Conclusions

- UVM : New method for quantifying volatility risk in derivative market-making
- Extremal prices are computed using the non-linear BSB equation
- Non-linear PDE quantifies portfolio risk-diversification (unlike linear models)
- UVM gives rise to risk-averse strategies in the presence of vol risk
- Method can be used to select a hedging portfolio of traded derivative instruments and hedge residual risk
- Competitive prices can be obtained by finding the "best fit" (option hedge) and hedging only residuals

References: M. Avellaneda, A. Levy and A. Parás: *" Pricing and Hedging Derivative Securities in Markets with Uncertain Volatilities, I and II "*, preprints, 1995.

P. Lewicki and M. Avellaneda, *"Pricing Interest Rate Claims in markets with Uncertain Volatilities "*, Working Paper, 1995.

UVM & Interest-Rate Derivatives

(M. Avellaneda & Pawel Lewicki, 1995)

One-factor, arbitrage-free model with uncertain vol.

$$R_t \;=\; f(t) \;+\; \int_0^t \sigma(t,s)\,\dot{\sigma}(t,s)\,ds \;+\; \int_0^t \dot{\sigma}(t,s)\,dZ_s$$

(Heath, Jarrow & Morton, 1990)

$\dot{\sigma}(t,s) \;=\; e^{-\alpha\,(s-t)}\;\sigma_t \;=\;$ forward rate vol

$f(t) \;=\;$ yield curve

New form of BSB equation for interest-rate derivatives:

$$\frac{\partial G^+}{\partial t} \;+\; \left(\, f'(t) \;+\; V \;-\alpha\; R\right) \frac{\partial G^+}{\partial R} \;-2\;\alpha V\;\frac{\partial G^+}{\partial V}$$

$$+ \sup_{\sigma_{min}\;\le\;\sigma\;\le\;\sigma_{max}} \sigma^2 \left\{\; \frac{1}{2}\,\frac{\partial^2 G^+}{\partial R^2} \;+\; \frac{\partial G^+}{\partial V} \;\right\} \;-\; R\,G^+ \;=\; 0.$$

$G^+ \;=\; G^+(R,V,t)$, $V \;=\;$ "accumulated vol"

Application to hedging caps & floors , etc.

Pricing the Put Option without Aug 22.50

"Arbitraging" the Aug 22.5 option

Idea: Since Aug 22.5 is cheap, buy it.

Hedge long position using the remaining listed calls

Cost of buying the option : \$7.3750

Optimal bid price (W^-): \$7.4384

Initial position

Delta= -0.3 TMX shares

Call hedge ratios

Mat.	Strike	Quantity (λ)
Aug	25	-0.09
Aug	30	+0.02
May	22.5	-0.18 *
May	25	-0.24 *
May	30	-0.04
Apr	22.5	-0.20 *
Apr	25	-0.05

Gain = 6 cents (just 1/16, too small!)

2. European put expiring on July 15 (0.3342 yr.) with strike price $K = \$\,20$

- Remove Mar 25 and Mar 40 calls
- Assume σ_{min}=0.3 and $\sigma_{max} = 1.5$
- Solve optimization problem with $M = 16$ (calls only).

Solution

Offer price (W^+) = -0.4427 IMVOL=NaN !

Delta (numb. of TMX) = -3.3457

Call hedge ratios

Mat.	Strike	Quantity (λ)
Aug	22.5	10.00 (max allowed)
Aug	25	-0.87
Aug	30	0.17
May	22.5	-1.71
May	25	-2.43
May	30	-0.47
Apr	22.5	-1.73
Apr	25	0.41

Arbitrage opportunity if vol stays in the band

Option hedge: Calendar spread of 2 synthetic puts

Without option hedge (just BSB): $W^+ = 0.5729$

1. European-style digital option

Terms

- Expiration on 3rd Friday of May
- Payoff = \$1 if TMX < 20 or \$0 if TMX > 20

Define UVM parameters

- Remove Mar 25 & Mar 40 calls
- Use all other calls with expiry up to May (M=11)
- Take $\sigma_{max} = 1.5$ $\sigma_{min} = 0.3$

