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# Penalty methods for American options with stochastic volatility

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**Abstract**

The American early exercise constraint can be viewed as transforming the original linear two dimensional stochastic volatility option pricing PDE into a PDE with a nonlinear source term. Several methods are described for enforcing the early exercise constraint by using a penalty source term in the discrete equations. The resulting nonlinear algebraic equations are solved using an approximate Newton iteration. The solution of the Jacobian is obtained using an incomplete LU (ILU) preconditioned conjugate gradient-like (PCG) method. Some example computations are presented for option pricing problems based on a stochastic volatility model, including an exotic American chooser option written on a put and call with discrete double knockout barriers and discrete dividends. 

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*Keywords:* PDE option pricing; Finite element; American constraint; Stochastic volatility

## 1. Introduction

A derivative is a security whose value depends on one or more underlying factors. Derivative markets are rapidly growing. For example, the total notional value of outstanding derivatives was \$5 trillion in 1990 and was over \$20 trillion in 1994.

Options are a form of derivative security that give the holder the right, but not the obligation, to buy or sell an asset for a specified exercise price at some future time. It is of great interest to financial institutions to be able to determine the value of an option as a function of the underlying factors and time.

Utilizing models of asset prices based on stochastic differential equations, a non-stochastic partial differential equation (PDE) for the price of the option can be derived [28]. This PDE has the familiar form of the multi-dimensional convection–diffusion equation.

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Many options have an early exercise feature. This allows the holder of the option to exercise the option at any time during its life [28]. An option with this early exercise feature is known as an American option. An option which cannot be exercised early is termed a European option. Assuming that investors act optimally, the value of an American option cannot fall below the value that would be obtained if it was exercised early. Effectively, this means that the American early exercise feature transforms the original linear European pricing equation into a nonlinear PDE.

If an implicit method is used to solve the basic option pricing PDE, then the nonlinear algebraic constraint (due to the early exercise feature) should, in general, also be handled implicitly. One method for incorporating the constraint is to view the problem as a linear complementarity problem [28] and then use projected Successive Overrelaxation (SOR) [6] to solve the discrete algebraic equations. However, in regions where it is not optimal to exercise the option early, this method simply reduces to unaccelerated SOR for solving the sparse linear system. Unaccelerated SOR iterative methods have, of course, been supplanted by the more robust preconditioned conjugate gradient (PCG-like) techniques [20, 23, 24, 26]. Projected SOR can be accelerated using a multigrid method [5]. While multigrid methods can sometimes be spectacularly successful, they must often be tuned to the problem at hand. Care must be taken with the choice of smoother, and the prolongation and restriction operators. For example, in [5], the smoother must be adjusted to fit early exercise and nonearly exercise parts of the computational domain. It is therefore a daunting task, at the present time, to produce black box option pricing software based on multigrid techniques which can be used in day-to-day financial applications. An alternative method based on linear programming [9] has recently been proposed. However, if the underlying PDE is more than one dimensional, then the linear programming method used in [9] may become computationally infeasible.

The objective of this article is to develop a general method for handling the American early exercise feature. We simply view the problem as a nonlinear PDE, where the early exercise constraint can be imposed using a penalty method. The resulting system of nonlinear algebraic equations is then solved using Newton iteration, where the nonsymmetric Jacobian at each nonlinear iteration is solved using PCG-like methods [23]. The advantages of this approach are

- Software can be developed based on black box *off-the-shelf* components. The sparse Jacobian is solved using a standard method. The Jacobian itself can be constructed using a variety of techniques.
- Since we regard the system as nonlinear right from the start, there is no difficulty incorporating more sophisticated discretization methods such as nonlinear flux limiters [19, 25]. In many cases, such as for an option pricing PDE based on a stochastic volatility model [15], the PDE has large regions which are convection dominated, and hence standard central or upstream weighting methods are inappropriate.
- Incorporation of other types of constraints (e.g. time dependent barriers [30]) can be done in a straightforward fashion, since the algorithm does not depend on the form of the constraint or the form of the PDE.

In this paper, we give examples of the use of this technique for pricing options based on a stochastic volatility model. To illustrate the flexibility of this approach, we include an example of an exotic American chooser option [16] written on a barrier put and call, which has a complex early exercise constraint.

## 2. Stochastic volatility

Recently, there has been some interest in models where the volatility of the underlying asset is random [17, 21, 15, 27]. Stochastic volatility models are considered to be a more realistic specification of stock price movement than models with constant volatility such as the classic Black–Scholes [3] analysis. Denote the underlying asset price by $s$ and its instantaneous variance by $v$, and following [15] assume that these evolve according to:

$$\begin{aligned} \mathrm{d}s &= \mu s\ \mathrm{d}t^* + \sqrt{v}s\ \mathrm{d}z_1, \\ \mathrm{d}v &= \kappa(\theta - v)\ \mathrm{d}t^* + \sigma\sqrt{v}\ \mathrm{d}z_2, \end{aligned} \tag{1}$$

where $\mu$ is the expected growth rate of the stock price, $\sqrt{v}$ is its instantaneous volatility, $\kappa$ is a parameter controlling how fast $v$ reverts to its mean level of $\theta$, $\sigma$ is the 'volatility of volatility' parameter, and $z_1, z_2$ are Wiener processes [28] with correlation parameter $\rho$.

We now briefly sketch how to derive the PDE governing the value of a derivative security such as an option in this context. Although this is standard material in the finance literature (the development below is a simple application of Appendix 13B of [16]), it is included here for the benefit of readers unfamiliar with the general methodology. We make the usual Black–Scholes assumptions that trading occurs continuously in frictionless markets (i.e. there are no taxes, transactions costs, short sale restrictions, etc.), that there is a single risk free interest rate for all maturities, denoted by $r$, and that there are no arbitrage opportunities (meaning that there is no way to earn a positive amount in the future without giving up something today).

In the constant volatility Black–Scholes framework, the basic idea is that it is possible to form a portfolio consisting of the underlying stock and the derivative security which is instantaneously risk free. To prevent arbitrage, this portfolio must earn a rate of return of $r$. Imposing this condition leads directly to the Black–Scholes PDE. In a stochastic volatility setting matters are complicated by the presence of a second source of risk, $v$, which is not a traded asset. This means that arbitrage considerations alone are insufficient – additional assumptions regarding investors' preferences are required.

