+j-k$$

and, calculating the second derivative,

$$\begin{aligned}\sum_{l=0}^{\infty} l^2\,P\Big(N_j=l\Big|N_k=m\Big) &= \Big(\frac{d^2}{d\sigma^2}+\frac{d}{d\sigma}\Big)\Big[\mathbb{E}\{\sigma^{N_j}|N_k=m\}\Big]_{\sigma=1}\\ &= m^2+4m(j-k)+2(j-k)^2+(j-k)\,.\end{aligned}$$

Altogether we get, using that $\sum_{m=0}^{\infty} m^3q^m = 6q^3/(1-q)^4 + 6q^2/(1-q)^3 + q/(1-q)^2$ for $q<1$, and a straightforward calculation

$$\begin{aligned}\mathbb{E}\Big\{X_iX_jX_k\Big\} &\leq \frac{1}{ijk^2}\sum_{m=0}^{\infty}\Big(1-\frac{1}{k}\Big)^m\cdot\Big[m^3+m^2\Big(4(j-k)+(i-j)\Big)\\ &\quad + m\Big(2(j-k)^2+(1+i-j)(j-k)\Big)\Big]\\ &= \frac{k(k-1)}{k^2ij}\Big(k(i-3-5j)+(j-1)^2+i(j-1)\Big) \leq 3\,.\end{aligned}$$

■

Lemma 3.6 *Almost surely, we have*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{X_i}{i} = 1\,.$$

Proof: By Lemma 3.5(a) we have

$$\lim_{n\to\infty} \mathbb{E}\Big\{\frac{1}{\log n} \sum_{i=1}^{n} \frac{X_i}{i}\Big\} = \lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{i-1}{i^2} = 1\,,$$

and for the variance we obtain from Lemma 3.5(b), for $n \geq 2$,

$$\begin{aligned}
\mathrm{Var}\Big\{\frac{1}{\log n} \sum_{i=1}^{n} \frac{X_i}{i}\Big\} &= \frac{1}{(\log n)^2} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{ij}\Big[\mathbb{E}\{X_iX_j\} - \mathbb{E}X_i\mathbb{E}X_j\Big] \\
&\leq \frac{2}{(\log n)^2} \sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{j-1}{j} \\
&\leq \frac{2}{(\log n)^2} \sum_{i=1}^{n} \frac{1}{i} \leq \frac{6}{\log n}\,.
\end{aligned}$$

Chebyshev's inequality yields, for sufficiently large n,

$$P\Big(\Big|\frac{1}{\log n} \sum_{i=1}^{n} \frac{X_i}{i} - 1\Big| > \varepsilon\Big) \leq P\Big(\Big|\frac{1}{\log n} \sum_{i=1}^{n} \frac{X_i}{i} - \mathbb{E}\Big\{\frac{1}{\log n} \sum_{i=1}^{n} \frac{X_i}{i}\Big\}\Big| > \varepsilon/2\Big) \leq \frac{24}{\varepsilon^2 \log n}\,.$$

Now let $s_n = 2^{(n^2)}$ and observe that, by the Borel–Cantelli Lemma, we have almost surely

$$\lim_{n\to\infty} \frac{1}{(\log 2)\, n^2} \sum_{i=1}^{s_n} \frac{X_i}{i} = 1\,.$$

If k is arbitrary, say with $s_{n-1} \leq k \leq s_n$, we get

$$\frac{1}{(\log 2)\, n^2} \sum_{i=1}^{s_{n-1}} \frac{X_i}{i} \leq \frac{1}{\log k} \sum_{i=1}^{k} \frac{X_i}{i} \leq \frac{1}{(\log 2)\, (n-1)^2} \sum_{i=1}^{s_n} \frac{X_i}{i}\,.$$

As both the lower and upper bound converge to 1 we get the result. ∎

We now return to the occupation measures. Using Lemma 3.4 we can approximate the density functions by means of the X_n.

Lemma 3.7 *For every $0 < r \leq 1$ we have, almost surely, as $n \to \infty$,*

$$r^2\, e^{-2b} \cdot X_{n+1} + o(1) \leq \frac{\mu(B(0,rt))}{\varphi(t)} \leq r^2\, e^{2b} \cdot X_n + o(1)\,,$$

uniformly for every $a_{n+1} \leq t \leq a_n$.

