Lecture 1: Stochastic Volatility and Local Volatility

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Abstract

In the course of these lectures, I will attempt to show by specific example that the financial engineer must carefully consider the dynamics of underlying variables when modelling contingent claims in addition to the requirement to match known market prices of other claims. In the process, I hope to help develop intuition and give some flavor of the types of problem faced in practical applications of financial theory. Finally, for those who end up specializing in the structuring, pricing and trading of exotic derivatives, the specific techniques introduced should prove useful.

*I am grateful to Peter Friz for carefully reading these notes, providing corrections and suggesting useful improvements.

1 Stochastic Volatility

1.1 Motivation

That it might make sense to model volatility as a random variable should be clear to the most casual observer of equity markets. To be convinced, you only need to remember the stock market crash of October 1987. It's not immediately obvious though what the benefits of making such a modelling choice might be given the success of the Black-Scholes model in parsimoniously describing market options prices.

Stochastic volatility models are useful because they explain in a self-consistent way why it is that options with different strikes and expirations have different Black-Scholes implied volatilities ("implied volatilities" from now on) – the "volatility smile". In particular, traders who use the Black-Scholes model to hedge keep having to change the volatility assumption in order to match market prices. Their hedge ratios change accordingly in an uncontrolled way. More interestingly for us, the prices of exotic options given by models based on Black-Scholes assumptions can be wildly wrong and dealers in such options are motivated to find models which can take the volatility smile into account when pricing these.

From Figure 1, we see that large moves follow large moves and small moves follow small moves (so called "volatility clustering"). From Figures 2 and 3 (which shows details of the tails of the distribution), we see that the distribution of stock price returns is highly peaked and fat-tailed relative to the Normal distribution. Fat tails and the high central peak are characteristics of mixtures of distributions with different variances. This motivates us to model variance as a random variable. The volatility clustering feature implies that volatility (or variance) is auto-correlated. In the model, this is a consequence of the mean reversion of volatility [1].

There is a simple economic argument which justifies the mean reversion of volatility (the same argument that is used to justify the mean reversion of interest rates). Consider the distribution of the volatility of IBM in one hundred years time say. If volatility were not mean-reverting (*i.e.* if the distribution of volatility were not stable), the probability of the volatility of IBM being between 1% and 100% would be rather low. Since we believe that it is overwhelmingly likely that the volatility of IBM would in fact lie

[1]Note that simple jump-diffusion models do not have this property. After a jump, the stock price volatility does not change.

Figure 1: SPX daily log returns from 1/1/1990 to 31/12/1999

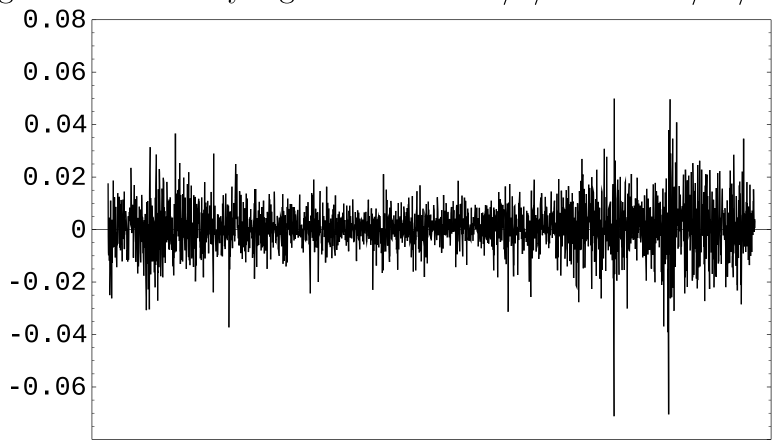
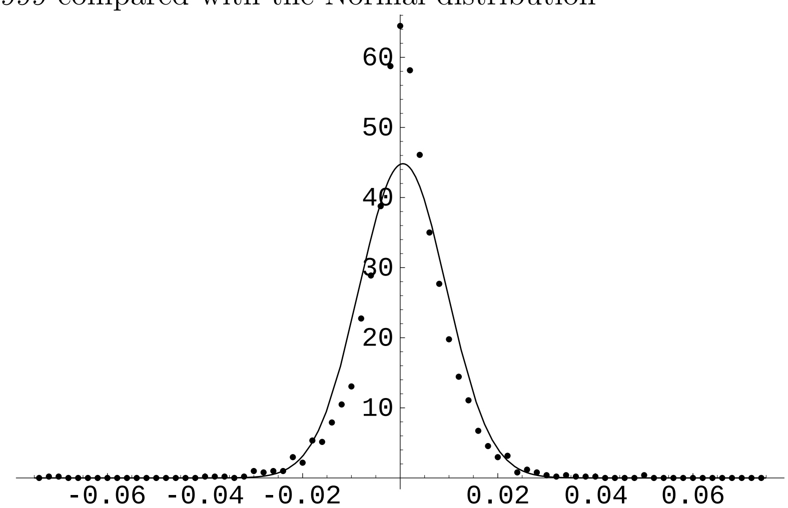