Denote the value of derivative security $j$ by $U^j(s, v, t^*)$. By Itô's lemma [28],

$$\mathrm{d}U^j = v^j U^j\ \mathrm{d}t^* + \zeta_1^j U^j\ \mathrm{d}z_1 + \zeta_2^j U^j\ \mathrm{d}z_2, \tag{2}$$

where

$$\begin{aligned} v^j U^j &= \frac{vs^2}{2} U^j_{ss} + \rho\sigma v s U^j_{sv} + \frac{\sigma^2 v}{2} U^j_{vv} + \mu s U^j_s + \kappa(\theta - v) U^j_v + U^j_{t^*}, \\ \zeta_1^j U^j &= \sqrt{v} s U^j_s, \\ \zeta_2^j U^j &= \sigma\sqrt{v} U^j_v. \end{aligned} \tag{3}$$

Form a portfolio $V$ with three derivative securities in it. Let the amount invested in security $j$ be $k_j$, so that $V = k_1 U^1 + k_2 U^2 + k_3 U^3$. Suppose that the $k_j$'s are chosen such that the portfolio is instantaneously risk free:

$$\begin{aligned} k_1\zeta_1^1 U^1 + k_2\zeta_1^2 U^2 + k_3\zeta_1^3 U^3 &= 0, \\ k_1\zeta_2^1 U^1 + k_2\zeta_2^2 U^2 + k_3\zeta_2^3 U^3 &= 0. \end{aligned} \tag{4}$$

Absence of arbitrage requires that this portfolio earns a rate of return of $r$, or $k_1 v^1 U^1 + k_2 v^2 U^2 + k_3 v^3 U^3 = rV$, which can be written as

$$k_1 U^1(v^1 - r) + k_2 U^2(v^2 - r) + k_3 U^3(v^3 - r) = 0. \tag{5}$$

Eqs. (4) and (5) have a nontrivial solution only if there are functions $\phi_1(s, v, t^*)$ and $\phi_2(s, v, t^*)$ such that

$$v^j - r = \phi_1(s, v, t^*)\zeta_1^j + \phi_2(s, v, t^*)\zeta_2^j. \tag{6}$$

These $\phi$ functions are referred to as the market prices of risk for the state variables $s$ and $v$. Substituting from (3) and suppressing the $j$ superscript gives the following PDE for the price of a derivative security:

$$\begin{aligned} &\frac{vs^2}{2} U_{ss} + \rho\sigma vs U_{sv} + \frac{\sigma^2 v}{2} U_{vv} + \mu s U_s + \kappa(\theta - v) U_v + U_{t^*} - rU \\ &\quad = \phi_1 \sqrt{v} s U_s + \phi_2 \sigma \sqrt{v} U_v, \end{aligned} \tag{7}$$

or

$$\begin{aligned} &\frac{vs^2}{2} U_{ss} + \rho\sigma vs U_{sv} + \frac{\sigma^2 v}{2} U_{vv} + (\mu - \phi_1\sqrt{v}) s U_s + (\kappa(\theta - v) - \phi_2 \sigma \sqrt{v}) U_v \\ &\quad + U_{t^*} - rU = 0 \end{aligned} \tag{8}$$

It is possible to eliminate $\phi_1$ as follows. Since the state variable $s$ is a traded asset, $U(s, v, t^*) = s$ must satisfy the PDE. In this case we have $U_{ss} = U_{vv} = U_{sv} = U_v = U_{t^*} = 0$, and $U_s = 1$ which implies $\mu - \phi_1\sqrt{v} = r$. Since the second state variable $v$ is not a traded asset, $\phi_2$ cannot be similarly eliminated. In order to specify $\phi_2$, additional assumptions are required regarding investors' preferences. A set of such assumptions is provided in [15] which allows $\phi_2\sigma\sqrt{v}$ to be written as $\lambda v$, where $\lambda$ is a constant parameter. Invoking these assumptions and eliminating $\phi_1$ as described above gives

$$\tfrac{1}{2} vs^2 U_{ss} + \rho\sigma vs U_{sv} + \tfrac{1}{2}\sigma^2 v U_{vv} + rs U_s + (\kappa(\theta - v) - \lambda v) U_v - rU + U_{t^*} = 0. \tag{9}$$

Eq. (9) is solved backward in time from the expiry date of the option $t^* = T$ to the current time $t^* = 0$. Eq. (9) can be converted to the familiar form of an equation forward in time by substituting $t = T - t^*$ to give

$$U_t = \tfrac{1}{2} vs^2 U_{ss} + \rho\sigma vs U_{sv} + \tfrac{1}{2}\sigma^2 v U_{vv} + rs U_s + (\kappa(\theta - v) - \lambda v) U_v - rU. \tag{10}$$

Following some algebraic manipulations, Eq. (10) can be put into the following form:

$$U_t + \boldsymbol{V} \cdot \nabla U = \nabla \cdot \boldsymbol{D} \cdot \nabla U - rU \tag{11}$$

where

$$\boldsymbol{D} = \frac{1}{2} \begin{pmatrix} vs^2 & \rho\sigma sv \\ \rho\sigma sv & \sigma^2 v \end{pmatrix}, \tag{12}$$

$$\boldsymbol{V} = -\begin{pmatrix} rs - vs - \rho\sigma s/2 \\ \kappa(\theta - v) - \lambda v - \sigma^2/2 - \rho\sigma v/2 \end{pmatrix}. \tag{13}$$

Eq. (11) has the form of the convection–diffusion equation. The initial conditions depend on the contractually agreed payoff function. For a vanilla (standard) put or call with an exercise price of $E$, the initial conditions (at $t = 0$ or equivalently at $t^* = T$) are

$$U(s,v,0) = \begin{cases} \max(s-E,0), & \text{call}, \\ \max(E-s,0), & \text{put}. \end{cases} \tag{14}$$

Other boundary conditions for this equation can be determined by examining the original equation (9). Letting $v, s \to 0$ we obtain

$$\begin{aligned} U_t &= rsU_s + \kappa\theta U_v - rU, \quad v \to 0, \\ U_t &= \tfrac{1}{2}\sigma^2 v U_{vv} + (\kappa(\theta - v) - \lambda v)U_v - rU, \quad s \to 0. \end{aligned} \tag{15}$$

For $s \to \infty$ we have

$$\begin{aligned} U &\simeq s \quad \text{call}, \\ U &\simeq 0 \quad \text{put}. \end{aligned}$$

