Lecture 5: Volatility and Variance Swaps

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12 Spanning Generalized European Payoffs

As usual, we assume that European options with all possible strikes and expirations are traded. In the spirit of the paper by Carr and Madan (1998), we now show that any twice-differentiable payoff at time T may be statically hedged using a portfolio of European options expiring at time T.

From Breeden and Litzenberger (1978), we know that we may write the *pdf* of the stock price S_T at time T as

$$p(S_T, T; S_t, t) = \left.\frac{\partial^2 \tilde{C}(S_t, K, t, T)}{\partial K^2}\right|_{K=S_T} = \left.\frac{\partial^2 \tilde{P}(S_t, K, t, T)}{\partial K^2}\right|_{K=S_T}$$

where $\tilde{C}$ and $\tilde{P}$ represent undiscounted call and put prices respectively.

Then, the value of a claim with a generalized payoff $g(S_T)$ at time T is given by

$$\begin{aligned}
\mathbf{E}\left[g(S_T)|\, S_t\right] &= \int_0^\infty dK\, p(K, T; S_t, t)\, g(K) \\
&= \int_0^F dK\, \frac{\partial^2 \tilde{P}}{\partial K^2}\, g(K) + \int_F^\infty dK\, \frac{\partial^2 \tilde{C}}{\partial K^2}\, g(K)
\end{aligned}$$

where F represents the time-T forward price of the stock. Integrating by parts twice and using the put-call parity relation $\tilde{C}(K) - \tilde{P}(K) = F - K$ gives

$$\begin{aligned}
\mathbf{E}\left[g(S_T)|\, S_t\right] &= \left.\frac{\partial \tilde{P}}{\partial K}\, g(K)\right|_0^F - \int_0^F dK\, \frac{\partial \tilde{P}}{\partial K}\, g'(K) + \left.\frac{\partial \tilde{C}}{\partial K}\, g'(K)\right|_F^\infty - \int_F^\infty dK\, \frac{\partial \tilde{C}}{\partial K}\, g(K) \\
&= g(F) - \int_0^F dK\, \frac{\partial \tilde{P}}{\partial K}\, g'(K) - \int_F^\infty dK\, \frac{\partial \tilde{C}}{\partial K}\, g'(K) \\
&= g(F) - \left.\tilde{P}(K) g'(K)\right|_0^F + \int_0^F dK\, \tilde{P}(K)\, g''(K) \\
&\quad - \left.\tilde{C}(K) g'(K)\right|_F^\infty + \int_F^\infty dK\, \tilde{C}(K)\, g''(K) \\
&= g(F) + \int_0^F dK\, \tilde{P}(K)\, g''(K) + \int_F^\infty dK\, \tilde{C}(K)\, g''(K) \qquad (49)
\end{aligned}$$

By letting $t \to T$ in Equation 49, we see that any European-style twice-differentiable payoff may be replicated using a portfolio of European options with strikes from 0 to ∞ with the weight of each option equal to the second derivative of the payoff at the strike price of the option. This portfolio of European options is a static hedge because the weight of an option with a particular strike depends only on the strike price and the form of the payoff function and not on time or the level of the stock price. Note further that Equation 49 is *completely model-independent*.

Example: European Options

In fact, using Dirac delta-functions, we can extend the above result to payoffs which are not twice-differentiable. Consider for example the portfolio of options required to hedge a single call option with payoff $(S_T - L)^+$. In this case $g''(K) = \delta(K - L)$ and Equation 49 gives

$$\begin{aligned}
\mathbf{E}\left[(S_T - L)^+\right] &= (F - L)^+ + \int_0^F dK\, \tilde{P}(K)\, \delta(K - L) + \int_F^\infty dK\, \tilde{C}(K)\, \delta(K - L) \\
&= \begin{cases} (F - L) + \tilde{P}(L) & \text{if} \quad L < F \\ \tilde{C}(L) & \text{if} \quad L \geq F \end{cases} \\
&= \tilde{C}(L)
\end{aligned}$$

with the last step following from put-call parity as before. In other words, the replicating portfolio for a European option is just the option itself.