Solution

Offer price $(W^{+}) = 0.142556$ IMVOL=0.7685

Delta = -0.2344 $\approx$ -0.2 shares   Cash=\$+4.5239

Call hedge-ratios

Mat.	Strike	Quantity (λ)
Apr	22.5	-0.15
Apr	25	0.05
May	22.5	+0.34

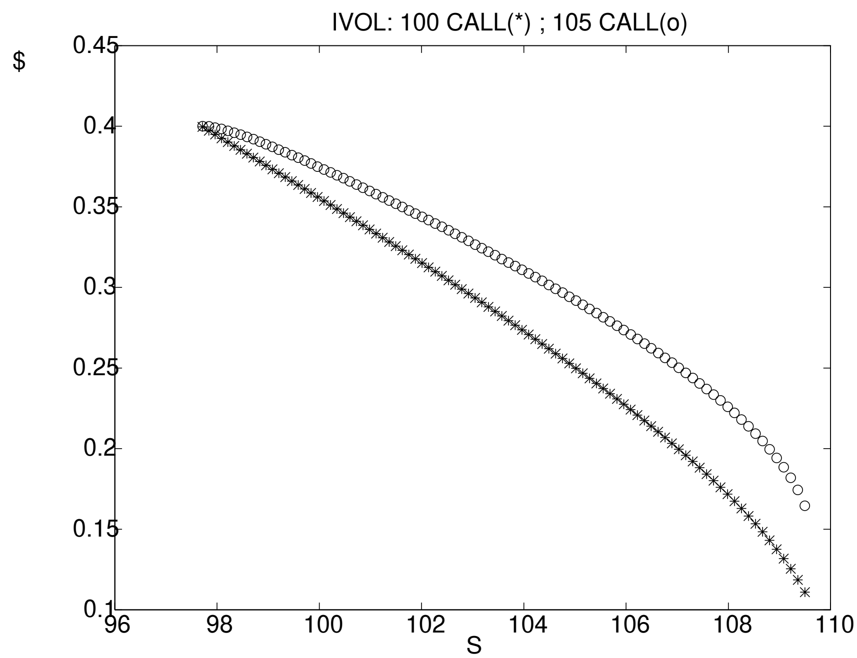
Example: using UVM to price and hedge options on TMX (Teléfonos de Mexico)

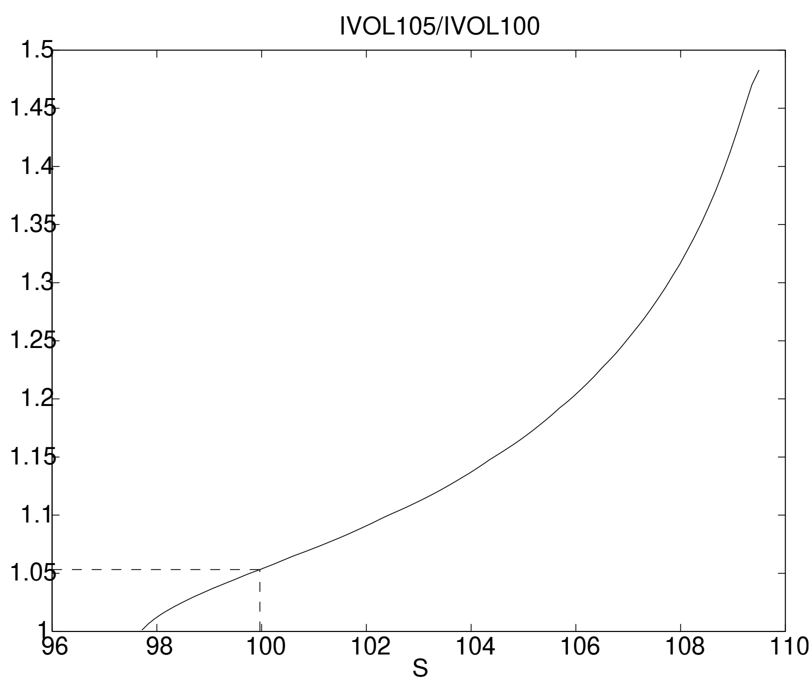
March 10 closing: TMX= 28.625 Assume rate= 7%