Proof: For any $a_{n+1} \le t \le a_n$ we have

$$a_{n+1}^2 \cdot bn + O(a_{n+1}^2) \le \varphi(t) \le a_n^2 \cdot bn + O(a_n^2)\,.$$

From this and Lemma 3.4 we infer

$$\begin{aligned} \frac{\mu(B(0,rt))}{\varphi(t)} &\le \frac{\mu(B(0,ra_n))}{a_{n+1}^2 \cdot bn} \cdot \Big(1 + O(\frac{1}{n})\Big) \\ &\le r^2 \Big(\frac{a_n}{a_{n+1}}\Big)^2 X_n + X_n \cdot O(\frac{1}{n}) + o(1) \\ &= r^2\, e^{2b} \cdot X_n + o(1)\,, \end{aligned}$$

and analogously for the opposite inequality. ∎

Lemma 3.8 *For every $0 < r \le 1$, almost surely,*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log|\log \varepsilon|} \int_\varepsilon^{1/e} \frac{\mu(B(0,rt))}{\varphi(t)} \, \frac{dt}{|t\, \log t|} = r^2\,.$$

Proof: For any $a_{n+1} \le t \le a_n$ we have, by Lemma 3.7,

$$r^2\, e^{-2b} \cdot X_{n+1} + o(1) \le \frac{\mu(B(0,rt))}{\varphi(t)} \le r^2\, e^{2b} \cdot X_n + o(1)\,.$$

Note that, as $k \to \infty$,

$$\int_{a_{k+1}}^{a_k} \frac{dt}{|t \log t|} = \log \Big|\frac{\log a_{k+1}}{\log a_k}\Big| \sim \log \frac{k+1}{k} \sim \frac{1}{k}\,. \tag{11}$$

Picking for every $\varepsilon > 0$ the index n such that $a_{n+1} \le \varepsilon \le a_n$, we get, using Lemma 3.6 and (11), almost surely,

$$\begin{aligned} &\liminf_{\varepsilon \downarrow 0} \frac{1}{\log|\log \varepsilon|} \int_\varepsilon^{1/e} \frac{\mu(B(0,rt))}{\varphi(t)} \, \frac{dt}{|t \log t|} \\ \ge\ & \liminf_{n\to\infty} \frac{1}{\log|\log a_{n+1}|} \int_{a_n}^{1/e} \frac{\mu(B(0,rt))}{\varphi(t)} \, \frac{dt}{|t \log t|} \\ =\ & \liminf_{n\to\infty} \frac{1}{\log(n+1)} \sum_{k=1}^{n-1} \int_{a_{k+1}}^{a_k} \frac{\mu(B(0,rt))}{\varphi(t)} \, \frac{dt}{|t \log t|} \\ \ge\ & e^{-2b}\, r^2 \Big(\lim_{n\to\infty} \frac{1}{\log(n+1)} \sum_{k=1}^{n-1} \frac{X_{k+1}}{k} \Big) = e^{-2b}\, r^2\,. \end{aligned}$$

Analogously, we get

$$\limsup_{\varepsilon \downarrow 0} \frac{1}{\log|\log \varepsilon|} \int_\varepsilon^{1/e} \frac{\mu(B(0,rt))}{\varphi(t)} \, \frac{dt}{|t \log t|} \le e^{2b}\, r^2\,,$$

and, altogether, letting $b \to 0$, almost surely,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log|\log \varepsilon|} \int_\varepsilon^{1/e} \frac{\mu(B(0,rt))}{\varphi(t)} \frac{dt}{|t \log t|} = r^2 .$$