Example: Amortizing Options

A useful variation on the payoff of the standard European option is given by the amortizing option with strike L with payoff

$$g(S_T) = \frac{(S_T - L)^+}{S_T}$$

Such options look particularly attractive when the volatility of the underlying stock is very high and the price of a standard European option is prohibitive . The payoff is effectively that of a European option whose notional amount declines as the option goes in-the-money. Then,

$$g''(K) = -\frac{2L}{{S_T}^3}\theta(S_T - L) + \frac{\delta(S_T - L)}{S_T}$$

Without loss of generality (but to make things easier), suppose $L > F$. Then substituting into Equation 49 gives

$$\begin{aligned}\mathbf{E}\left[\frac{(S_T - L)^+}{S_T}\right] &= \int_F^\infty dK\, \tilde{C}(K)\, g''(K) \\ &= \frac{C(L)}{L} - 2L \int_L^\infty \frac{dK}{K^3}\, \tilde{C}(K)\end{aligned}$$

and we see that an Amortizing call option struck at L is equivalent to a European call option struck at L minus an infinite strip of European call options with strikes from L to ∞.

12.1 The Log Contract

Now consider a contract whose payoff at time T is $\log(\frac{S_T}{F})$. Then $g''(K) = -\frac{1}{{S_T}^2}$ and it follows from Equation 49 that

$$\mathbf{E}\left[\log\left(\frac{S_T}{F}\right)\right] = -\int_0^F \frac{dK}{K^2}\, \tilde{P}(K) - \int_F^\infty \frac{dK}{K^2}\, \tilde{C}(K)$$

Rewriting this equation in terms of the log-strike variable $y \equiv \log\left(\frac{K}{F}\right)$, we get the promising-looking expression

$$\mathbf{E}\left[\log\left(\frac{S_T}{F}\right)\right] = -\int_{-\infty}^0 dy\, p(y) - \int_0^\infty dy\, c(y) \qquad (50)$$

with $c(y) \equiv \frac{\tilde{C}(Fe^y)}{Fe^y}$ and $p(y) \equiv \frac{\tilde{P}(Fe^y)}{Fe^y}$ representing option prices expressed in terms of percentage of the strike price.

13 Variance and Volatility Swaps

We now revert to our usual assumption of zero interest rates and dividends. In this case, $F = S_0$ and applying Itô's Lemma, path-by-path

$$\begin{aligned}\log\left(\frac{S_T}{F}\right) &= \log\left(\frac{S_T}{S_0}\right) \\ &= \int_0^T d\log(S_t) \\ &= \int_0^T \frac{dS_t}{S_t} - \int_0^T \frac{{\sigma_{S_t}}^2}{2} dt \qquad (51)\end{aligned}$$

The second term on the RHS of Equation 51 is immediately recognizable as half the total variance W_T over the period $\{0, T\}$. The first term on the RHS represents the payoff of a hedging strategy which involves maintaining a constant dollar amount in stock (if the stock price increases, sell stock; if the stock price decreases, buy stock so as to maintain a constant dollar value of stock). Since the log payoff on the LHS can be hedged using a portfolio of European options as noted earlier, it follows that the total variance W_T may be replicated in a completely model-independent way so long as the stock price process is a diffusion. In particular, volatility may be stochastic or deterministic and Equation 51 still applies.

13.1 Variance Swaps

Although variance and volatility swaps are relatively recent innovations, there is already a significant literature describing these contracts and the practicalities of hedging them including articles by Chriss and Morokoff (1999) and Demeterfi, Derman, Kamal, and Zou (1999).