Mat.	Strike	Call	Ivol	Put
Mar	25	3.6250	0.0000(∗)	0.25
Mar	30	0.4375	0.5938	2.0000
Mar	35	0.0625	0.8345	6.250
Mar	40	0.0625	1.3221	12.6250
Apr	22.5	6.6250	0.7028	0.4375
Apr	25	4.5000	0.6306	0.875
Apr	30	1.5000	0.5642	2.750
Apr	35	0.3750	0.5675	6.50
May	22.5	6.8750	0.5888	0.6875
May	25	5.1250	0.6193	1.25
May	30	2.3125	0.5794	3.50
May	35	3.8125	0.5478	7.250
May	40	0.3750	0.5959	12.50
Aug	22.5	7.3750	0.4251	1.25
Aug	25	5.8750	0.4764	2.125
Aug	30	3.2500	0.4689	4.250
Aug	35	1.7500	0.4777	7.625
Aug	40	0.8750	0.4761	14.00

The picture in terms of implied volatilies

(100 Call = \$10, 105 call computed)





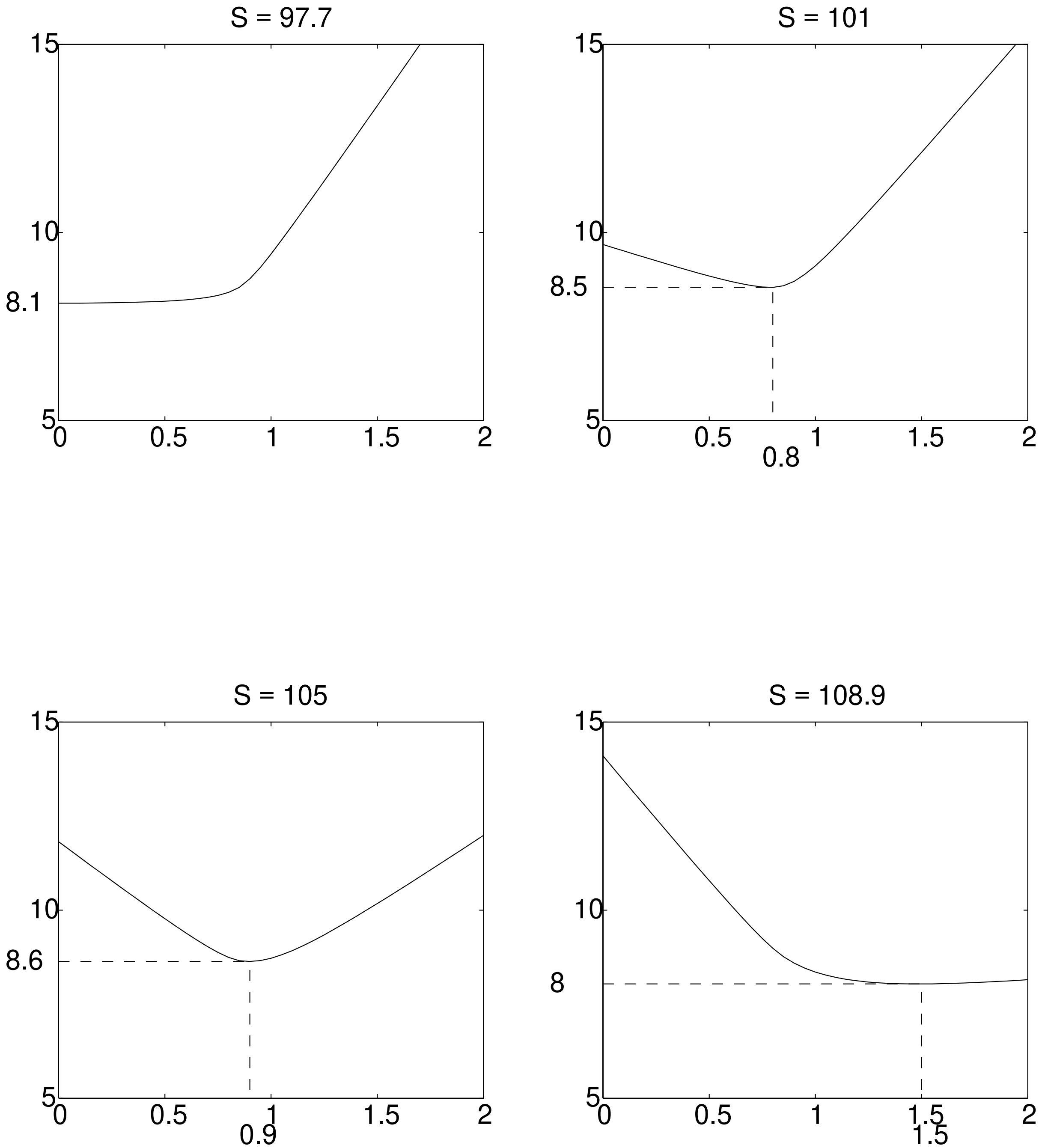
At the money (S=100) IVOL 100 = 0.35, IVOL 105 = 0.367