Finally, noting that as $v \to \infty$ then $U_v \to 0$, so

$$U_t = \tfrac{1}{2} v s^2 U_{ss} + rsU_s - rU, \quad v \to \infty. \tag{16}$$

Eq. (11) is valid for a European option. For an American option, we have the additional constraint that at any time

$$U(s,v,t) \geqslant U(s,v,0). \tag{17}$$

Consequently, in any region where it is optimal to exercise the option early, we have [28]

$$U = U(s,v,0), \qquad U_t + \boldsymbol{V} \cdot \nabla U - \nabla \cdot \boldsymbol{D} \cdot \nabla U + rU > 0 \tag{18}$$

while in those regions where it is not optimal to exercise early, we have

$$U > U(s,v,0), \qquad U_t + \boldsymbol{V} \cdot \nabla U - \nabla \cdot \boldsymbol{D} \cdot \nabla U + rU = 0. \tag{19}$$

Eqs. (18), (19) can be combined into one equation which is valid everywhere (in both early exercise and no-exercise regions)

$$U_t + \boldsymbol{V} \cdot \nabla U - \nabla \cdot \boldsymbol{D} \cdot \nabla U + rU = q', \tag{20}$$

where $q'$ is defined so as to ensure that $U(s,v,t) \geqslant U(s,v,0)$. Consequently, from Eqs. (18), (19), it can be seen that $q'$ satisfies the conditions:

$$\begin{aligned} q' &= 0 \;\text{ if }\; U > U(s,v,0), \\ q' &> 0 \;\text{ if }\; U = U(s,v,0). \end{aligned} \tag{21}$$

In the following, we will determine $q'$ so that $q' = 0$ in those regions where it is not optimal to exercise early. In regions where it is optimal to exercise early, then $q'$ will designed to force the solution of Eq. (20) to be $U = U(s,v,0)$. Consequently, we can regard Eq. (20) as valid in the entire computational domain.

A more intuitive understanding of the role of $q'$ can be obtained by noting that the value of an American option is always worth more than a European option in regions where it is optimal to exercise early. Consequently, the source term $q'$ is positive (it adds value) in optimal early exercise regions.

## 3. Discretization

We will now discretize Eq. (20) using a standard Galerkin finite element method for the diffusion terms. For the convective terms, we will use a finite volume approach. Formally, a finite volume discretization can be considered to be a Galerkin method with a special quadrature rule [13], so that in a mathematical sense, a Galerkin finite element method is being used for all terms in the equation. However, it is more intuitively appealing to use a geometric finite volume approach for discretizing the convective term.

Consider a discrete two dimensional computational domain $R$ which is tiled by triangles. Let $N_i$ be the usual $C^0$ Lagrange basis functions defined on triangles. Then,

$$N_i = \begin{cases} 1 & \text{at node } i, \\ 0 & \text{at all other nodes,} \end{cases}$$
$$\sum_j N_j = 1 \quad \text{everywhere in the solution domain.} \tag{22}$$

If $U^n = \sum_j U_j^n$ where $U_j^n = U(s_j, v_j, t^n)$ is the value of $U$ at $(s_j, v_j, t^n)$, then the discretization of Eq. (11) is given by

$$\begin{aligned} A_i\left(\frac{U_i^{n+1} - U_i^n}{\Delta t}\right) = {} & (1-\beta)\left(\sum_{j\in\eta_i} \gamma_{ij}(U_j^{n+1} - U_i^{n+1}) + \sum_{j\in\eta_i} \boldsymbol{L_{ij}} \cdot \boldsymbol{V}_i U_{ij+1/2}^{n+1} - A_i r U_i^{n+1}\right) \\ & +\beta\left(\sum_{j\in\eta_i} \gamma_{ij}(U_j^n - U_i^n) + \sum_{j\in\eta_i} \boldsymbol{L_{ij}} \cdot \boldsymbol{V}_i U_{ij+1/2}^n - A_i r U_i^n\right) \\ & +q_i^{n+1}, \end{aligned} \tag{23}$$

where

$$\begin{aligned} A_i &= \int N_i \, \mathrm{d}R \\ \Delta t &= \text{ timestep} \\ \beta &= \text{ timeweighting} \\ \beta &= \begin{cases} 0 & \text{fully implicit} \\ 1 & \text{explicit} \\ 1/2 & \text{Crank–Nicolson} \end{cases} \\ U_i^{n+1} &= U(s_i, v_i, t^{n+1}) \\ \gamma_{ij} &= -\int_R \nabla N_i \cdot \boldsymbol{D} \cdot \nabla N_j \, \mathrm{d}R \\ \eta_i &= \text{ set of neighbours of node } i \end{aligned}$$

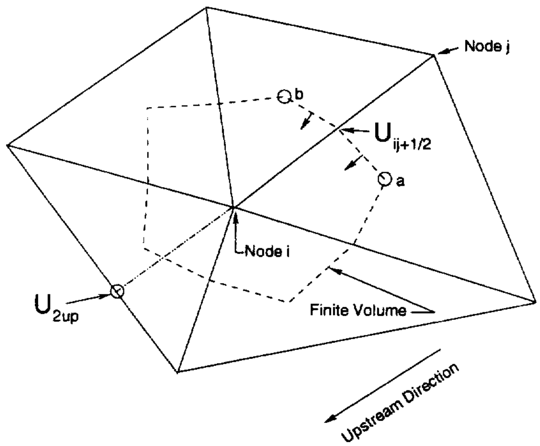


Fig. 1. Finite volume surrounding node i. Points a and b are the centroids of their respective triangles. The line segments from a and b pass through the midpoint of the triangle edge $i-j$.

$q_i =$ discrete form of the source/sink term $q'$ (Eq. (20)) used to force the American constraint

$U^{n+1}_{ij+1/2} =$ value of $U$ at the face between node $i$ and node $j$ (24)

We have also used mass lumping for the time derivative term. Other details concerning this discretization method can be found in [11, 13]. Note that $A_i$ can be considered to be the area of the *cell* or finite volume surrounding node $i$. The finite volume surrounding node $i$ is shown in Fig. 1. The finite volume is constructed by joining the midpoint of each edge of a triangle to the centroid of the triangle [1, 12, 18]. The vector length $\boldsymbol{L}_{ij}$ in Eq. (23) is given by

$$\boldsymbol{L}_{ij} = \int_a^b \hat{n} \, \mathrm{d}s, \tag{25}$$

where the points $a, b$ are shown in Fig. 1, and $\hat{n}$ is the inward pointing normal to the face between node $i$ and node $j$. An alternative choice of finite volume can be based on the perpendicular bisectors of triangle edges [12].