In fact, a variance swap is not really a swap at all but a forward contract on the realized annualized variance. The payoff at time T is

$$N \times A \times \left\{ \frac{1}{N} \sum_{i=1}^{N} \left\{ \log \left(\frac{S_i}{S_{i-1}} \right) \right\}^2 - \left\{ \frac{1}{N} \log \left(\frac{S_N}{S_0} \right) \right\}^2 \right\} - N \times K_{var}$$

where N is the notional amount of the swap, A is the annualization factor and K_{var} is the strike price. Annualized variance may or may not be defined as mean-adjusted in practice so the corresponding drift term in the above payoff may or may not appear.

From a theoretical perspective, the beauty of a variance swap is that it may be replicated perfectly assuming a diffusion process for the stock price as shown in the previous section. From a practical perspective, market operators may express views on volatility using variance swaps without having to delta hedge.

Variance swaps took off as a product in the aftermath of the LTCM meltdown in late 1998 when implied stock index volatility levels rose to unprecedented levels. Hedge funds took advantage of this by paying variance in swaps (selling the realized volatility at high implied levels). The key to their willingness to pay on a variance swap rather than sell options was that a variance swap is a pure play on realized volatility – no labor-intensive delta hedging or other path dependency is involved. Dealers were happy to buy vega at these high levels because they were structurally short vega (in the aggregate) through sales of guaranteed equity-linked investments to retail investors and were getting badly hurt by high implied volatility levels.

13.2 Variance Swaps in the Heston Model

Recall that in the Heston model, instantaneous variance v follows the process:

$$dv(t) = -\lambda(v(t) - \bar{v})dt + \eta \sqrt{v(t)}\, dZ$$

It follows that the expectation of the total variance W_T is given by

$$\begin{aligned}
\mathbf{E}\left[W_T\right] &= \mathbf{E}\left[\int_0^T v_t\, dt\right] \\
&= \int_0^T \hat{v}_t\, dt \\
&= \frac{1 - e^{-\lambda T}}{\lambda}\,(v - \bar{v}) \,+\, \bar{v}T
\end{aligned}$$

The expected annualized variance V_T is given by

$$V_T \,\equiv\, \frac{1}{T}\mathbf{E}\left[W_T\right] \,=\, \frac{1 - e^{-\lambda T}}{\lambda T}\,(v - \bar{v}) \,+\, \bar{v}$$

We see that the expected variance in the Heston model depends only on v, $\bar{v}$ and λ. It does not depend on the volatility of volatility η. Since the value of a variance swap depends only on the prices of European options, it follows that a variance swap would be priced identically by both Heston and our local volatility approximation to Heston.

13.3 Dependence on Skew and Curvature

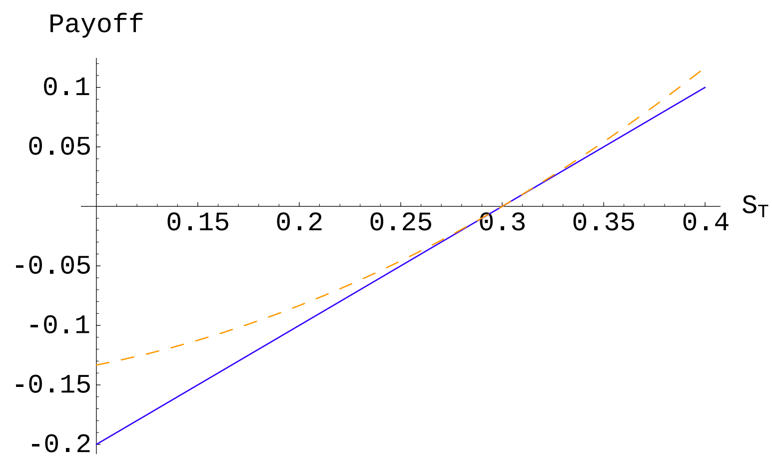
We know that the implied volatility of an at-the-money forward option in the Heston model is lower than the square root of the expected variance (just think of the shape of the implied distribution of the final stock price in Heston). In practice, we start with a strip of European options of a given expiration and we would like to know how we should expect the price of a variance swap to relate to the at-the-money-forward implied volatility, the volatility skew and the volatility curvature (smile).