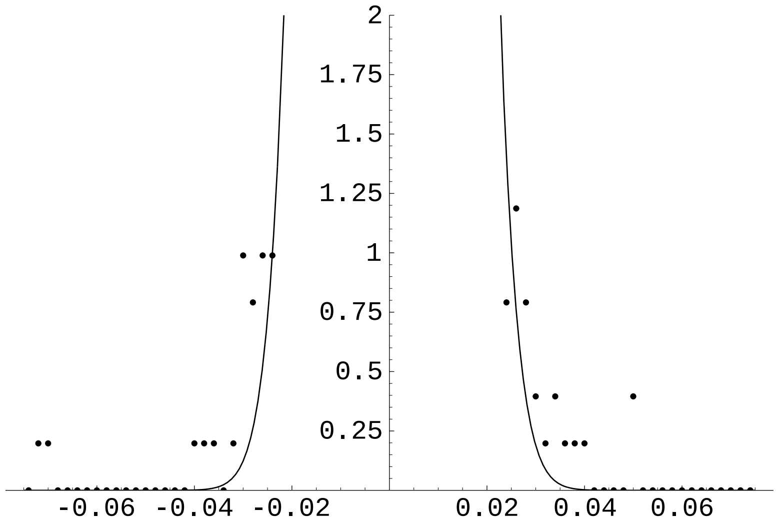
Figure 2: Frequency distribution of SPX daily log returns from 1/1/1990 to 31/12/1999 compared with the Normal distribution



in that range, we deduce that volatility must be mean-reverting.

Having motivated the description of variance as a mean-reverting random variable, we are now ready to derive the valuation equation.

Figure 3: Tails of SPX frequency distribution



1.2 Derivation of the Valuation Equation

In this section, we follow Wilmott (1998) closely. We suppose that the stock price S and its variance v satisfy the following SDEs:

$$dS(t) = \mu(t)S(t)dt + \sqrt{v(t)}S(t)dZ_1 \tag{1}$$

$$dv(t) = \alpha(S, v, t)dt + \eta\,\beta(S, v, t)\sqrt{v(t)}dZ_2 \tag{2}$$

with

$$\langle dZ_1\, dZ_2 \rangle = \rho\, dt$$

where $\mu(t)$ is the (deterministic) instantaneous drift of stock price returns, η is the volatility of volatility and ρ is the correlation between random stock price returns and changes in $v(t)$. dZ_1 and dZ_2 are Wiener processes.

The stochastic process (1) followed by the stock price is equivalent to the one assumed in the derivation of Black and Scholes (1973). This ensures that the standard time-dependent volatility version of the Black-Scholes formula (as derived in section 8.6 of Wilmott (1998) for example) may be retrieved in the limit $\eta \to 0$. In practical applications, this is a key requirement of a stochastic volatility option pricing model as practitioners' intuition for the behavior of option prices is invariably expressed within the framework of the Black-Scholes formula.

In the Black-Scholes case, there is only one source of randomness – the stock price, which can be hedged with stock. In the present case, random changes in volatility also need to be hedged in order to form a riskless portfolio. So we set up a portfolio Π containing the option being priced whose value we denote by $V(S, v, t)$, a quantity $-\Delta$ of the stock and a quantity $-\Delta_1$ of another asset whose value V_1 depends on volatility. We have

$$\Pi = V - \Delta\, S - \Delta_1\, V_1$$

The change in this portfolio in a time dt is given by

$$\begin{aligned} d\Pi \;=\;& \left\{ \frac{\partial V}{\partial t} + \frac{1}{2} v\, S^2(t) \frac{\partial^2 V}{\partial S^2} + \rho\eta\, v\beta\, S(t) \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2}\eta^2 v\beta^2 \frac{\partial^2 V}{\partial v^2} \right\} dt \\ & -\Delta_1 \left\{ \frac{\partial V_1}{\partial t} + \frac{1}{2} v\, S^2(t) \frac{\partial^2 V_1}{\partial S^2} + \rho\eta\, v\beta\, S(t) \frac{\partial^2 V_1}{\partial v \partial S} + \frac{1}{2}\eta^2 v\beta^2 \frac{\partial^2 V_1}{\partial v^2} \right\} dt \\ & + \left\{ \frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta \right\} dS \\ & + \left\{ \frac{\partial V}{\partial v} - \Delta_1 \frac{\partial V_1}{\partial v} \right\} dv \end{aligned}$$