■

To finish the proof of the first part of Theorem 2.1, we have to pass from the measure $\mu = \mu[0,\tau]$ to the measure $\mu[X-T,X]$. Given any $\delta > 0$, we can choose B so small that $\tau < X$ with probability at least $1-\delta$ or so large that $\tau > X$ with probability at least $1-\delta$. The monotonicity of $t \to \mu[0,t]$ and the independence of the average density of the choice of B then yield, that the statement of Lemma 3.8 holds with probability exceeding $1-2\delta$ for the measure $\mu[0,X]$. Analogously, we can make sure that the statement holds for the measure $\mu[X-T,0]$ with probability more than $1-2\delta$. By adding these statements we get that, with probability exceeding $1-4\delta$,

$$\lim_{\varepsilon \to 0} \frac{1}{\log|\log \varepsilon|} \int_\varepsilon^{1/e} \frac{\mu[X-T,X](B(0,rt))}{\varphi(t)} \frac{dt}{|t\ \log t|} = r^2 . \tag{12}$$

Finally, as δ was arbitrary, (12) holds almost surely and, choosing $r=1$, this implies the first statement of Theorem 2.1 by means of Corollary 3.2.

Now let us look at the second statement of Theorem 2.1. Suppose that the average density of order two of the measure $\mu[X-T,X]$ at the origin exists with positive probability. As the Brownian motion almost surely never returns to the origin, the average density of order two of the measures $\mu[X-T,-t]$ and $\mu[t,X]$ at the origin vanish almost surely, for any $t>0$. This implies, by Blumenthal's 01–law, that the order–two density of $\mu[X-T,X]$ at the origin exists with probability one and is constant, say equal to $2C \geq 0$. We conclude further that, for every $t>0$, the order–two density of $\mu[-t,t]$ at the origin equals $2C$ almost surely. As the Brownian motions $(B_t)_{t\geq 0}$ and $(B_{-t})_{t \geq 0}$ are independent, the order–two density of $\mu[0,t]$ at the origin is equal to C for every $t>0$ and, with the same argument as above, for every $B>0$, the average density of order two at the origin for the occupation measure $\mu[0,\tau]$ of the Brownian motion stopped upon crossing the circle of radius B is equal to C almost surely. For every $b>0$ we define, as before, the Markov chain X_n. Application of Lemma 3.7 yields, almost surely,

$$C \cdot e^{-2b} \leq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^n X_i \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^n X_i \leq C \cdot e^{2b} .$$

By Lemma 3.3 the distribution of the sums $(1/n)\sum_{i=1}^n X_i$ is independent of the choice of $b>0$ and hence the assumption implies that $\frac{1}{n}\sum_{i=1}^n X_i$ converges to C almost surely.

Lemma 3.9 *We have*

$$\lim_{n\to\infty} \operatorname{Var}\Big\{ \frac{1}{n} \sum_{i=1}^n X_i \Big\} = 1 , \tag{13}$$

and also

$$\sup_n \mathbb{E}\Big\{ \Big(\frac{1}{n} \sum_{i=1}^n X_i \Big)^3 \Big\} < \infty . \tag{14}$$

Proof: We obtain for the variance, by Lemma 3.5b),

$$\begin{aligned}\lim_{n\to\infty} \mathrm{Var}\Big\{\frac{1}{n}\sum_{i=1}^n X_i\Big\} &= \lim_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \Big\{\mathbb{E}\{X_iX_j\} - \mathbb{E}X_i\mathbb{E}X_j\Big\} \\ &= \lim_{n\to\infty} \frac{2}{n^2}\sum_{i=1}^n \frac{1}{i}\sum_{j=1}^{i-1}(j-1) \\ &= \lim_{n\to\infty} \frac{2}{n^2}\sum_{i=1}^n \frac{i^2-3i+2}{2i} = 1\,.\end{aligned}$$

By Lemma 3.5c) we have

$$\mathbb{E}\Big\{\Big(\frac{1}{n}\sum_{i=1}^n X_i\Big)^3\Big\} \le \frac{1}{n^3}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n \mathbb{E}\{X_iX_jX_k\} \le 3\,.$$

■

Now, by (14), the process $(\frac{1}{n}\sum_{i=1}^n X_i)^2$ is uniformly integrable and hence, if $\frac{1}{n}\sum_{i=1}^n X_i$ converges to C, the variances converge to 0, which is a contradiction to (13). Therefore, with probability one, the average density of order two of $\mu[X-T,X]$ at the origin fails to exist and, using the Palm property as in Corollary 3.2, we infer that, with probability one, the average density of order two of $\mu[0,T]$ fails to exist at almost every point.