13.4 Volatility Swaps

Realized volatility Σ_T is the square root of realized variance V_T and we know that the expectation of the square root of a random variable is less than (or equal to) the square root of the expectation. The difference between $\sqrt{V_T}$ and Σ_T is known as the *convexity adjustment*.

Figure 1 shows how the payoff of a variance swap compares with the payoff of a volatility swap.

Figure 1: Payoff of a variance swap (dashed line) and volatility swap (solid line) as a function of realized volatility. Both swaps are stuck at 30% volatility. Σ_T



Intuitively, the magnitude of the convexity adjustment must depend on the volatility of realized volatility. Note that volatility does not have to be stochastic for realized volatility to be volatile; realized volatility Σ_T varies according to the path of the stock price even in a local volatility model.

In fact, there is no replicating portfolio for a volatility swap and the magnitude of the convexity adjustment is highly model-dependent. As a consequence, market makers' prices for volatility swaps are both wide (in terms of bid-offer) and widely dispersed. As in the case of live-out options, price takers such as hedge funds may occasionally have the luxury of being able to cross the bid-offer – that is, buy on one dealer's offer and sell on the other dealer's bid.

Assuming no jumps however (Matytsin (1999) discusses the impact of jumps), the convexity adjustment is not so model dependent. We will now compute it for the Heston model.

13.5 Convexity Adjustment in the Heston Model

To compute the expectation of volatility in the Heston model we use the following trick:

$$\mathbf{E}\left[\sqrt{V_T}\right] \;=\; \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1-\mathbf{E}\left[e^{-\psi V}\right]}{\psi^{3/2}}\,d\psi \tag{52}$$

From Cox, Ingersoll, and Ross (1985), the Laplace transform of the total variance $W_T = \int_0^\infty v_t\,dt$ is given by

$$\mathbf{E}\left[e^{-\psi W_T}\right] \;=\; A\,e^{-\psi v B}$$

where

$$\begin{aligned} A &= \left\{\frac{2\phi\,e^{(\phi+\lambda)T/2}}{(\phi+\lambda)(e^{\phi T}-1)+2\phi}\right\}^{2\lambda\bar{v}/\eta^2} \\ B &= \frac{2\,(e^{\phi T}-1)}{(\phi+\lambda)(e^{\phi T}-1)+2\phi} \end{aligned}$$

with $\phi = \sqrt{\lambda^2 + 2\psi\eta^2}$.

With some tedious algebra, we may verify that

$$\begin{aligned} \mathbf{E}\left[W_T\right] &= -\frac{\partial}{\partial\psi}\mathbf{E}\left[e^{-\psi W_T}\right]\Bigg|_{\psi=0} \\ &= \frac{1-e^{-\lambda T}}{\lambda}\,(v-\bar{v}) \;+\; \bar{v}T \end{aligned}$$

as we found earlier in Section 13.2.

Computing the integral in Equation 52 numerically using the usual parameters from Homework 2 ($v = 0.04, \bar{v} = 0.04, \lambda = 10.0, \eta = 1.0$), we get the graph of the convexity adjustment as a function of time to expiration shown in Figure 2.

Using Bakshi, Cao and Chen parameters ($v = 0.04, \bar{v} = 0.04, \lambda = 1.15, \eta = 0.39$), we get the graph of the convexity adjustment as a function of time to expiration shown in Figure 3.

Figure 2: Annualized Heston convexity adjustment as a function of T with parameters from Homework 2.