To make the portfolio instantaneously risk-free, we must choose

$$\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta = 0$$

to eliminate dS terms, and

$$\frac{\partial V}{\partial v} - \Delta_1 \frac{\partial V_1}{\partial v} = 0$$

to eliminate dv terms. This leaves us with

$$\begin{aligned} d\Pi \;=\;& \left\{ \frac{\partial V}{\partial t} + \frac{1}{2} v\, S^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta v\beta\, S \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2}\eta^2\, v\beta^2 \frac{\partial^2 V}{\partial v^2} \right\} dt \\ & -\Delta_1 \left\{ \frac{\partial V_1}{\partial t} + \frac{1}{2} v\, S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\eta v\beta\, S \frac{\partial^2 V_1}{\partial v \partial S} + \frac{1}{2}\eta^2\, v\beta^2 \frac{\partial^2 V_1}{\partial v^2} \right\} dt \\ =\;& r\, \Pi\, dt \\ =\;& r(V - \Delta S - \Delta_1 V_1)\, dt \end{aligned}$$

where we have used the fact that the return on a risk-free portfolio must equal the risk-free rate r which we will assume to be deterministic for our purposes. Collecting all V terms on the left-hand side and all V_1 terms on the right-hand side, we get

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2} v\, S^2 \frac{\partial^2 V}{\partial S^2} + \rho \eta\, v\, \beta\, S \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2} \eta^2 v \beta^2 \frac{\partial^2 V}{\partial v^2} + r S \frac{\partial V}{\partial S} - r V}{\frac{\partial V}{\partial v}}$$

$$= \frac{\frac{\partial V_1}{\partial t} + \frac{1}{2} v\, S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho \eta\, v \beta\, S \frac{\partial^2 V_1}{\partial v \partial S} + \frac{1}{2} \eta^2 v \beta^2 \frac{\partial^2 V_1}{\partial v^2} + r S \frac{\partial V_1}{\partial S} - r V_1}{\frac{\partial V_1}{\partial v}}$$

The left-hand side is a function of V only and the right-hand side is a function of V_1 only. The only way that this can be is for both sides to be equal to some function f of the *independent* variables S, v and t. We deduce that

$$\frac{\partial V}{\partial t} + \frac{1}{2} v\, S^2 \frac{\partial^2 V}{\partial S^2} + \rho \eta\, v\, \beta\, S \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2} \eta^2 v \beta^2 \frac{\partial^2 V}{\partial v^2} + r S \frac{\partial V}{\partial S} - r V \;=\; -\, (\alpha - \varphi\, \beta) \frac{\partial V}{\partial v} \tag{3}$$

where, without loss of generality, we have written the arbitrary function f of S, v and t as $(\alpha - \varphi\, \beta)$. Conventionally, $\varphi(S, v, t)$ is called the market price of volatility risk because it tells us how much of the expected return of V is explained by the risk (*i.e.* standard deviation) of v in the Capital Asset Pricing Model framework.

2 Local Volatility

2.1 History

Given the computational complexity of stochastic volatility models and the extreme difficulty of fitting parameters to the current prices of vanilla options, practitioners sought a simpler way of pricing exotic options consistently with the volatility skew. Since before Breeden and Litzenberger (1978), it was understood that the risk-neutral *pdf* could be derived from the market prices of European options. The breakthrough came when Dupire (1994) and Derman and Kani (1994) noted that under risk-neutrality, there was a unique diffusion process consistent with these distributions. The corresponding unique state-dependent diffusion coefficient $\sigma_L(S, t)$ consistent with current European option prices is known as the local volatility function.

It is unlikely that Dupire, Derman and Kani ever thought of local volatility as representing a model of how volatilities actually evolve. Rather, it is likely that they thought of local volatilities as representing some kind of average over all possible instantaneous volatilities in a stochastic volatility world (an "effective theory"). Local volatility models do not therefore really represent a separate class of models; the idea is more to make a simplifying assumption that allows practitioners to price exotic options consistently with the known prices of vanilla options.

As if any proof had been needed, Dumas, Fleming, and Whaley (1998) performed an empirical analysis which confirmed that the dynamics of the implied volatility surface were not consistent with the assumption of constant local volatilities.

In section 2.5, we will show that local volatility is indeed an average over instantaneous volatilities, formalizing the intuition of those practitioners who first introduced the concept.

2.2 A Brief Review of Dupire's Work

For a given expiration T and current stock price S_0 , the collection $\{C\left(S_0, K, T\right); K \in (0, \infty)\}$ of undiscounted option prices of different strikes yields the risk neutral density function φ of the final spot S_T through the relationship

$$C\left(S_0, K, T\right) = \int_K^\infty dS_T \, \varphi\left(S_T, T; S_0\right) \left(S_T - K\right)$$

Differentiate this twice with respect to K to obtain

$$\varphi\left(K, T; S_0\right) = \frac{\partial^2 C}{\partial K^2}$$

so the Arrow-Debreu prices for each expiration may be recovered by twice differentiating the undiscounted option price with respect to K. This process will be familiar to any option trader as the construction of an (infinite size) infinitesimally tight butterfly around the strike whose maximum payoff is one.

