Lecture 4: Barrier Options

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11 Barrier Options

Unlike previous sections where every problem presented came with a satisfactory solution, this section generally confines itself to presenting intuition for problems without providing convincing solutions. That's because convincing solutions are thin on the ground. In fact, prices quoted for certain kinds of barrier option can vary so much between dealers that customers can sometimes cross the bid-offer (that is, buy on one dealer's offer and sell on another dealer's bid for a profit. So there is still plenty of scope for the ambitious modeler.

Barrier options are important building blocks for structured products but their valuation can be highly model-dependent. Consequently much has been written on the subject notably by Taleb (1996), Wilmott (1998) and Carr and Chou (1997).

By considering two limiting cases, we will see that barrier option values are not always so model-dependent. Developing intuition is therefore particularly important not only to be able to estimate the value of a barrier option but also to know whether the output of a model should be trusted or not.

As usual, we suppose that European options of all strikes and expirations are traded in the market and our objective is to price barrier options consistently with these European option prices.

11.1 Definitions

A *knock-out* option is an option which becomes worthless when a pre-specified "barrier" level is reached.

A *live-out* option is a special case of a knock-out option which is significantly in-the-money when it knocks out.

A *knock-in* option is an option which can only be exercised if a barrier level is reached prior to exercise. Obviously, a knock-in option is just a portfolio of short a knock-out option and long a European.

An amount of money paid to a barrier option buyer if the barrier is hit is termed a *rebate*. This rebate may be paid when the barrier is hit or at expiration.

11.2 Limiting Cases

Limit Orders

Suppose we sell a knock-out call option with barrier B equal to the strike price K below the current stock price S. Suppose further that we hedge this position buy buying one stock per option and we charge $S_0 - K$ as the premium. If interest rates and dividends are zero, it is clear that this hedge is perfect. To see this, suppose first that the barrier is never hit: the buyer of the knock-out call option exercises the option and we deliver the stock. Net proceeds are $-(S_T - K) + (S_0 - K) + (S_T - S_0) = 0$. On the other hand, if the barrier is hit, we lose $S_0 - K$ on our purchase of stock which is perfectly offset by the premium we charged.

In this special case, a knock-out option has no optionality whatsoever. Delta is one, gamma is zero and vega is zero. The result is completely model-independent; the only requirement is to have no carry on the stock for this construction to work.

Now consider what this portfolio really is. So long as the stock price remains above the barrier level, we are net flat. When the barrier is hit, the option knocks out and we are left long of the stock we bought to hedge. This is exactly the position we would be in if the option buyer had left us a stop-loss order to sell stock if the price ever reached the barrier level B. There is however a big difference between the two contracts – a barrier option like this guarantees execution at the barrier level but a conventional stop-loss order would get filled at the earliest opportunity after the barrier is hit (usually a bit below the barrier). If we could really trade continuously as models conventionally assume, there would be no difference between the two contracts. In the real world, a knock-out option needs to be priced more highly than the model price to compensate for the risk of the stock price gapping through the barrier level. Practitioners compensate for gap risk when pricing options by moving the barrier by some amount related to the expected gap in the stock price when the barrier is hit.

In summary, in this special case when $K = B < S_0$, the price of a knock-out call is given by the difference $S_0 - K$ between the current stock price and the strike price plus a bit to compensate for gap risk.

Now, if the strike price K and the barrier level B are not equal but not so far apart with $B \leq K \leq S_0$, it is natural to expect that neither gamma nor vega would be very high relative to the European option with the same strike K. Nor would we expect the price of such a knock-out option to be very sensitive to the model used to value it (assuming of course that this model prices consistently with all European options). Investigation shows that this is indeed the case.

European Capped Calls

The next limiting case we consider is that of the European capped call. This option is a call struck at K with barrier $B > S_0$ such that if the stock price reaches B before expiration, the option expires and pays out intrinsic of $B - K$.

If the barrier is far away from the current stock price S_0, the price of such an option cannot be very different from the price of a conventional European option. To see this, consider a portfolio consisting of long a European option struck at K (not too different from S_0) and short the capped call. If the barrier is not hit, this portfolio pays nothing. If the barrier is hit, the portfolio will be long a European option and short cash in the amount of the intrinsic value $B-K$. The time value of this European option cannot be very high because, by assumption, $B \gg S_0$ and moreover, the barrier is most likely to be hit close to expiration. Since the value of the capped call must be close to the value of a conventional European call, the value of the capped call cannot be very model-dependent and should be well approximated by a model using Black-Scholes assumptions (no volatility skew) and the implied volatility of the corresponding European option.