Example: hedging a call with a listed call

Single listed derivative: 100 Call with 6 mo. maturity

Objective : Price/ hedge a 105 Call with 6 mo. maturity

100 call @ \$10 , $\sigma_{min} = 0.1$, $\sigma_{max} = 0.4$, $r = 0$



Hedging with other derivative securities (cont.)

Optimal bidding price, given market

$$W^-(S,t,C_1 \ldots, C_M) \;=\; \underset{\lambda_i}{Sup}\;\; \Psi\,[\lambda_1\,\lambda_2\,\ldots,\lambda_M\;]$$

$$\Psi[\lambda_1\;,\;\lambda_2\;,\ldots\lambda_M\;] \;=$$

$$\underset{P}{Inf}\;\; \mathrm{E}_t^P\left[F \;-\; \sum_{i=1}^{M}\,\lambda_i\;e^{-r(t_i-t)}\,G_i(S_{t_i})\;\right] \;+\; \sum_{i=1}^{M}\,\lambda_i\;C_i$$

- $\Phi\,[\lambda_1\,\lambda_2\,\ldots,\lambda_M\;]$ is convex
- $\Psi\,[\lambda_1\,\lambda_2\,\ldots,\lambda_M\;]$ is concave
- Solutions to Inf or Sup problems exist **unless** there is an arbitrage oppportunity in the market prices $C_1\;,\;C_2\ldots,C_M$ with the *a-priori* bounds σ_{min} and σ_{max}
- Φ and Ψ are evaluated using the BSB equation
- The minimizer (maximizer) $\lambda_1^*\,\lambda_2^*\,\ldots,\lambda_M^*$ gives the optimal hedge using marketed derivatives
- Balance **risk** and **cost**

Application of UVM: hedging derivative portfolios using listed derivative securities

Hedge derivatives position by

- offsetting the position with marketed derivatives
- pricing & hedging the "residual" using UVM .

Assume M marketed (listed) derivatives

Payoffs: $G_i(S_{t_i})$, $i = 1, ..., M$

Market prices: C_i , $1 = 1, .., M$

$$F \;=\; \sum_{j=1}^{N} e^{-r(t_j - t)}\, F_j \;=\; \text{discounted payoff of OTC derivative or book}$$

Optimal asking price, given market

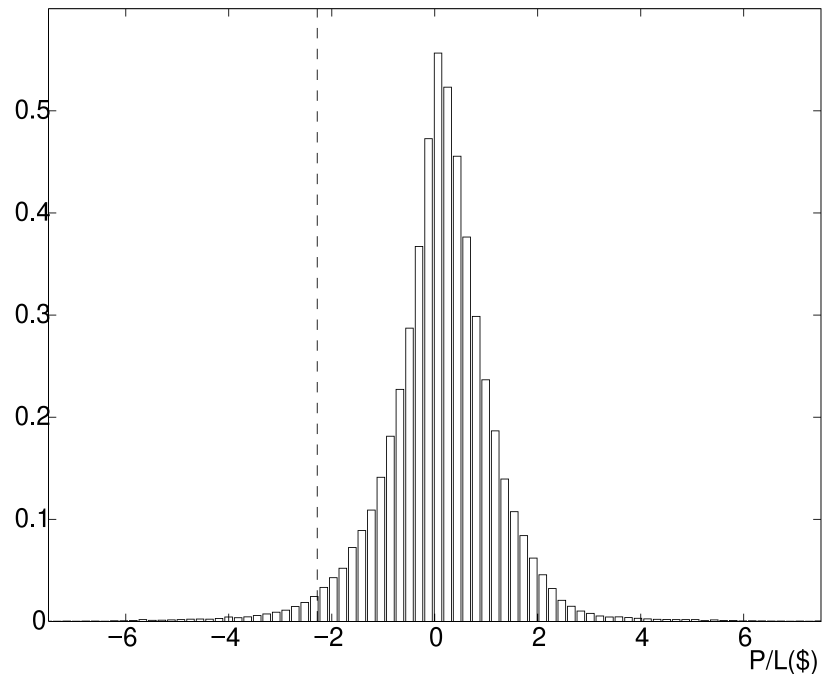
$$W^+(S, t, C_1 ..., C_M) \;=\; \underset{\lambda_i}{Inf}\;\; \Phi\,[\lambda_1\, \lambda_2\; ..., \lambda_M\;]$$