There are various choices for the terms $U_{ij+1/2}$, e.g., second order central weighting for $U_{ij+1/2}$ is given by

$$U_{ij+1/2} = \tfrac{1}{2}(U_i + U_j) \tag{26}$$

while first order upstream weighting is given by

$$U_{ij+1/2} = \begin{cases} U_i & \text{if } \boldsymbol{L}_{ij} \cdot \boldsymbol{V}_i < 0, \\ U_j & \text{otherwise.} \end{cases} \tag{27}$$

Note that Eq. (10) becomes first order hyperbolic as $v \to 0$. First order upstream weighting is usually too diffusive for accurate solutions, while central weighting may cause spurious oscillations in convection dominated regions. Recently, nonlinear flux limiters have been used to obtain accurate solutions without causing oscillations. Essentially, these methods use a more accurate (usually second order) method as much as possible, but reduce to lower order accuracy only where necessary to avoid spurious oscillations [29, 30]. One popular method uses a van Leer limiter [4, 19, 25]. With reference to Fig. 1, assume that node $i$ is upstream of node $j$ (the upstream directions are given by Eq. (27)). Point $2up$ is the value of $U$ which is upstream of node $i$, interpolated using the two nearest nodes where $U$ is known (see Fig. 1). The value of $U_{ij+1/2}$ is then extrapolated to the face $(ij+\frac{1}{2})$ using the values at $U_i$ and $U_{2up}$ [14]. A nonlinear limiter is applied to avoid spurious oscillations in the solution [2, 12, 19]. In this work we will use the van Leer limiter. Other possibilities include the smooth MUSCL limiter described in [2].

## 4. Solution of the discrete equations

The discrete equations (23) are in general nonlinear. This is due to the use of a nonlinear flux limiter for the convection term, and also due to the application of the American constraint. The method used to apply this constraint, in an implicit fashion, will be described in a subsequent section. An approximate Newton iteration will be used to solve the discrete equations. The complete Jacobian is constructed with the exception of all derivatives with respect to the second upstream points $U_{2up}$, which are ignored. The iteration for a given timestep is deemed to have converged when

$$\max_i \frac{|(U_i^{n+1})^{k+1} - (U_i^{n+1})^k|}{\max(|(U_i^{n+1})^{k+1}|, |(U_i^n)|, 1.0)} < tol, \tag{28}$$

where $(U_i^{n+1})^k$ is the *kth* iterate for $U_i^{n+1}$. The Jacobian is solved using an incomplete $LU$ [7, 8] preconditioned CGSTAB iteration [26]. An automatic timestep selection method is also used [22].

## 5. American options

American options are easily handled in a fully implicit fashion, through suitable definition of the source/sink term in Eq. (23). Two approaches will be discussed in this paper. Effectively, these are penalty methods for forcing the discrete problem to satisfy the early exercise constraint.

### 5.1. Constraint switching

If an American option is to be priced, then we define two possible states for a node {ON, OFF}. The source term (Eq. (23)) is then defined as

$$\begin{aligned}
&\text{IF( state}_i = \text{ON) then} \\
&\qquad q_i^{n+1} = \frac{A_i}{\Delta t}(U_i^* - U_i^{n+1}) \times Large \\
&\text{ELSE} \\
&\qquad q_i^{n+1} = 0 \\
&\text{ENDIF}
\end{aligned} \tag{29}$$

where $U_i^*$ is the value of the option if exercised immediately. For a vanilla American put, this is given by Eq. (14). In Eq. (29), *Large* is a suitably defined large number.

After each nonlinear iteration, the state of each node can be switched:

$$
\begin{array}{l}
\text{IF}\,(\text{state}_i = \text{ON}) \\
\quad \text{IF}\,(U_i^{n+1} > U_i^*)\,\text{then} \\
\quad\quad \text{state}_i := \text{OFF} \\
\quad \text{ENDIF} \\
\text{ELSE} \\
\quad \text{IF}(U_i^{N+1} < U_i^*)\,\text{then} \\
\quad\quad \text{state}_i := \text{ON} \\
\quad \text{ENDIF} \\
\text{ENDIF}
\end{array}
\tag{30}
$$

Note that when $state_i = \text{ON}$, then we must have $q_i^{n+1} > 0$ (since the American constraint adds value). Consequently, as $Large \to \infty$, then for nodes with $state_i = \text{ON}$, then $U_i^{n+1} \to U_i^* - \varepsilon$, where $\varepsilon = 0(1/Large)$. This error in enforcing the constraint can be made arbitrarily small by making *Large* sufficiently large.

The transition rules in Eq. (30) are based on the assumption that a minimum constraint is being imposed. In the case of an option with both maximum and minimum type constraints (e.g., callable convertible bonds [28]), there would be three possible states for a node, with the obvious changes to the transition rules.

### 5.2. Quadratic source term

If Newton iteration is used to solve the nonlinear discrete equations which result from use of the constraint switching method in (30), then the Jacobian has a discontinuous derivative at $U_i^{n+1} = U_i^*$, which might cause some difficulties. An alternative approach uses a smoother method of implementing the constraint. The source/sink term in Eq. (23) can be defined as

$$q_i^{n+1} = \frac{A_i}{\Delta t}(\min(U_i^{n+1} - U_i^*, 0))^2 \times Large, \tag{31}$$

where *Large* is a suitably defined large number and $U_i^*$ is the value of the option if exercised immediately.

Imagine solving the discrete equations with the source term (31), by a Newton iteration. Suppose the initial guess for the solution at $U_i^{n+1}$ uses the value of $U_i^n$, and suppose that this value is above the value obtained by early exercise. Consequently, on the first iteration, the source term (31) is zero. If after the first iteration, $U_i^{n+1} > U_i^*$, then it is not optimal to exercise early, and the iteration terminates. However, if the first iteration produces $U_i^{n+1} < U_i^*$, then the source term becomes nonzero, and then forces another nonlinear iteration. Since the source term is positive, the next iteration will produce a larger value for $U_i^{n+1}$. The quadratic form for the source term will cause a monotonic approach to a value of $U_i^{n+1} = U_i^* - \varepsilon$ with $\varepsilon \ll 1$. The size of $\varepsilon$ will be determined by the size of *Large*. The larger the value of this constant, the smaller $\varepsilon$, but in general the number of nonlinear iterations will increase as *Large* increases in magnitude.