With this understanding of the pricing of capped calls, we are in a position to develop intuition for the pricing of live-out calls. To get a live-out call from a capped call, we need only omit the rebate at the barrier. We would then have a call option struck at K which goes deep-in-the-money as the stock price approaches the barrier $B \gg K$ and knocks-out when the stock price reaches B (with no rebate). So to get intuition for the pricing and hedging of live-out options, we need only study the pricing and hedging of the rebate (or *one-touch* option).

11.3 The Reflection Principle

We suppose that the stock price is driven by a constant volatility stochastic process with zero log-drift. That is

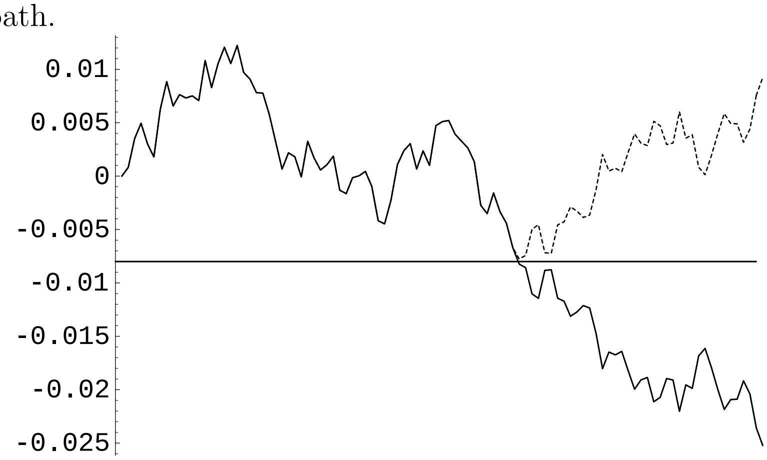
$$dx = \sigma dZ \tag{48}$$

with $x \equiv \log\left(\frac{S}{K}\right)$.

In this special case, there is a very simple relationship between the price of a European binary option struck at B and the value of the one-touch option struck at B.

Consider the realization of the zero log-drift stochastic process (48) given by the solid line in Figure 1. From the symmetry of the problem, the dashed path has the same probability of being realized as the original solid path. We deduce that the probability of hitting the barrier B is exactly twice the probability of ending up below the barrier at expiration. Putting this another way, the value of a one-touch option is precisely twice the value of a European binary put.

Figure 1: A realization of the zero log-drift stochastic process and the reflected path.

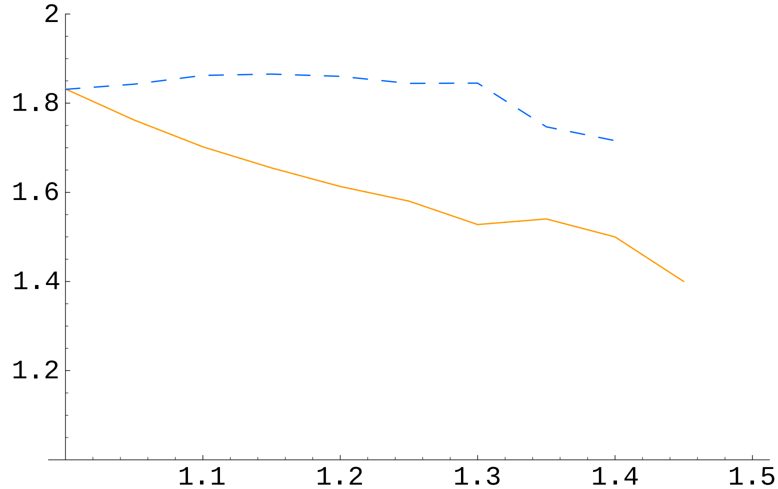


To make this result appear plausible note that an at-the-money barrier has 100% chance of getting hit but there is only 50% chance of ending up below the barrier at expiration in this special case. Guessing at a generalization, we might suppose that the ratio of the fair value of a one-touch option should be given by $B(S_0)^{-1}$ where $B(K)$ represents the value of a European binary put struck at K.