Given the distribution of final spot prices S_T for each time T conditional on some starting spot price S_0, Dupire shows that there is a unique risk neutral diffusion process which generates these distributions. That is, given the set of all European option prices, we may determine the functional form of the diffusion parameter (local volatility) of the unique risk neutral diffusion process which generates these prices. Noting that the local volatility will in general be a function of the current stock price S_0, we write this process as

$$\frac{dS}{S} = \mu\,(t)\;dt + \sigma\,(S,t;S_0)\;dZ$$

Application of Itô's Lemma together with risk neutrality, gives rise to a partial differential equation for functions of the stock price which is a straightforward generalization of Black-Scholes. In particular, the pseudo probability densities $\varphi\,(K,T;S_0) = \frac{\partial^2 C}{\partial K^2}$ must satisfy the Fokker-Planck equation. This leads to the following equation for the undiscounted option price C in terms of the strike price K:

$$\frac{\partial C}{\partial T} = \frac{\sigma^2 K^2}{2}\frac{\partial^2 C}{\partial K^2} + (r(T) - D(T))\left(C - K\frac{\partial C}{\partial K}\right) \tag{4}$$

where $r(t)$ is the risk-free rate, $D(t)$ is the dividend yield and C is short for $C\,(S_0,K,T)$. See the Appendix for a derivation of this equation.

Were we to express the option price as a function of the forward price $F_T = S_0 \exp\left\{\int_0^T \mu(t)dt\right\}$ [2], we would get the same expression minus the drift term. That is

$$\frac{\partial C}{\partial T} = \frac{1}{2}\,\sigma^2\,K^2\,\frac{\partial^2 C}{\partial K^2}$$

where C now represents $C\,(F_T,K,T)$. Inverting this gives

$$\sigma^2(K,T,S_0) \;=\; \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}\;K^2\;\frac{\partial^2 C}{\partial K^2}} \tag{5}$$

The right hand side of Equation 5 can be computed from known European option prices. So, given a complete set of European option prices for all strikes and expirations, local volatilities are given uniquely by Equation 5.

We can view Equation 5 as a *definition* of the local volatility function regardless of what kind of process (stochastic volatility for example) actually governs the evolution of volatility.

[2] From now on, $\mu(T)$ represents the risk-neutral drift of the stock price process which is the risk-free rate $r(T)$ minus the dividend yield $D(T)$

2.3 Transforming to Black-Scholes Implied Volatility Space

Market prices of options are quoted in terms of Black-Scholes implied volatility $\sigma_{BS}(K, T; S_0)$. In other words, we may write

$$C(S_0, K, T) = C_{BS}(S_0, K, \sigma_{BS}(S_0, K, T), T)$$

It will be more convenient for us to work in terms of two dimensionless variables: the Black-Scholes implied total variance w defined by

$$\mathrm{w}(S_0, K, T) \equiv \sigma_{BS}^2(S_0, K, T)\ T$$

and the log forward price x defined by

$$x = \ln\left(\frac{F_T}{K}\right)$$

where $F_T = S_0 \exp\left\{\int_0^T dt\, \mu(t)\right\}$ gives the forward price of the stock at time 0. In terms of these variables, the Black-Scholes formula for the future value of the option price becomes

$$\begin{aligned} C_{BS}(F_T, x, \mathrm{w}) &= F_T\left\{N(d_1) - e^{-x} N(d_2)\right\} \\ &= F_T\left\{N\left(\frac{x}{\sqrt{\mathrm{w}}} + \frac{\sqrt{\mathrm{w}}}{2}\right) - e^{-x} N\left(\frac{x}{\sqrt{\mathrm{w}}} - \frac{\sqrt{\mathrm{w}}}{2}\right)\right\} \end{aligned} \tag{6}$$

and the Dupire equation (Equation 4) becomes

$$\frac{\partial C}{\partial T} = \frac{v_L}{2}\left\{\frac{\partial^2 C}{\partial x^2} + \frac{\partial C}{\partial x}\right\} + \mu(T)\ C \tag{7}$$

with $v_L = \sigma^2(S_0, K, T)$ representing the local variance. Now, by taking derivatives of the Black-Scholes formula, we obtain

$$\begin{aligned} \frac{\partial^2 C_{BS}}{\partial \mathrm{w}^2} &= \left(-\frac{1}{8} - \frac{1}{2\,\mathrm{w}} + \frac{x^2}{2\,\mathrm{w}^2}\right)\frac{\partial C_{BS}}{\partial \mathrm{w}} \\ \frac{\partial^2 C_{BS}}{\partial x \partial \mathrm{w}} &= \left(-\frac{1}{2} - \frac{x}{\mathrm{w}}\right)\frac{\partial C_{BS}}{\partial \mathrm{w}} \\ \frac{\partial^2 C_{BS}}{\partial x^2} + \frac{\partial C_{BS}}{\partial x} &= 2\,\frac{\partial C_{BS}}{\partial \mathrm{w}} \end{aligned} \tag{8}$$