### 5.3. Equivalence of penalty method and linear complementarity formulation

Let the vector with components $U_i^{n+1}$ be denoted by $\boldsymbol{U}^{n+1}$. Similarly

$$(\boldsymbol{q}^{n+1})_i = q_i^{n+1}, \qquad (\boldsymbol{U}^*)_i = U_i^*. \tag{32}$$

Let $\alpha$ be the the region of the computational domain $D$ where

$$q_i^{n+1} > 0 \text{ if } (x_i, y_i) \in \alpha. \tag{33}$$

In other words, $\alpha$ is the region of $D$ where it is optimal to exercise the option early. Let

$$\begin{aligned}(\boldsymbol{L}\boldsymbol{U}^{n+1})_i = {} & A_i\left(\frac{U_i^{n+1} - U_i^n}{\Delta t}\right) \\ & - \left\{(1-\beta)\left(\sum_{j\in\eta_i}\gamma_{ij}(U_j^{n+1} - U_i^{n+1}) + \sum_{j\in\eta_i}\boldsymbol{L}_{ij}\cdot \boldsymbol{V}_i U_{ij+1/2}^{n+1} - A_i r U_i^{n+1}\right)\right\} \\ & - \left\{\beta\left(\sum_{j\in\eta_i}\gamma_{ij}(U_j^n - U_i^n) + \sum_{j\in\eta_i}\boldsymbol{L}_{ij}\cdot \boldsymbol{V}_i U_{ij+1/2}^n - A_i r U_i^n\right)\right\}.\end{aligned} \tag{34}$$

For simplicity, we consider only the constraint switching method in the following. From Eq. (29) as *Large* $\to \infty$ we have

$$\begin{aligned}(U_i^{n+1} - U_i^*) &= -\left|O\left(\frac{1}{Large}\right)\right| \quad (x_i, y_i) \in \alpha \\ &> 0 \qquad\qquad (x_i, y_i) \in D - \alpha\end{aligned} \tag{35}$$

From Eqs. (29), (34), (35) it follows that

$$\begin{aligned}(\boldsymbol{L}\boldsymbol{U}^{n+1})_i &= O\left(\frac{A_i}{\Delta t}\right) \qquad (x_i, y_i) \in \alpha, \\ &= 0 \qquad\qquad (x_i, y_i) \in D - \alpha.\end{aligned} \tag{36}$$

Eqs. (35), (36) then imply that

$$\begin{aligned}&(\boldsymbol{U}^{n+1} - \boldsymbol{U}^*) \geqslant -\left|O\left(\frac{1}{Large}\right)\right| && (x_i, y_i) \in D, \\ &\boldsymbol{L}\boldsymbol{U} \geqslant 0 && (x_i, y_i) \in D, \\ &|(\boldsymbol{U}^{n+1} - \boldsymbol{U}^*)\cdot \boldsymbol{L}\boldsymbol{U}| \leqslant O\left(\frac{1}{\mathrm{Large}}\right)\frac{\mathrm{A_D}}{\Delta \mathrm{t}} && (x_i, y_i) \in D, \\ &A_D = \sum_i A_i.\end{aligned} \tag{37}$$

Table 1
Data for American put with stochastic volatility

| | |
|---|---|
| $\sigma$ | 0.9 |
| $\rho$ | 0.1 |
| $\kappa$ | 5.0 |
| $\theta$ | 0.16 |
| $\lambda$ | 0.0 |
| $r$ | 0.10 |
| Time to expiry | 0.25 years |
| Exercise price ($E$) | \$10 |
| $U_i^*$ | $\max(E - s_i, 0)$ |

Therefore, as *Large* $\to \infty$, Eq. (37) can be regarded as an approximation to

$$\begin{aligned} &(U^{n+1} - U^*) \geqslant 0, \\ &LU \geqslant 0, \\ &(U^{n+1} - U^*) \cdot LU = 0 \end{aligned} \tag{38}$$

for all $(x, y) \in D$. This is a discrete form of the linear complementarity formulation of the American constraint [28]. The linear complementarity approach is an identical numerical problem to a discrete variational inequality [28]. Consequently, it is possible to demonstrate, in some cases, that a unique solution exists, and that the discrete solution converges to a solution having $C^1$ continuity across the early exercise boundary [10].

## 6. Clark and Parrot Problem [5]

An American put option with stochastic volatility was extensively studied in [5]. The data for this problem are given in Table 1.

Table 2 gives the values of the American put computed on an $89 \times 52$ grid, and a refined grid formed by inserting a node between each node in the coarse grid ($176 \times 102$). The timestep selector parameters [22] were halved for the fine grid as well. The values of *Large* and the convergence Newton iteration tolerance *tol* are given in Table 2. For comparison, the finest grid results from [5] are also given. A Crank–Nicolson timestepping method was used.

The results are in general agreement with those in [5], but there are some differences. Note that in this work, the computations on the fine grid used smaller timesteps than the coarse grid results. Hence, the results in Table 2 (for this work) reflect both time and space truncation errors. In contrast, in [5], a constant timestep was used on all grids. As well, interpolation was used in [5] to obtain the values shown in Table 2 (a coordinate transformation was used in [5] to obtain discrete equations more suitable for a multigrid approach). These effects probably account for the differences between this work and the results in [5].

Table 3 compares the results for the above problem with various values for *tol* and *Large*. The coarse $89 \times 52$ grid was used for these computations. The constraint switching method (see Section 5.1) was used. This table should be viewed as comparing the effects of using different values of

Table 2
Convergence of American put with stochastic volatility. Constraint switching method used with $tol = 10^{-5}$, $Large = 10^5$. Table values are the value of an American put with the stochastic volatility model, in dollars at the initial time $t^* = 0$ $(t = T)$

| $v$ | S | | | | |
|---|---|---|---|---|---|
| | 8.0 | 9.0 | 10.0 | 11.0 | 12.0 |
| 89 × 52 grid | | | | | |
| 0.0625 | 2.0000 | 1.1078 | 0.5206 | 0.2142 | 0.0823 |
| 0.08 | 2.0000 | 1.1317 | 0.5512 | 0.2380 | 0.0962 |
| 0.12 | 2.0103 | 1.1838 | 0.6163 | 0.2908 | 0.1294 |
| 0.16 | 2.0279 | 1.2326 | 0.6760 | 0.3414 | 0.1637 |
| 0.20 | 2.0492 | 1.2790 | 0.7316 | 0.3901 | 0.1987 |
| 0.24 | 2.0723 | 1.3231 | 0.7837 | 0.4370 | 0.2339 |
| 0.25 | 2.0784 | 1.3338 | 0.7963 | 0.4485 | 0.2427 |
| 177 × 103 grid, smaller timesteps | | | | | |
| 0.0625 | 2.0000 | 1.1076 | 0.5202 | 0.2138 | 0.0821 |
| 0.08 | 2.0003 | 1.1316 | 0.5507 | 0.2376 | 0.0961 |
| 0.12 | 2.0103 | 1.1836 | 0.6159 | 0.2904 | 0.1293 |
| 0.16 | 2.0280 | 1.2326 | 0.6758 | 0.3412 | 0.1637 |
| 0.20 | 2.0493 | 1.2789 | 0.7314 | 0.3900 | 0.1987 |
| 0.24 | 2.0724 | 1.3230 | 0.7835 | 0.4369 | 0.2340 |
| 0.25 | 2.0784 | 1.3337 | 0.7961 | 0.4483 | 0.2428 |
| Results in [5], finest grid | | | | | |
| 0.0625 | 2.0000 | 1.1080 | 0.5316 | 0.2261 | 0.0907 |
| 0.25 | 2.0733 | 1.3290 | 0.7992 | 0.4536 | 0.2502 |