4 The density distribution of planar Brownian motion

The proof of Theorem 2.2 will be given by means of Laplace transforms. As in the arguments before we first study a stopped Brownian motion. For this purpose fix $B>0$ and denote

$$\tau = \inf\{t\ge 0 \,:\, |B_t|\ge B\} \text{ and } \sigma = \sup\{t \le 0 \,:\, |B_t| \ge B\}\,.$$

Let $\mu_1=\mu[0,\tau]$, $\mu_2=\mu[\sigma,0]$ and $\mu=\mu[\sigma,\tau]$.

Lemma 4.1 *Almost surely, for every $\kappa,\lambda\ge 0$ and $0<r\le 1$,*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log|\log\varepsilon|}\int_\varepsilon^{1/e} \exp\Big(-\kappa\Big[\frac{\mu_1(B(0,rt))}{\varphi(t)}\Big] - \lambda\Big[\frac{\mu_2(B(0,rt))}{\varphi(t)}\Big]\Big)\,\frac{dt}{|t\,\log t|} = \Big(\frac{1}{1+\kappa r^2}\Big)\Big(\frac{1}{1+\lambda r^2}\Big)\,.$$

Proof: We use the same approximation technique as in the previous section. For every $b>0$ denote $a_n = Be^{-bn}$ and define the Markov chain X_n as before for the Brownian motion $(B_t)_{t\ge 0}$ with stopping time τ and in the same way an independent Markov chain Y_n for the Brownian motion $(B_{-t})_{t\ge 0}$ with the stopping time $-\sigma$. Recall from Lemma 3.7 that, almost surely for all $a_{n+1}\le t\le a_n$,

$$\begin{aligned} e^{-2b}r^2\Big(-\kappa X_n - \lambda Y_n\Big) + o(1) &\le -\kappa\Big[\frac{\mu_1(B(0,rt))}{\varphi(t)}\Big] - \lambda\Big[\frac{\mu_2(B(0,rt))}{\varphi(t)}\Big] \\ &\le e^{2b}r^2\Big(-\kappa X_{n+1} - \lambda Y_{n+1}\Big) + o(1)\,,\end{aligned}$$

and therefore it suffices to show that, for fixed $\kappa, \lambda > 0$,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{e^{-\kappa X_i} e^{-\lambda Y_i}}{i} = \Big(\frac{1}{1+\kappa}\Big)\Big(\frac{1}{1+\lambda}\Big).$$

We start by calculating the limit of the expectations. By (5),

$$\mathbb{E}\Big\{e^{-\kappa X_i}\Big\} = \sum_{k=0}^{\infty} \frac{1}{i}\Big(\frac{i-1}{i}\Big)^k e^{-\kappa(k/i)} = \frac{1}{i-(i-1)e^{-\kappa/i}} \longrightarrow \frac{1}{1+\kappa}.$$

Using the independence of X_i and Y_i we thus get

$$\lim_{n\to\infty} \mathbb{E}\Big\{\frac{1}{\log n} \sum_{i=1}^{n} \frac{e^{-\kappa X_i} e^{-\lambda Y_i}}{i}\Big\} = \Big(\frac{1}{1+\kappa}\Big)\Big(\frac{1}{1+\lambda}\Big).$$

Our next aim is to estimate the variance. For this purpose let $i \geq j$ and use (6) to see

$$\begin{aligned}
\mathbb{E}\Big\{e^{-\kappa X_i} e^{-\kappa X_j}\Big\} &= \sum_{k=0}^{\infty} \frac{1}{j}\Big(\frac{j-1}{j}\Big)^k e^{-\kappa k/j} \frac{\Big[e^{-\kappa/i} + (1-e^{-\kappa/i})(i-j)\Big]^k}{\Big[1+(1-e^{-\kappa/i})(i-j)\Big]^{k+1}} \\
&= \Big[j\Big(1+(1-e^{-\kappa/i})(i-j)\Big) - e^{-\kappa/j}\Big(e^{-\kappa/i} + (1-e^{-\kappa/i})(i-j)\Big)(j-1)\Big]^{-1}.
\end{aligned}$$