We may transform Equation 7 into an equation in terms of implied variance by making the substitutions

$$\begin{aligned}
\frac{\partial C}{\partial x} &= \frac{\partial C_{BS}}{\partial x} + \frac{\partial C_{BS}}{\partial \mathrm{w}} \frac{\partial \mathrm{w}}{\partial x} \\
\frac{\partial^2 C}{\partial x^2} &= \frac{\partial^2 C_{BS}}{\partial x^2} + 2 \frac{\partial^2 C_{BS}}{\partial x \partial \mathrm{w}} \frac{\partial \mathrm{w}}{\partial x} + \frac{\partial^2 C_{BS}}{\partial \mathrm{w}^2} \left(\frac{\partial \mathrm{w}}{\partial x} \right)^2 + \frac{\partial C_{BS}}{\partial \mathrm{w}} \frac{\partial^2 \mathrm{w}}{\partial x^2} \\
\frac{\partial C}{\partial T} &= \frac{\partial C_{BS}}{\partial T} + \frac{\partial C_{BS}}{\partial \mathrm{w}} \frac{\partial \mathrm{w}}{\partial T} = \frac{\partial C_{BS}}{\partial \mathrm{w}} \frac{\partial \mathrm{w}}{\partial T} + \mu(T)\, C_{BS}
\end{aligned}$$

where the last equality follows from the fact that the only explicit dependence of the option price on T in Equation 6 is through the forward price $F_T = S_0 \exp\left\{ \int_0^T dt\, \mu(t) \right\}$. Equation 4 now becomes (cancelling $\mu(T)\, C$ terms on each side)

$$\begin{aligned}
& \frac{\partial C_{BS}}{\partial \mathrm{w}} \frac{\partial \mathrm{w}}{\partial T} \\
= \; & \frac{v_L}{2} \left\{ \frac{\partial C_{BS}}{\partial x} + \frac{\partial^2 C_{BS}}{\partial x^2} + \frac{\partial C_{BS}}{\partial \mathrm{w}} \frac{\partial \mathrm{w}}{\partial x} + 2 \frac{\partial^2 C_{BS}}{\partial x \partial \mathrm{w}} \frac{\partial \mathrm{w}}{\partial x} + \frac{\partial^2 C_{BS}}{\partial \mathrm{w}^2} \left(\frac{\partial \mathrm{w}}{\partial x} \right)^2 + \frac{\partial C_{BS}}{\partial \mathrm{w}} \frac{\partial^2 \mathrm{w}}{\partial x^2} \right\} \\
= \; & \frac{v_L}{2} \frac{\partial C_{BS}}{\partial \mathrm{w}} \left\{ 2 + \frac{\partial \mathrm{w}}{\partial x} + 2 \left(-\frac{1}{2} - \frac{x}{\mathrm{w}} \right) \frac{\partial \mathrm{w}}{\partial x} + \left(-\frac{1}{8} - \frac{1}{2\mathrm{w}} + \frac{x^2}{2\mathrm{w}^2} \right) \left(\frac{\partial \mathrm{w}}{\partial x} \right)^2 + \frac{\partial^2 \mathrm{w}}{\partial x^2} \right\}
\end{aligned}$$

Then, taking out a factor of $\frac{\partial C_{BS}}{\partial \mathrm{w}}$ and simplifying, we get

$$\frac{\partial \mathrm{w}}{\partial T} = v_L \left\{ 1 - \frac{x}{\mathrm{w}} \frac{\partial \mathrm{w}}{\partial x} + \frac{1}{4} \left(-\frac{1}{4} - \frac{1}{\mathrm{w}} + \frac{x^2}{\mathrm{w}^2} \right) \left(\frac{\partial \mathrm{w}}{\partial x} \right)^2 + \frac{1}{2} \frac{\partial^2 \mathrm{w}}{\partial x^2} \right\}$$

Inverting this gives our final result:

$$v_L = \frac{\frac{\partial \mathrm{w}}{\partial T}}{1 - \frac{x}{\mathrm{w}} \frac{\partial \mathrm{w}}{\partial x} + \frac{1}{4} \left(-\frac{1}{4} - \frac{1}{\mathrm{w}} + \frac{x^2}{\mathrm{w}^2} \right) \left(\frac{\partial \mathrm{w}}{\partial x} \right)^2 + \frac{1}{2} \frac{\partial^2 \mathrm{w}}{\partial x^2}}$$

2.4 Special Case: No Skew

If the skew $\frac{\partial \mathrm{w}}{\partial x}$ is zero[3] , we must have

$$v_L = \frac{\partial \mathrm{w}}{\partial T}$$

[3]Note that this implies that $\frac{\partial}{\partial K} \sigma_{BS}(S_0, K, T)$ is zero

So the local variance in this case reduces to the forward Black-Scholes implied variance. The solution to this is of course

$$\mathrm{w}\left(T\right)=\int_{0}^{T}v_{L}\left(t\right)\,dt$$

2.5 Local Variance as a Conditional Expectation of Instantaneous Variance

In this section, we review the elegant derivation of Derman and Kani (1998). We assume the same stochastic process for the stock price as in Equation 1 but write it in terms of the forward price $F_{t,T}=S_t\exp\left\{\int_t^T ds\,\mu_s\right\}$.