nonlinear convergence tolerance *tol* and *Large* (see Eqs. (29)–(30)), for a given grid size and timestep sequence.

Note that if a tolerance of $tol = 10^{-k}$ is desired, then the value of *Large* should be $\simeq 10^k$. Examination of Table 3 shows that, as expected, there is no change in the solution to five figures for $k \geqslant 4$.

Table 4 shows similar results, but this time the quadratic source term (see Section 5.2) was used. For a quadratic penalty term, if an accuracy of $tol = 10^{-k}$ is desired, then *Large* should be $\simeq 10^{2k}$. Although moderate accuracy can be obtained with the quadratic source term, difficulties were observed when requesting very tight convergence tolerances (note the nonconvergence for $tol = 10^{-5}$ in Table 4).

The results shown here are representative of our observations for many problems. It appears that the constraint switching method is more efficient and reliable than the quadratic source method. This appears surprising at first glance, since the quadratic term would seem to be more easily solved using Newton iteration. However, the timesteps required for reasonable levels of time discretization

Table 3
Constraint switching method used with indicated values of *Large* and convergence tolerance *tol*. 89 × 52 grid used. Problem from [5]

| $S$ | | | | | |
|---|---|---|---|---|---|
| 8.0 | 10.0 | 12.0 | 8.0 | 10.0 | 12.0 |
| $v$=0.0625 | | | $v$=0.25 | | |
| $tol = 10^{-3}$, $Large = 10^3$ Total nonlinear iterations = 100 | | | | | |
| 2.0000 | 0.5206 | 0.0823 | 2.0783 | 0.7963 | 0.2427 |
| $tol = 10^{-4}$, $Large = 10^4$ Total nonlinear iterations = 117 | | | | | |
| 2.0000 | 0.5206 | 0.0823 | 2.0784 | 0.7963 | 0.2427 |
| $tol = 10^{-5}$, $Large = 10^5$ Total nonlinear iterations = 125 | | | | | |
| 2.0000 | 0.5206 | 0.0823 | 2.0784 | 0.7963 | 0.2427 |
| $tol = 10^{-6}$, $Large = 10^6$ Total nonlinear iterations = 133 | | | | | |
| 2.0000 | 0.5206 | 0.0823 | 2.0784 | 0.7963 | 0.2427 |
| $tol = 10^{-8}$, $Large = 10^8$ Total nonlinear iterations = 154 | | | | | |
| 2.0000 | 0.5206 | 0.0823 | 2.0784 | 0.7963 | 0.2427 |

Table 4
Quadratic source method used with indicated values of *Large* and convergence tolerance *tol*. 89 × 52 grid used. Problem from [5]

| $S$ | | | | | |
|---|---|---|---|---|---|
| 8.0 | 10.0 | 12.0 | 8.0 | 10.0 | 12.0 |
| $v$=0.0625 | | | $v$=0.25 | | |
| $tol = 10^{-3}$, $Large = 10^6$ Total nonlinear iterations = 98 | | | | | |
| 1.9999 | 0.5205 | 0.0823 | 2.0783 | 0.7963 | 0.2427 |
| $tol = 10^{-4}$, $Large = 10^8$ Total nonlinear iterations = 140 | | | | | |
| 2.0000 | 0.5206 | 0.0823 | 2.0783 | 0.7963 | 0.2427 |
| $tol = 10^{-5}$, $Large = 10^{10}$ Total nonlinear iterations = * * ** | | | | | |
| Not Converged | | | | | |

error are quite small, so that the discontinuity in the derivative of the constraint switching source term does not appear to have serious consequences.

To isolate the effect of the American constraint on the nonlinear iterations, Table 5 shows the total number of nonlinear iterations required for solution of the above problem with various discretization methods.

If pure upstream weighting is used (Eq. (27)), then the only nonlinearity in the discrete equations is due to the American constraint. In this case, Table 5 indicates that about five nonlinear iterations

Table 5
Constraint switching with upstream weighting and flux limiter, convergence tolerance $10^{-5}$, *Large* $= 10^5$. For upstream weighting, the only nonlinearity is due to the American constraint. Problem from [5]

| Method | Number of nonlinear iterations | Number of timesteps |
|---|---|---|
| American upstream | 122 | 27 |
| European with flux limiter | 104 | 27 |
| American flux limiter | 125 | 27 |

per timestep is required to resolve the American constraint to five figure accuracy. In contrast, solving a European problem using the flux limiter requires about four iterations per timestep. Finally, use of the flux limiter with the American constraint requires almost the same number of total nonlinear iterations as with the American constraint with upstream weighting. Of course, the solutions to all these problems are not identical, so the comparisons are not perfectly valid. Nevertheless, it appears that the cost of using the nonlinear equation approach for the early exercise constraint, coupled with the flux-limited discretization, is not much more expensive than using the flux-limiter alone. However, if a European option is being priced using upstream weighting, then this is a purely linear problem, and only one iteration per timestep is necessary. Consequently, the cost of solving an American option with a flux-limited discretization is about five times greater than for a European option with upstream weighting, for this quite severe convergence criteria.