A straightforward calculation yields

$$\frac{\mathbb{E}\Big\{e^{-\kappa X_i}\Big\}\,\mathbb{E}\Big\{e^{-\kappa X_i}\Big\}}{\mathbb{E}\Big\{e^{-\kappa X_i} e^{-\kappa X_j}\Big\}} = 1 - \frac{(1-e^{-\kappa/i})\Big(j(1-e^{-\kappa/j}) - j^2(1-e^{-\kappa/j})\Big)}{\Big(i-(i-1)e^{-\kappa/i}\Big)\Big(j-(j-1)e^{-\kappa/j}\Big)}.$$

The modulus of the fraction on the right is bounded by a constant multiple of j/i. Using the independence of X_i and Y_i we infer that, for some constant $C > 0$,

$$\frac{\mathbb{E}\Big\{e^{-\kappa X_i} e^{-\lambda Y_i}\Big\}\,\mathbb{E}\Big\{e^{-\kappa X_j} e^{-\lambda Y_j}\Big\}}{\mathbb{E}\Big\{e^{-\kappa X_i} e^{-\lambda Y_i} e^{-\kappa X_j} e^{-\lambda Y_j}\Big\}} \geq 1 - C \cdot (j/i),$$

and, because the denominator is bounded by 1,

$$\mathbb{E}\Big\{e^{-\kappa X_i} e^{-\lambda Y_i} e^{-\kappa X_j} e^{-\lambda Y_j}\Big\} - \mathbb{E}\Big\{e^{-\kappa X_i} e^{-\lambda Y_i}\Big\}\,\mathbb{E}\Big\{e^{-\kappa X_j} e^{-\lambda Y_j}\Big\} \leq C \cdot (j/i).$$

This implies, for $n \geq 2$,

$$\begin{aligned}
&\mathrm{Var}\Big\{\frac{1}{\log n} \sum_{i=1}^{n} \frac{e^{-\kappa X_i} e^{-\lambda Y_i}}{i}\Big\} \\
&= \frac{1}{(\log n)^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{ij}\Big(\mathbb{E}\Big\{e^{-\kappa X_i} e^{-\lambda Y_i} e^{-\kappa X_j} e^{-\lambda Y_j}\Big\} - \mathbb{E}\Big\{e^{-\kappa X_i} e^{-\lambda Y_i}\Big\}\,\mathbb{E}\Big\{e^{-\kappa X_j} e^{-\lambda Y_j}\Big\}\Big) \\
&\leq \frac{2C}{(\log n)^2} \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{1}{i^2} \leq \frac{6C}{\log n}.
\end{aligned}$$

Chebyshev's inequality yields, for sufficiently large n,

$$P\Big(\Big|\frac{1}{\log n}\Big(\sum_{i=1}^{n}\frac{e^{-\kappa X_i}e^{-\lambda Y_i}}{i}\Big)-\Big(\frac{1}{1+\kappa}\Big)\Big(\frac{1}{1+\lambda}\Big)\Big|>\varepsilon\Big)\le\frac{24C}{\varepsilon^2\log n}.$$

By the Borel–Cantelli lemma we get, for $s_n=2^{(n^2)}$, almost surely

$$\lim_{n\to\infty}\frac{1}{\log s_n}\sum_{i=1}^{s_n}\frac{e^{-\kappa X_i}e^{-\lambda Y_i}}{i}=\Big(\frac{1}{1+\kappa}\Big)\Big(\frac{1}{1+\lambda}\Big)$$

and, for k arbitrary, say with $s_{n-1}\le k\le s_n$, we can repeat previous arguments and sandwich the sum between two sums which both converge to the same limit. This finishes the proof. ∎

By the continuity theorem for Laplace transforms (see e.g. Kallenberg (1983), 15.5.2) we infer from the case $r=1$ in Lemma 4.1 that the distribution of the φ–density function at the origin with respect to μ_1 or μ_2 converges almost surely to a standard exponential distribution and that the distribution with respect to μ converges almost surely to the distribution of the sum of two independent standard exponentially distributed random variables, which is the gamma distribution with parameter two.

The result is now proved for the density distribution at the origin of the occupation measure μ_1 and μ of the stopped Brownian motion. The argument needed to extend this result and obtain the statement of Theorem 2.2 is exactly as in the previous section.

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