For the model and parameters we chose in Homework 4 ($v = 0.04, \bar{v} = 0.04, \lambda = 10, \eta = 1, \rho = -1$), $B(S_0) = 0.54614$ and the ratio of the one-touch price to the European binary price should be around $B(S_0)^{-1} = 1.831$

if our guess is correct. Figure 2 shows how this ratio is, as Taleb (1996) emphasizes, very sensitive to modelling assumptions. Although our guess was pretty accurate for the local volatility case, it is very inaccurate in the stochastic volatility case.

Figure 2: The ratio of the value of a one-touch call to the value of a European binary call under stochastic volatility and local volatility assumptions as a function of strike. The solid line is stochastic volatility and the dashed line is local volatility.



For comparison, consider the effect of modelling assumptions on the price of a European binary call. Figure 3 shows that modelling assumptions have no effect – the price of a European binary is independent of modelling assumptions and depends only on the given prices of conventional European options (being a limit of a call spread in this case).

Finally, we graph the value of the one-touch option as a function of strike under stochastic volatility and local volatility assumptions in Figure 4.

11.4 The Lookback Hedging Argument

A closely-related useful hedging argument originally given by Goldman, Sosin, and Gatto (1979) is used to estimate the price and hedge portfolio of a lookback option. For our purposes, we will define a lookback call to be an option that pays $(\tilde{S} - K)^+$ at expiration where $\tilde{S}$ is the maximum stock price over the life of the option and K is the strike price.

Figure 3: The value of a European binary call under stochastic volatility and local volatility assumptions as a function of strike. The solid line is stochastic volatility and the dashed line is local volatility.

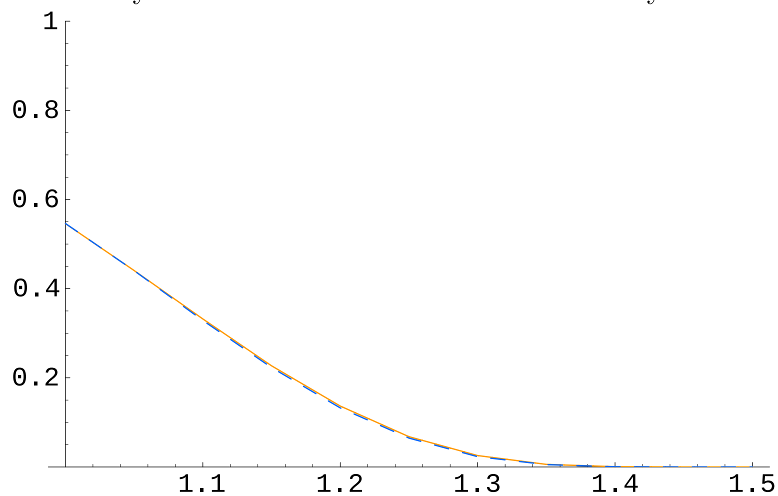
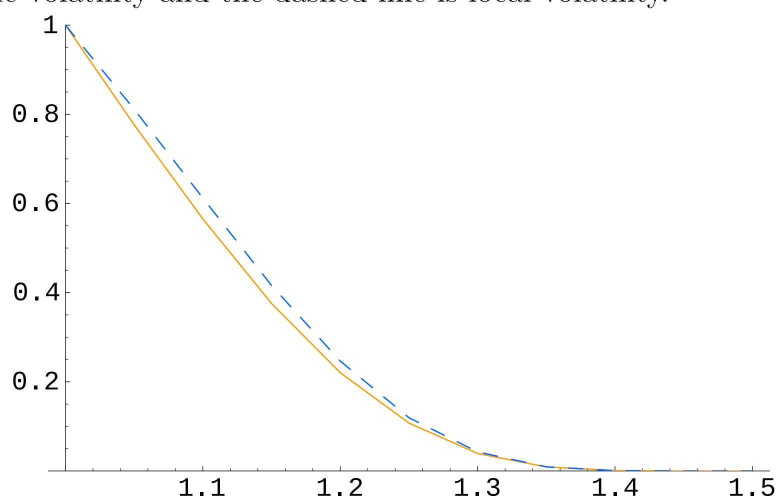


Figure 4: The value of a one-touch call under stochastic volatility and local volatility assumptions as a function of barrier level. The solid line is stochastic volatility and the dashed line is local volatility.



Once again, assuming zero log-drift and constant volatility, suppose we hedge a short position in this lookback call by holding two conventional European options struck at K. If the stock price never reaches K, both the lookback and the European option expire worthless. If and when the stock price does reach K and increases by some small increment ΔK, the value of the lookback option must increase by ΔK (since $K + \Delta K$ is now the new maximum). The new lookback option must pay $\Delta K + (\tilde{S} - (K + \Delta K))^+$ – the payoff of another lookback option with a higher strike price plus a fixed cashflow ΔK.