$$dF_{t,T}=\sqrt{v_t}F_{t,T}dZ \tag{9}$$

Note that $dF_{T,T}=dS_T$. The undiscounted value of a European option with strike K expiring at time T is given by

$$C\left(S_0,K,T\right)\;=\;\mathbf{E}\left[\left(S_T-K\right)^{+}\right]$$

Differentiating once with respect to K gives

$$\frac{\partial C}{\partial K}=-\mathbf{E}\left[\theta\left(S_T-K\right)\right]$$

where $\theta(\cdot)$ is the Heaviside function. Differentiating again with respect to K gives

$$\frac{\partial^2 C}{\partial K^2}=\mathbf{E}\left[\delta\left(S_T-K\right)\right]$$

where $\delta(\cdot)$ is the Dirac δ function.

Now, a formal application of Itô's Lemma to the terminal payoff of the option (and using $dF_{T,T}=dS_T$) gives the identity

$$d\left(S_T-K\right)^{+}=\theta\left(S_T-K\right)\,dS_T+\frac{1}{2}\,v_T\,S_T^2\,\delta\left(S_T-K\right)\,dT$$

Taking conditional expectations of each side, and using the fact that $F_{t,T}$ is a Martingale, we get

$$dC\;=\;d\mathbf{E}\left[\left(S_T-K\right)^{+}\right]=\;\frac{1}{2}\,\mathbf{E}\left[v_T\,S_T^2\,\delta\left(S_T-K\right)\right]\,dT$$

Also, we can write

$$\begin{aligned}\mathbf{E}\left[v_T S_T^2 \, \delta\left(S_T - K\right)\right] &= \mathbf{E}\left[v_T \left| S_T = K\right.\right] \, \frac{1}{2} K^2 \, \mathbf{E}\left[\, \delta\left(S_T - K\right)\right] \\ &= \mathbf{E}\left[v_T \left| S_T = K\right.\right] \, \frac{1}{2} K^2 \, \frac{\partial^2 C}{\partial K^2}\end{aligned}$$

Putting this together, we get

$$\frac{\partial C}{\partial T} = \mathbf{E}\left[v_T \left| S_T = K\right.\right] \, \frac{1}{2} K^2 \, \frac{\partial^2 C}{\partial K^2}$$

Comparing this with the definition of local volatility (Equation 5), we see that

$$\sigma^2(K, T, S_0) \; = \; \mathbf{E}\left[v_T \left| S_T = K\right.\right]$$

That is, local variance is the risk-neutral expectation of the instantaneous variance conditional on the final stock price S_T being equal to the strike price K.

3 The Heston Model

3.1 The Model

The Heston model (Heston (1993)) corresponds to choosing $\alpha(S, v(t), t) = -\lambda(v(t) - \bar{v})$ and $\beta(S, v, t) = 1$ in Equations 1 and 2. These equations then become

$$dS(t) = \mu(t) S(t) dt + \sqrt{v(t)} S(t) dZ_1 \tag{10}$$

and

$$dv(t) = -\lambda(v(t) - \bar{v}) dt + \eta \sqrt{v(t)} dZ_2 \tag{11}$$

with

$$\langle dZ_1 \, dZ_2 \rangle = \rho \, dt$$

where λ is the speed of reversion of $v(t)$ to its long term mean $\bar{v}$.

The process followed by $v(t)$ may be recognized as a version of the square root process described by Cox, Ingersoll, and Ross (1985). It is a (jump-free) special case of a so-called *affine jump diffusion (AJD)* which is roughly speaking a jump-diffusion process for which the drifts and covariances and jump intensities are linear in the state vector (which is $\{x, v\}$ in this case with

$x = \log(S)$). Duffie, Pan, and Singleton (1999) show that AJD processes are analytically tractable in general. The solution technique involves computing an "extended transform" which in the Heston case is a conventional Fourier transform.

We now substitute the above values for $\alpha(S, v, t)$ and $\beta(S, v, t)$ into the general valuation equation (Equation 3). We obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2} v\, S^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta\, v\, S \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2}\eta^2 v \frac{\partial^2 V}{\partial v^2} + rS\frac{\partial V}{\partial S} - rV = (\lambda(v - \bar{v}) - \varphi)\frac{\partial V}{\partial v} \tag{12}$$

Now, to be able to use the AJD results, the market price of volatility risk also needs to be affine. Various economic arguments can be made (see for example Wiggins (1987)) that the market price of volatility risk φ should be proportional to the variance v. Then, let $\varphi = \theta v$ for some constant θ.