## 7. An American chooser based on European barrier options

A chooser option gives the holder the right to either a call or a put at maturity [16]. The payoff for a chooser is given by

$$\begin{aligned}
&\text{Payoff} = \max(C(s, v, T_C - T, E_C), P(s, v, T_P - T, E_P)),\\
&C = \text{ value of call},\\
&P = \text{ value of put},\\
&s = \text{ asset price},\\
&v = (\text{volatility})^2\\
&E_C = \text{ call exercise price},\\
&E_P = \text{ put exercise price},\\
&T_C = \text{ maturity date of call},\\
&T_P = \text{ maturity date of put},\\
&T = \text{ maturity date of chooser}.
\end{aligned}$$

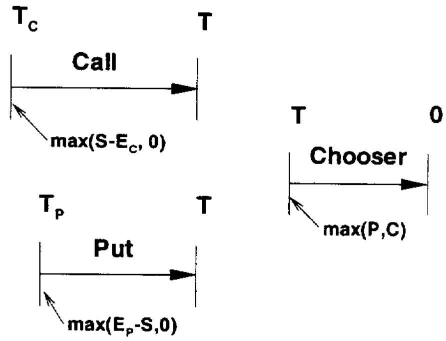


Fig. 2. Schematic of Chooser option. Put $P$ and call $C$ are solved over different periods $(T_C, T_P)$, with exercise prices $E_C, E_P$, for underlying asset price $S$. Then the maximum of the put and call values at each node gives the terminal payoff of the chooser, which is then solved over the life of the chooser.

The value of a chooser is determined by solving for a put over the period $T_P \to T$, and then solving for a call over the period $T_C \to T$. The payoff $U(s_i, v_i, t^* = T) = U_i^0$ for the chooser at node $i$ is then given by

$$U_i^0 = \max(P_i, C_i). \tag{39}$$

This provides the initial condition for equation (10), which is then solved over the life of the chooser option. This is illustrated in Fig. 2.

In this example, we will also be specifying discrete dollar dividends. Discrete dollar dividends are easily handled with an unstructured grid. If $t^+$ and $t^-$ represent the times just before and after the dividend dates (recall that $t = T - t^*$), then

$$U(s, v, t^+) = U(s - D^*, v, t^-), \tag{40}$$

where $D$ is the dividend payment and

$$D^* = \min(D, s). \tag{41}$$

Eq. (41) prevents the unrealistic phenomenon of dividend payments being larger than the asset price. The value of $U(s - D^*, v, t^-)$ is interpolated using linear interpolation on the triangular mesh.

We give an example for an American chooser option written on a European put and call. The European put and call have double knockout barriers, which are observed weekly. More formally, the knockout barriers are defined as

$$U(s, v, t^+) = \begin{cases} U(s, v, t^-) & \text{if } 80 \leqslant s \leqslant 110, \\ 0 & \text{otherwise,} \end{cases} \tag{42}$$

where $t^+, t^-$ are the times just after and just before application of the barrier. Note that Eq. (42) imposes a jump discontinuity on the solution after each barrier observation date. Barriers are used to reduce the cost of an option, which is desirable for purchasers of the option if they believe that the underlying asset is likely to trade only within a restricted range. The data for the European put and call are given in Table 6. The data for the chooser, with initial condition given by Eq. (39) is given in Table 7. No barriers are applied to the chooser option.

Table 6
Data for the stochastic volatility put and call, which are the basis for the chooser option. These are European options with discretely observed barriers

| | |
|---|---|
| $\sigma$ | 0.5 |
| $\rho$ | −0.5 |
| $\kappa$ | 0.2 |
| $\theta$ | 0.04 |
| $\lambda$ | 0.0 |
| $r$ | 0.05 |
| Time to expiry | 0.5 years |
| Exercise price: put | $100 |
| Exercise price: call | $90 |
| Dividend | $1.00 quarterly |
| Knockout barriers at | $80, $110 |
| Barriers observed | Weekly |
| Early exercise | No |

Table 7
Data for the stochastic volatility chooser. This is an American chooser (i.e., it can be exercised at any time)

| | |
|---|---|
| $\sigma$ | 0.5 |
| $\rho$ | −0.5 |
| $\kappa$ | 0.2 |
| $\theta$ | 0.04 |
| $\lambda$ | 0.0 |
| $r$ | 0.05 |
| Time to expiry | 1.0 years |
| Dividend | $1.00 quarterly |
| Early exercise | Yes |

The early exercise constraint is implemented using constraint switching, with the value of $U_i^*$ (Eq.(29)) given by

$$U_i^* = U_i^0, \tag{43}$$

where $U_i^0$ is given in Eq. (39).

These problems were solved on a $123 \times 76$ grid. For the European put and call, a fully implicit timestepping method was used. This is necessary to avoid spurious oscillations, as discussed in [30] when pricing discrete barrier options. Crank–Nicolson timestepping was used for the chooser computation.

Fig. 3 shows the results for the put and call at $t^* = T$ (see Fig. 2). This data is used for the initial condition for the chooser (Eq. (39)) and the American constraint (Eq. (43)).

Fig. 4 gives the results for the chooser option at the initial time ($t^* = 0$). For comparison, the results are also given for a chooser based on the same initial data, but without the American early exercise feature. Note the regions near $v = 0.04$, Asset Price =$ 95, where the American chooser has significantly more value than the European version. Grid and timestep reduction studies show that the discretization errors in the region of interest are < $ 0.01.

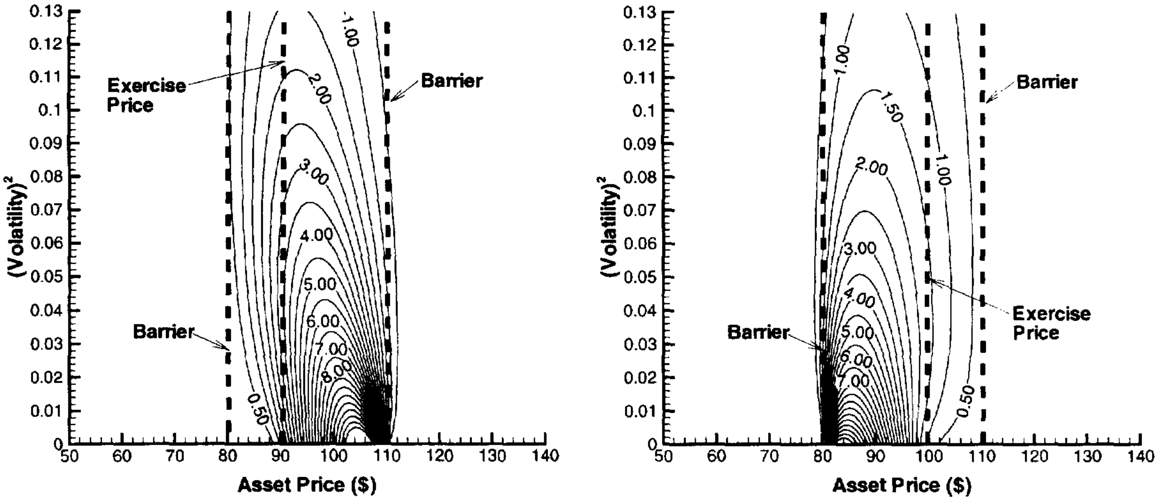


Fig. 3. Value of European put and call at $t^* = T$. The put and call have discrete double knockout barriers at \$80 and \$110. Left: call, right: put.