Assuming we were right to hedge with two calls in the first place, the new hedge portfolio must be two calls struck at $K + \Delta K$. So we must rebalance our hedge portfolio by selling two calls struck at K and buying two calls struck at $K + \Delta K$. The profit generated by rebalancing is

$$\begin{aligned} 2\,C(K+\Delta K, K) - 2\,C(K+\Delta K, K+\Delta K) &\approx -2\left.\frac{\partial C}{\partial K}\right|_{S=K} \Delta K \\ &= 2\,N(d_2)|_{S=K} \\ &= \Delta K \end{aligned}$$

using the fact that $N(d_2)|_{S=K} = \frac{1}{2}$ when the log-drift is zero.

The profit generated by rebalancing is exactly what is needed to generate the required payoff of the lookback option and our hedge is perfect.

Now reconsider the value of a one-touch call option struck at B. It is the probability that the maximum stock price is greater than B. We can generate this payoff by taking the limit of a lookback call spread as the difference between the strikes gets very small. Because a lookback call has the same value as two European calls, a lookback call spread must have the same value as two European call spreads. Put another way, a one-touch option is worth two European binary options when the log-drift is zero.

11.5 Put-Call Symmetry

We now assume zero interest rates and dividends and constant volatility again (as opposed to zero log-drift). In this case, by inspection of the Black-Scholes formula, we have:

$$C\left(\frac{B^2}{S}, K\right) = \frac{K}{S} P\left(S, \frac{B^2}{K}\right)$$

From one of the many references containing closed-form formulae for knock-out options, we may deduce that

$$\begin{aligned} DO\,(S,K,B) &= C\,(S,K) - \frac{S}{B}\,C\left(\frac{B^2}{S},K\right) \\ &= C\,(S,K) - \frac{K}{B}\,P\left(S,\frac{B^2}{K}\right) \end{aligned}$$

where $DO(.)$ represents the value of a down-and-out call.

By letting $S = B$ in the above formula, we see that $DO\,(B,K,B) = 0$ as we would expect. So, in this special case, there is a static hedge for a down-and-out call option which consists of long a European call with the same strike and short $\frac{K}{B}$ European puts struck at the reflection of the log-strike in the log-barrier ($K' = \frac{B^2}{K}$).

The reason this static hedge works is that the value of the call we are long always exactly offsets the value of the put we are short when the stock price reaches the barrier B.

A special case of this special case is when $B = K$. In this case, we have

$$DO\,(S,K,K) \;=\; C\,(S,K) -\; P\,(S,K) \;=\; S - K$$

and we see again that there is no optionality – the down-and-out call option is worth only intrinsic value and has the same payoff as a portfolio of long the stock and short K bonds as we already argued in Section 11.2.

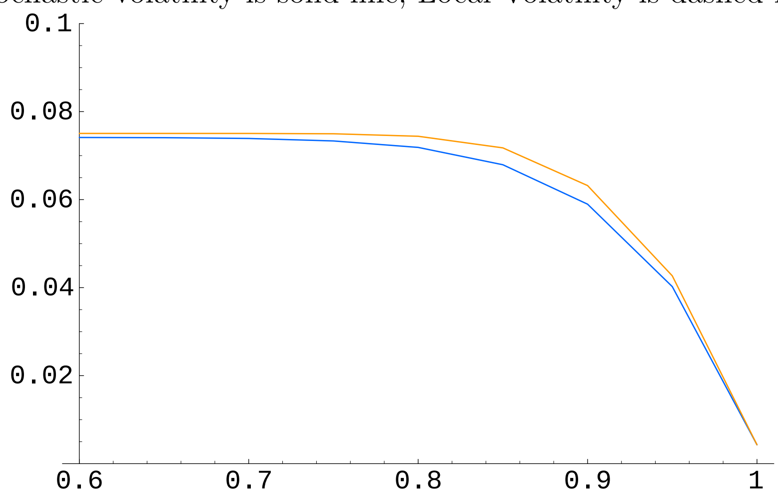
11.6 Static Hedging

We can generalize the above procedure to other cases where interest rates, dividends and volatility have arbitrary structure. Although there is no exact static hedge in the general case, we can construct a portfolio which has rather small payoffs under all reasonable scenarios. A sophisticated version of this procedure known as the Lagrangian Uncertain Volatility Model is described by Avellaneda, Levy, and Parás (1995). In this model, volatility is bounded but uncertain; volatility is assumed to be high when the portfolio is short gamma and low when the portfolio is long gamma (worst case). Thus, different prices are generated depending on whether an option position is long or short (a bid-offer spread is generated). By minimizing the bid-offer spread of a given portfolio of exotic options (such as barrier options) and European options with respect to the weights of the European options, we can determine an optimal hedge and the minimal bid-offer spread that would be required to guarantee profitability assuming that volatility does indeed remain within the assumed bounds.