Now define the risk-adjusted parameters λ' and $\bar{v}'$ through $\lambda' = \lambda - \theta,\ \lambda'\bar{v}' = \lambda\bar{v}$. Substituting this into equation (12) gives

$$\frac{\partial V}{\partial t} + \frac{1}{2} v\, S^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta\, v\, S \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2}\eta^2 v \frac{\partial^2 V}{\partial v^2} + rS\frac{\partial V}{\partial S} - rV = (\lambda'(v - \bar{v}'))\frac{\partial V}{\partial v} \tag{13}$$

Note that equation (13) is now identical to equation (12) with no explicit risk preference related parameters except that the parameters λ' and $\bar{v}'$ are now risk adjusted. From now on we will drop the primes on λ' and $\bar{v}'$ and assume that we are dealing with the risk-adjusted parameters.

3.2 The Heston Solution for European Options

This section repeats the derivation of the Heston formula for the value of a European-style option first presented in Heston (1993) but with rather more detail than is provided in that paper.

Before solving equation (13) with the appropriate boundary conditions, we can simplify it by making some suitable changes of variable. Let K be the strike price of the option, T be its expiry date and $F(t, T)$ the forward price of the stock index to expiry. Then let

$$x(t) = \ln\left(\frac{F(t,T)}{K}\right)$$

Further, suppose that we consider only the future value to expiration C of the European option price rather than its value today and define $\tau = T - t$. Then equation (13) simplifies to

$$-\frac{\partial C}{\partial \tau} + \frac{1}{2} v(t)\, C_{11} + \frac{1}{2} v(t)\, C_1 + \frac{1}{2}\eta^2 v(t) C_{22} + \rho\eta\, v(t)\, C_{12} - \lambda(v(t) - \bar{v})\, C_2 = 0 \quad (14)$$

where the subscripts 1 and 2 refer to differentiation with respect to x and v respectively.

According to Duffie, Pan, and Singleton (1999), the solution of equation 14 has the form

$$C(x,v,\tau) = e^x\, P_1(x,v,\tau) - P_0(x,v,\tau) \quad (15)$$

where the first term represents the pseudo-expectation of the final index level given that the option is in-the-money and the second term represents the pseudo-probability of exercise.

Substituting the proposed solution (15) into equation (14) shows that P_0 and P_1 must satisfy the equation

$$-\frac{\partial P_j}{\partial \tau} + \frac{1}{2} v \frac{\partial P_j}{\partial x^2} - \left(\tfrac{1}{2} - j\right) v \frac{\partial P_j}{\partial x} + \frac{1}{2}\eta^2 v \frac{\partial^2 P_j}{\partial v^2} + \rho\eta v \frac{\partial^2 P_j}{\partial x \partial v} + (a - b_j v) \frac{\partial P_j}{\partial v} = 0 \quad (16)$$

for $j = 0, 1$ where

$$a = \lambda\, \bar{v}, \quad b_j = \lambda - j\rho\eta$$

subject to the terminal condition

$$\lim_{\tau \to 0} P_j(x, v, \tau) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x \le 0 \end{cases} \equiv \theta(x) \quad (17)$$

We solve equation (16) subject to the condition (17) using a Fourier transform technique. To this end define the Fourier transform of P_j through

$$\tilde{P}(k, v, \tau) = \int_{-\infty}^{\infty} dx\, e^{-ikx}\, P(x, v, \tau)$$

Then

$$\tilde{P}(k, v, 0) = \int_{-\infty}^{\infty} dx\, e^{-ikx}\, \theta(x) = \frac{1}{ik}$$

The inverse transform is given by

$$P(x,v,\tau)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\;e^{ikx}\,\tilde{P}(k,v,\tau) \tag{18}$$

Substituting this into equation (16) gives

$$\begin{aligned}-\frac{\partial\tilde{P}_j}{\partial\tau} &- \frac{1}{2}k^2\,v\,\tilde{P}_j-(\tfrac{1}{2}-j)\,ik\,v\,\tilde{P}_j\\ &+ \frac{1}{2}\eta^2\,v\frac{\partial^2\tilde{P}_j}{\partial v^2}+\rho\eta\,ikv\frac{\partial\tilde{P}_j}{\partial v}+(a-b_jv)\frac{\partial\tilde{P}_j}{\partial v}=0\end{aligned} \tag{19}$$

Now define

$$\begin{aligned}\alpha &= -\frac{k^2}{2}-\frac{ik}{2}+ijk\\ \beta &= \lambda-\rho\eta j-\rho\eta ik\\ \gamma &= \frac{\eta^2}{2}\end{aligned}$$

Then equation (19) becomes

$$v\left\{\alpha\,\tilde{P}_j-\beta\frac{\partial\tilde{P}_j}{\partial v}+\gamma\frac{\partial^2\tilde{P}_j}{\partial v^2}\right\}+a\,\frac{\partial\tilde{P}_j}{\partial v}-\frac{\partial\tilde{P}_j}{\partial\tau}=0 \tag{20}$$