$$\Phi[\lambda_1\;,\;\lambda_2\;, ...\lambda_M\;] \;=$$

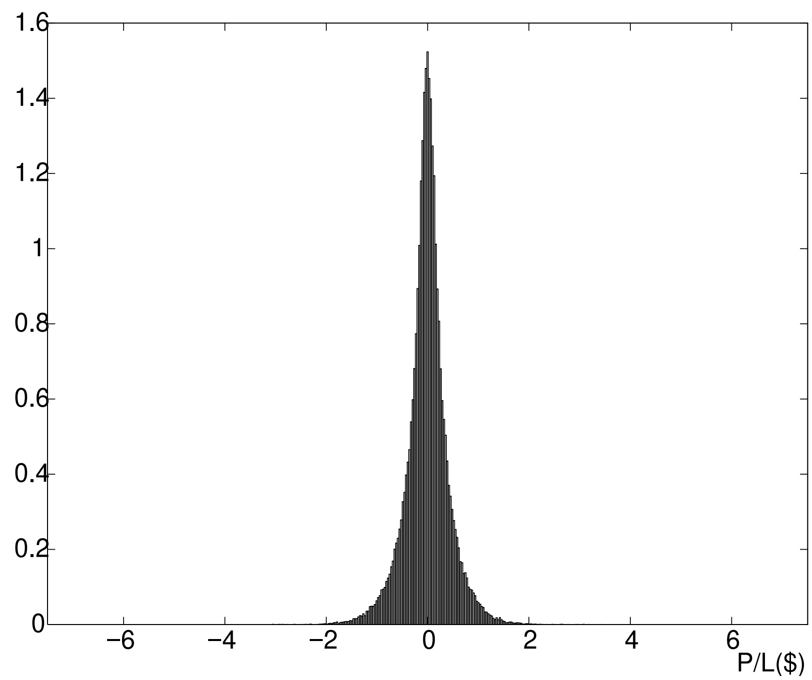
$$\underset{P}{Sup}\;\; \mathrm{E}_t^P\left[F \;-\; \sum_{i=1}^{M}\; \lambda_i\; e^{-r(t_i - t)}\, G_i(S_{t_i})\;\right] \;+\; \sum_{i=1}^{M}\; \lambda_i\; C_i$$