11.7 Qualitative Discussion

From the above, we would guess that the pricing of out-of-the-money knock-out options would not be very model-dependent. This guess is supported by the graphs in Figures 5 and 6.

Figure 5: Values of knockout call options struck at 1 as a function of barrier level. Stochastic volatility is solid line; Local Volatility is dashed line.



On the other, given the sensitivity of the one-touch to modelling assumptions and the insensitivity of the capped call, we would expect that live-out values would be sensitive to modelling assumptions. This guess is supported by the graph in Figure 7.

The stochastic volatility price of the live-out call is always above the local volatility price of the same option with our parameters. This is a reflection of our earlier observation that the value of the one-touch under stochastic volatility is strictly lower than the value of the same option under local volatility assumptions with our parameters. Note that the difference in valuation between the two modelling assumptions can be very substantial.

Figure 6: Values of knockout call options struck at 0.9 as a function of barrier level. Stochastic volatility is solid line; Local Volatility is dashed line.

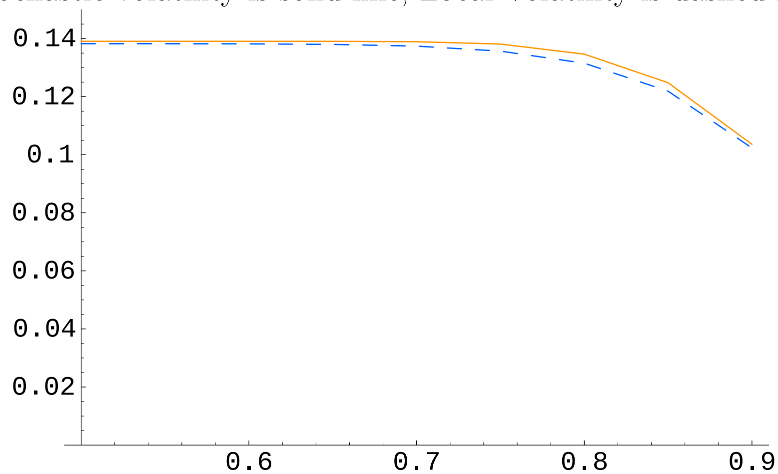
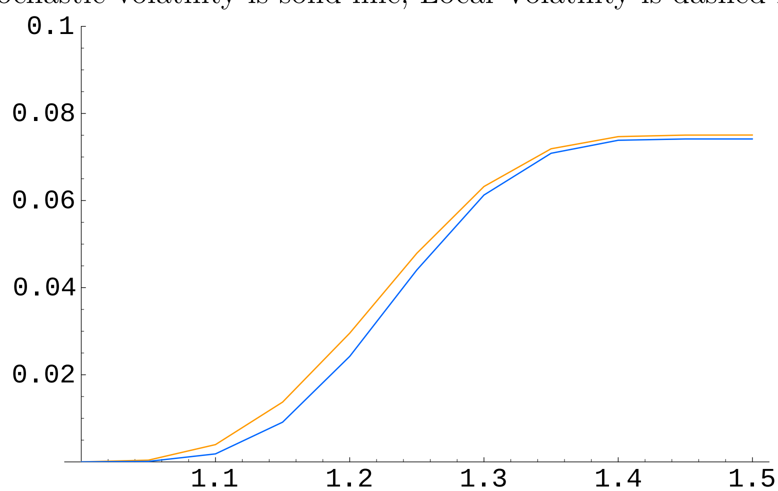


Figure 7: Values of live-out call options struck at 1 as a function of barrier level. Stochastic volatility is solid line; Local Volatility is dashed line.



Adjusting for Discrete Barriers

A practical point that is worth noting is that the discreteness effect for barrier options is very significant. Often barrier option contracts specify that the barrier is only to be monitored at the market close. How can we estimate the magnitude of the effect of this on the value of a barrier option?