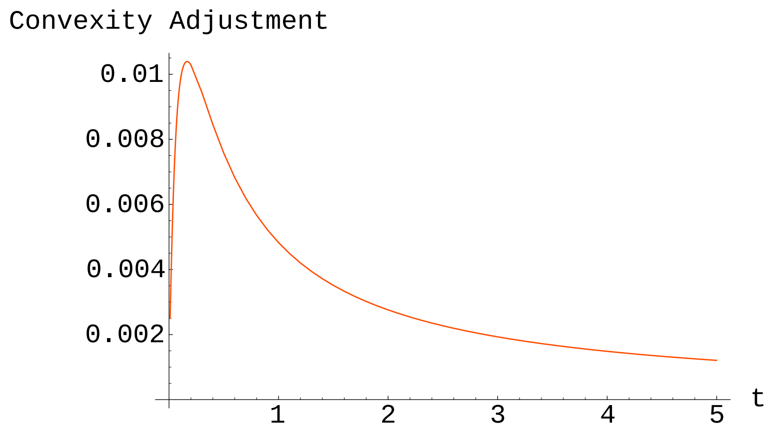
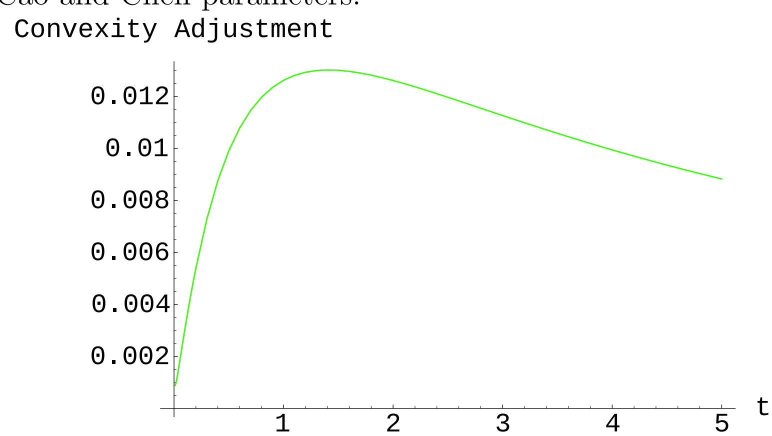


Figure 3: Annualized Heston convexity adjustment as a function of T with Bakshi, Cao and Chen parameters.



To get intuition for what is going on here, compute the limit of the variance of V_T as $T \to \infty$ with $v = \bar{v}$ using

$$\begin{aligned}
\mathrm{var}\left[\mathrm{W_T}\right] &= \mathbf{E}\left[{W_T}^2\right] - \left\{\mathbf{E}\left[W_T\right]\right\}^2 \\
&= \frac{\partial^2}{\partial\psi^2}\mathbf{E}\left[e^{-\psi W_T}\right]\Big|_{\psi=0} - \left\{\frac{\partial}{\partial\psi}\mathbf{E}\left[e^{-\psi W_T}\right]\Big|_{\psi=0}\right\}^2 \\
&= \bar{v}T\frac{\eta^2}{\lambda^2} + O(T^0)
\end{aligned}$$

Then, as $T \to \infty$, the standard deviation of *annualized* variance has the leading order behavior $\frac{\bar{v}}{\sqrt{T}} \frac{\eta}{\lambda}$. The convexity adjustment should be of the order of the standard deviation of annualized volatility over the life of the contract. From the last result, we expect this to scale as $\frac{\eta}{\lambda}$. Comparing Bakshi, Cao and Chen (BCC) parameters with Homework 2 parameters, we deduce that the convexity adjustment should be roughly 3.39 times greater for BCC parameters and that's what we see in the graphs.

14 Epilog

I hope that this series of lectures has given students an insight into how financial mathematics is used in the derivatives industry. It should be apparent that modelling is an art in the true sense of the word – not a science, although when it comes to implementing the chosen solution approach, science becomes necessary. We have seen several examples of claims which may be priced differently under different modelling assumptions even though the models generate identical prices for European options. The importance of lateral thinking outside the framework of a given model cannot be overemphasized. For example, what is the impact of jumps? of stochastic volatility? of skew? and so on. Finally, intuition together with the ability to express this intuition clearly is ultimately what counts; without this intuition, financial mathematics is useless in practice given that the ultimate users of models are overwhelmingly non-mathematicians.

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