Now substitute

$$\begin{aligned}\tilde{P}_j(k,v,\tau) &= \exp\left\{C(k,\tau)\,\bar{v}+D(k,\tau)\,v\right\}\;\tilde{P}_j(k,v,0)\\ &= \frac{1}{ik}\exp\left\{C(k,\tau)\,\bar{v}+D(k,\tau)\,v\right\}\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\partial\tilde{P}_j}{\partial\tau} &= \left\{\bar{v}\frac{\partial C}{\partial\tau}+v\frac{\partial D}{\partial\tau}\right\}\tilde{P}_j\\ \frac{\partial\tilde{P}_j}{\partial v} &= D\,\tilde{P}_j\\ \frac{\partial^2\tilde{P}_j}{\partial v^2} &= D^2\,\tilde{P}_j\end{aligned}$$

Then equation (20) is satisfied if

$$\begin{aligned} \frac{\partial C}{\partial \tau} &= \lambda D \\ \frac{\partial D}{\partial \tau} &= \alpha - \beta\, D + \gamma\, D^2 \\ &= \gamma (D - r_+)(D - r_-) \end{aligned} \tag{21}$$

where we define

$$r_\pm = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\gamma} \equiv \frac{\beta \pm d}{\eta^2}$$

Integrating (21) with the terminal conditions $C(k, 0) = 0$ and $D(k, 0) = 0$ gives

$$\begin{aligned} D(k,\tau) &= r_- \frac{1 - e^{-d\tau}}{1 - g e^{-d\tau}} \\ C(k,\tau) &= \lambda \left\{ r_- \tau - \frac{2}{\eta^2} \ln \left(\frac{1 - g e^{-d\tau}}{1 - g} \right) \right\} \end{aligned}$$

where we define

$$g \equiv \frac{r_-}{r_+}$$

Taking the inverse transform using equation (18) and performing the complex integration carefully gives the final form of the pseudo-probabilities P_j in the form of an integral of a real-valued function.

$$\boxed{P_j(x, v, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty dk \, \mathrm{Re} \left\{ \frac{\exp\{C_j(k,\tau)\, \bar{v} + D_j(k,\tau)\, v + ikx\}}{ik} \right\}}$$

This integration may be performed using standard numerical methods.

As a final point, it is worth noting that taking derivatives of the Heston formula with respect to x or v in order to derive risk parameters is extremely straightforward because the functions $C(k, \tau)$ and $D(k, \tau)$ are independent of x and v.

A Derivation of the Dupire Equation

Suppose the stock price diffuses with risk-neutral drift $\mu(t)$ and local volatility $\sigma(S,t)$ according to the equation:

$$\frac{dS}{S} = \mu(t)\, dt + \sigma(S,t)\, dZ$$

The undiscounted risk-neutral value $C(S_0,K,T)$ of a European option with strike K and expiration T is given by

$$C(S_0,K,T) = \int_K^\infty dS_T\, \varphi(S_T,T;S_0)\,(S_T - K) \tag{22}$$

Here $\varphi(S_T,T;S_0)$ is the pseudo probability density of the final spot at time T. It evolves according to the Fokker-Planck equation[4] :

$$\frac{1}{2}\frac{\partial^2}{\partial S_T^2}\left(\sigma^2 S_T^2 \varphi\right) - S\frac{\partial}{\partial S_T}(\mu S_T \varphi) = \frac{\partial \varphi}{\partial T}$$

Differentiating with respect to K gives

$$\begin{aligned}\frac{\partial C}{\partial K} &= -\int_K^\infty dS_T\, \varphi(S_T,T;S_0)\\ \frac{\partial^2 C}{\partial K^2} &= \varphi(K,T;S_0)\end{aligned}$$

Now, differentiating (22) with respect to time gives

$$\begin{aligned}\frac{\partial C}{\partial T} &= \int_K^\infty dS_T \left\{\frac{\partial}{\partial T}\varphi(S_T,T;S_0)\right\}(S_T-K)\\ &= \int_K^\infty dS_T \left\{\frac{1}{2}\frac{\partial^2}{\partial S_T^2}\left(\sigma^2 S_T^2\varphi\right) - \frac{\partial}{\partial S_T}(\mu S_T \varphi)\right\}(S_T-K)\end{aligned}$$

Integrating by parts twice gives:

$$\begin{aligned}\frac{\partial C}{\partial T} &= \frac{\sigma^2 K^2}{2}\varphi + \int_K^\infty dS_T\, \mu S_T \varphi\\ &= \frac{\sigma^2 K^2}{2}\frac{\partial^2 C}{\partial K^2} + \mu(T)\left(C - K\frac{\partial C}{\partial K}\right)\end{aligned}$$

[4]See Section 5 of Robert Kohn's PDE for Finance Class Notes for a very readable account of this topic

which is the Dupire equation when the underlying stock has risk-neutral drift μ. That is, the forward price of the stock at time T is given by

$$F(T) = S_0 \exp\left\{\int_0^T dt\, \mu(t)\right\}$$

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