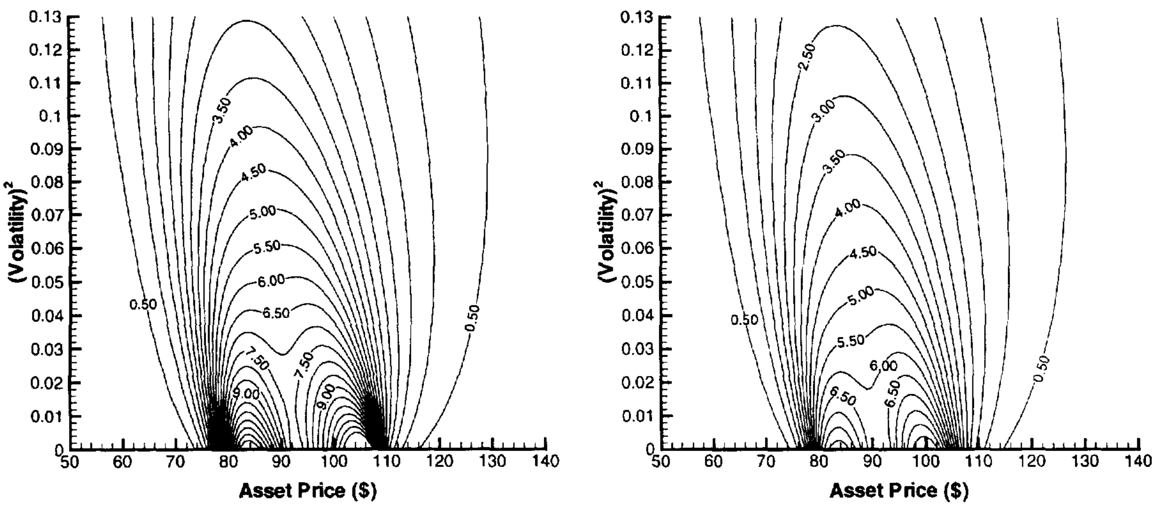


Fig. 4. Value of a chooser option, written on a European put and call at $t^* = 0$. The put and call have double knockout barriers at \$80 and \$110. Left: American chooser, right: European chooser.

The optimal early exercise regions (at $t^* = 0$ and $t^* = 0.5$) are shown in Fig. 5. These regions are determined from

$$U_i < U_i^* - \varepsilon, \quad \varepsilon \ll 1. \tag{44}$$

In these cases, the optimal early exercise regions are multiply connected, which causes no particular difficulty for the penalty method of satisfying the American constraint.

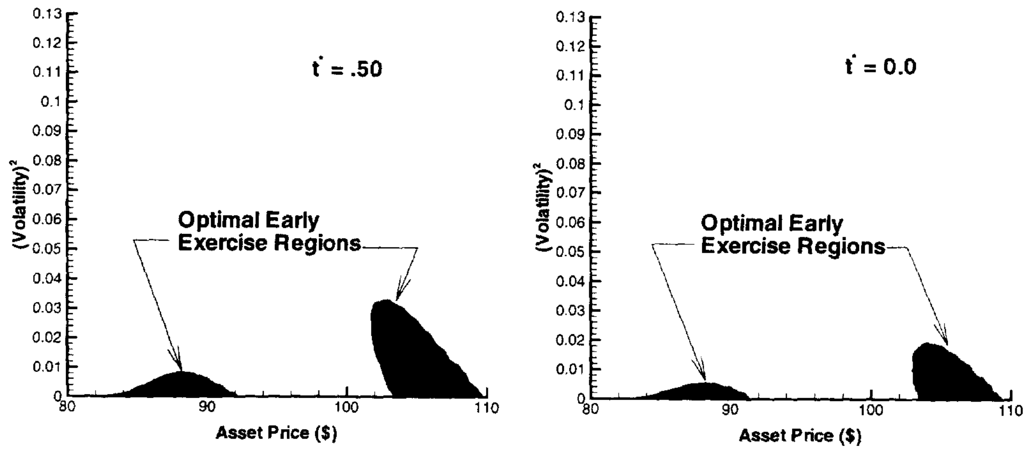


Fig. 5. Optimal early exercise regions for the American chooser option, left: $t^* = 0.5$; right: $t^* = 0.0$.

## 8. Conclusions

The American early exercise constraint for option pricing problems can be viewed as simply transforming the original linear convection diffusion equation into a PDE with a nonlinear penalty term. Since option pricing constraints are typically nonlinear, the resulting set of nonlinear discretized equations can be solved by approximate Newton iteration. This approach allows for the use of modern, robust methods for iterative solution of the Jacobian matrix.

There are various ways to impose the early exercise constraint. A smoothly differentiable penalty method was compared with a constraint switching technique. The constraint switching method does not have a continuous derivative at points where the constraint is switched. Somewhat surprisingly, the constraint switching method was superior to the smooth penalty technique.

The constraint switching method for computing American options was demonstrated on some option pricing problems based on a stochastic volatility model (which gives rise to a problem in two space-like dimensions). Even very complex American constraints (e.g. an American chooser written on discrete barrier options with discrete dividends) with multiply connected early exercise regions were easily handled.

The method used here to impose early exercise constraint is very straightforward to implement. Other types of constraints (e.g. callable convertible bonds) are easily modelled. As long as an efficient sparse matrix solution method is used, there are no restrictions on using this technique for higher dimensional problems. Note that the computationally intensive part of these computations, the solution of the sparse Jacobian, is completely decoupled from the details of any particular model, which permits the use of modern sparse matrix software.

Since most stochastic models of the underlying assets for option pricing will result in a convection–diffusion problem, and virtually any type of constraint can be forced using a suitable definition of the discrete source/sink term, this means that it is possible to construct a modular library for pricing a wide variety of options. This is because the basic discrete equations are formally identical for a large number of different types of options. Use of modern object-oriented approaches to software development thus permit the user to develop complex new pricing models simply by writing a small number of virtual functions.

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