Residual (UVM) Market

Hedging: Stochastic vol vs. constant vol BS models



Stochastic vol

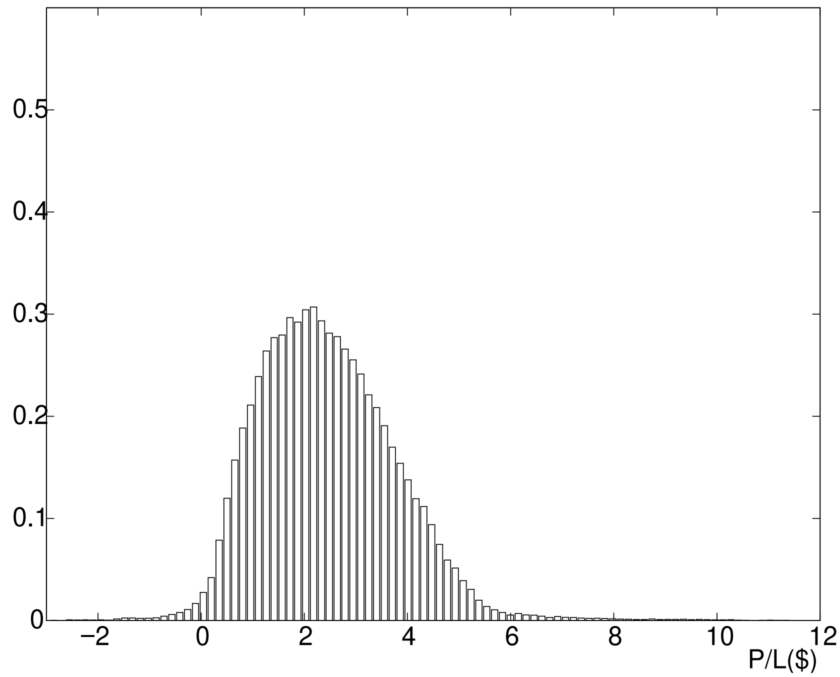


$\sigma = 0.2214$

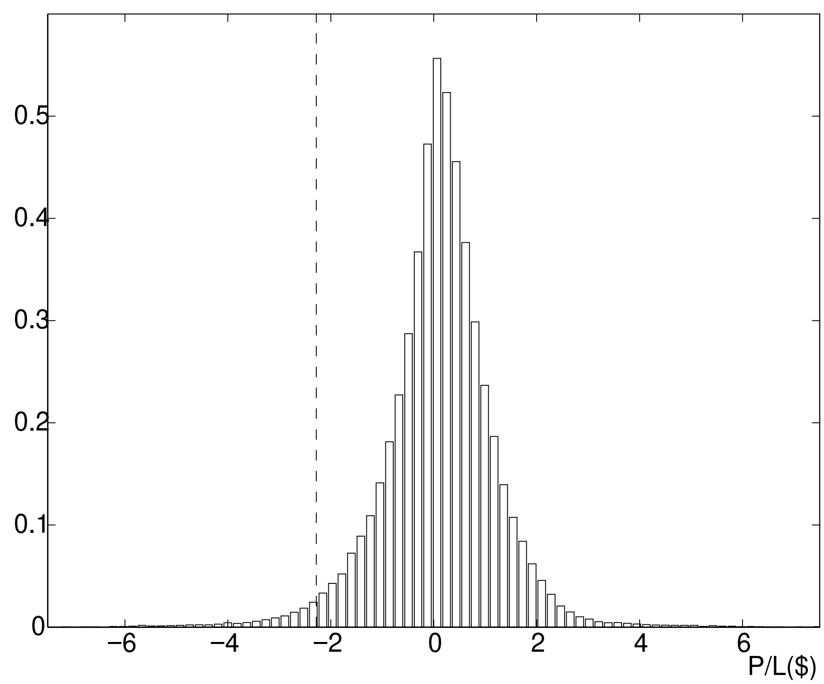
BSB hedge: optimal control on losses

Hedging a Call-Spread: BSB vs. Black-Scholes w/ constant $\sigma = 0.2214$

6-mo. maturity, $S_t = \$88$. $100,000$ Monte Carlo runs



BSB hedge



BS hedge

Hedging with Stochastic Volatility

Log-normal, mean-reverting volatility paths.

$$dS_t \;=\; S_t \left(\sigma_t \; dZ_t \;+\; r \; dt \right)$$

$$\sigma_t \;=\; \exp\left(X_t \right) \;\; ,$$

$$d\,X_t \;=\; \alpha\left(\gamma \;-\; X_t \right) dt \;+\; \rho\, dW_t$$

$\alpha = 0.693 = \ln 2$, $\gamma = -1.609 = \ln 0.2$,
$\rho \;=\; 2 \times\; (\ln 2)^{1/2} / Z_{95} \;=\; 0.702$, $< \sigma_t >= 0.2$,
$< \sigma_t^2 >^{1/2} = 0.2214$.

- $0.1 - 0.4 \;=\;$ centered 90% confidence interval for vol.
- $0.1 - 0.4 \;\approx\;$ 96% conf. int. for 6-month periods

Hedging

Solution to BSB equation gives rise to **risk-averse** delta-hedging strategy

<u>Short position</u>

- charge (or book) initial premium $W^+(S_t, t)$
- borrow/lend $W^+(S_t, t) \; - \; \frac{\partial W^+(S_t,t)}{\partial S} \; S_t$ \$ (m. mkt.)
- buy/sell $\frac{\partial W^+(S_t,t)}{\partial S}$ shares
- dynamical hedge ratio $= \frac{\partial W^+(S_\tau,\tau)}{\partial S}$ for $t \; \leq \; \tau \; \leq \; T = t_N$

$\Delta_\tau \; = \;$ no. shares held , $B_\tau \; =$ money mkt. acct.