To answer this question, we apply the lookback hedging argument.

Consider the day on which the stock price is first over the barrier level at the market close. It is highly likely that the stock price was over this level intra-day prior to the close. We approximate the value of the discretely monitored barrier option by the value of a continuously monitored barrier option whose barrier level is adjusted by the average difference between the intraday high and the close (which must by assumption be greater than the previous close).

We may compute the expected difference between the highest intra-day stock price $\tilde{S}$ and the stock price at the market close S_1, conditional on the close exceeding the previous day's close S_0 as follows:

$$\begin{aligned}\mathbf{E}\left[\tilde{S} - S_1 \,|S_1 > S_0\right] &= \mathbf{E}\left[\tilde{S} - S_0 - (S_1 - S_0)^+ \,|S_1 > S_0\right] \\ &= \mathbf{E}\left[\tilde{S} - S_0 \,|S_1 > S_0\right] - C(S_0)\end{aligned}$$

where $C(S_0)$ is the value of a European option priced at t_0 and expiring at t_1. Assuming the monitoring interval $t_1 - t_0$ to be small, by symmetry we must have:

$$\mathbf{E}\left[\tilde{S} - S_0 \,|S_1 > S_0\right] \approx \mathbf{E}\left[\tilde{S} - S_1 \,|S_1 \leq S_0\right]$$

Then

$$\begin{aligned}\mathbf{E}\left[\tilde{S} - S_1 \,|S_1 > S_0\right] &\approx \frac{1}{2}\left\{\mathbf{E}\left[\tilde{S} - S_1 \,|S_1 > S_0\right] + \mathbf{E}\left[\tilde{S} - S_1 \,|S_1 \leq S_0\right] - C(S_0)\right\} \\ &\approx \frac{1}{2}\left\{2\,\mathbf{E}\left[\tilde{S} - S_1\right] - C(S_0)\right\} \\ &\approx \frac{3}{2}C(S_0) \\ &\approx \frac{3}{2}\frac{\sigma\sqrt{\Delta t}}{\sqrt{2\pi}}\end{aligned}$$

where we have used the fact from Section 11.4 that a lookback option is worth approximately twice a European option and also that an at-the-money European option expiring in time Δt is worth roughly $\sigma\sqrt{\Delta t}/\sqrt{2\pi}$.

The value of a barrier option whose barrier is monitored at an interval Δt is therefore given approximately by the value of a continuously monitored barrier option whose barrier is offset by an amount $0.5984\,\sigma\sqrt{\Delta t}$. This may significantly affect the price of a barrier option. For example, with $\sigma = 0.32$ and daily monitoring ($\sqrt{\Delta T} \approx 1/16$), the adjustment would be around $\frac{0.32\times 0.6}{16} = .012$ (1.2% of the barrier level).

Broadie, Glasserman, and Kou (1997) show using a more careful argument that the appropriate correction is in fact $\beta\sigma\sqrt{\Delta T}$ where $\beta \approx 0.5826$.

11.8 Some Applications of Barrier Options

Ladders

Consider a strip of capped calls with strikes B_i strictly increasing and greater than the initial stock price S_0. The cap of the option with strike B_i is B_{i+1} so a rebate of $B_{i+1} - B_i$ is paid when the barrier at B_{i+1} is hit. The buyer of such an option locks-in his gain each time a barrier is crossed. This gain is not lost if the stock price subsequently falls. Not surprisingly, this structure is very popular with retail investors. In the limit where the caps are very close to the strikes, a ladder approximates a lookback option (every time the stock price increases, the gain is locked in) and the value of the ladder would be approximately twice the value of a European option. Typically though, barriers would be every 10% or so and the value of the ladder would be around 1.5 times the value of the corresponding European option.

Ranges

Another popular investment is one that pays a high coupon for each day that the stock price remains within a range but ceases paying a coupon as soon as one of the boundaries is hit. This is a just a one-touch double barrier construction.

11.9 Conclusion

Barrier option values can be very sensitive to modelling assumptions and prices must be adjusted to take this into account. Nevertheless, by understanding limiting cases which are well understood, we can gain a good qualitative understanding of the appropriate valuation and hedge portfolio for any given barrier option. Market practitioners are often reluctant to quote on any barrier option given the potential valuation uncertainty and the hedging complexity. What we have shown is that this reluctance is not always justified – sometimes a barrier option is much less risky and easier to price than its European equivalent.

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