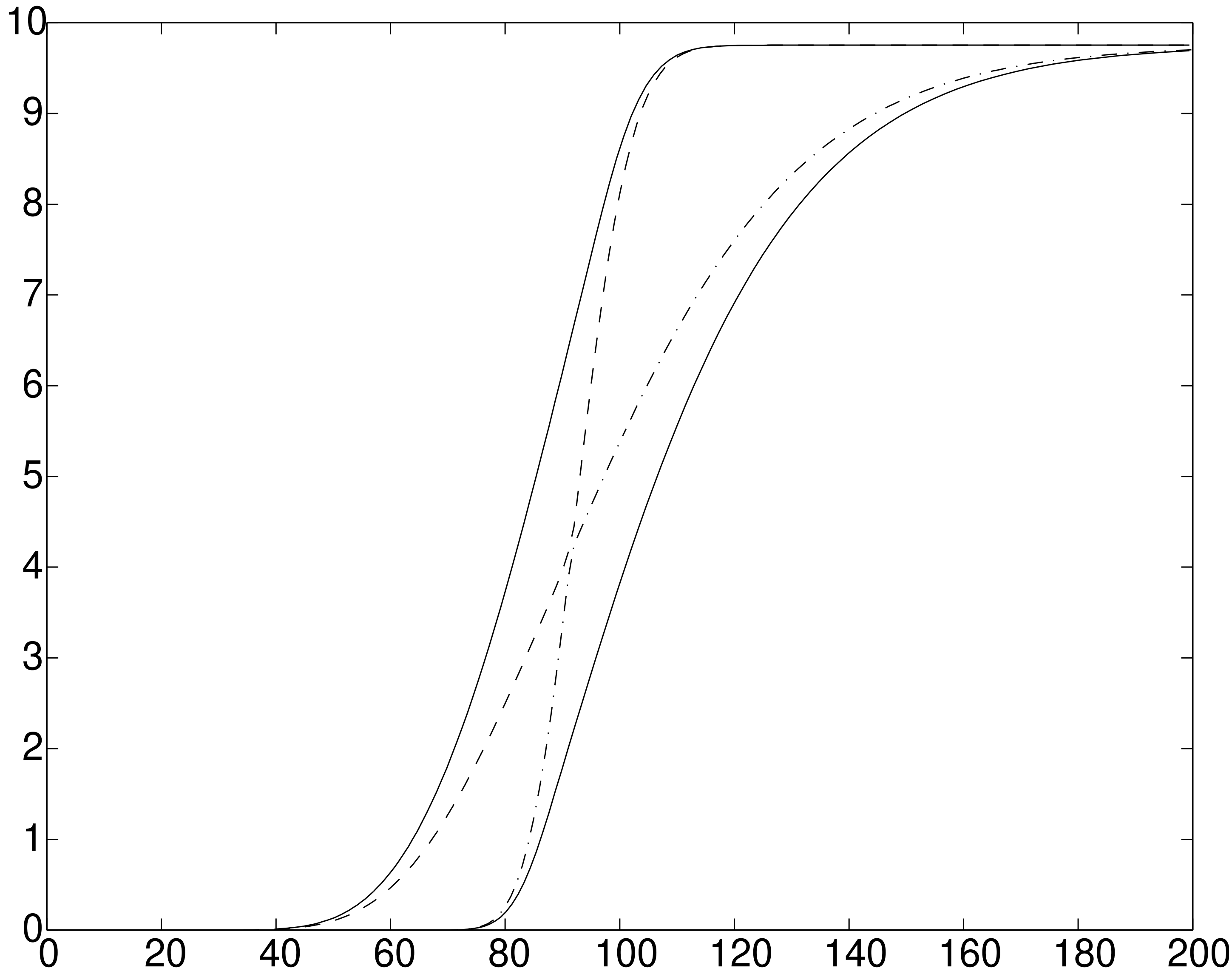
<u>value of the portfolio</u> : $V_\tau = \; \Delta_\tau \; S_\tau \; + \; B_\tau$

$\left(\; V_\tau \; \geq \; W^+(S_\tau, \tau) \; \right)$

After paying out all cash-flows, $V_T \; \geq \; 0$

Call-spread: BSB pricing vs. pooled pricing using constant volatility and Black-Scholes

Agent has volatility risk if he or she uses any constant σ in the range $\sigma_{min} \leq \sigma \leq \sigma_{max}$

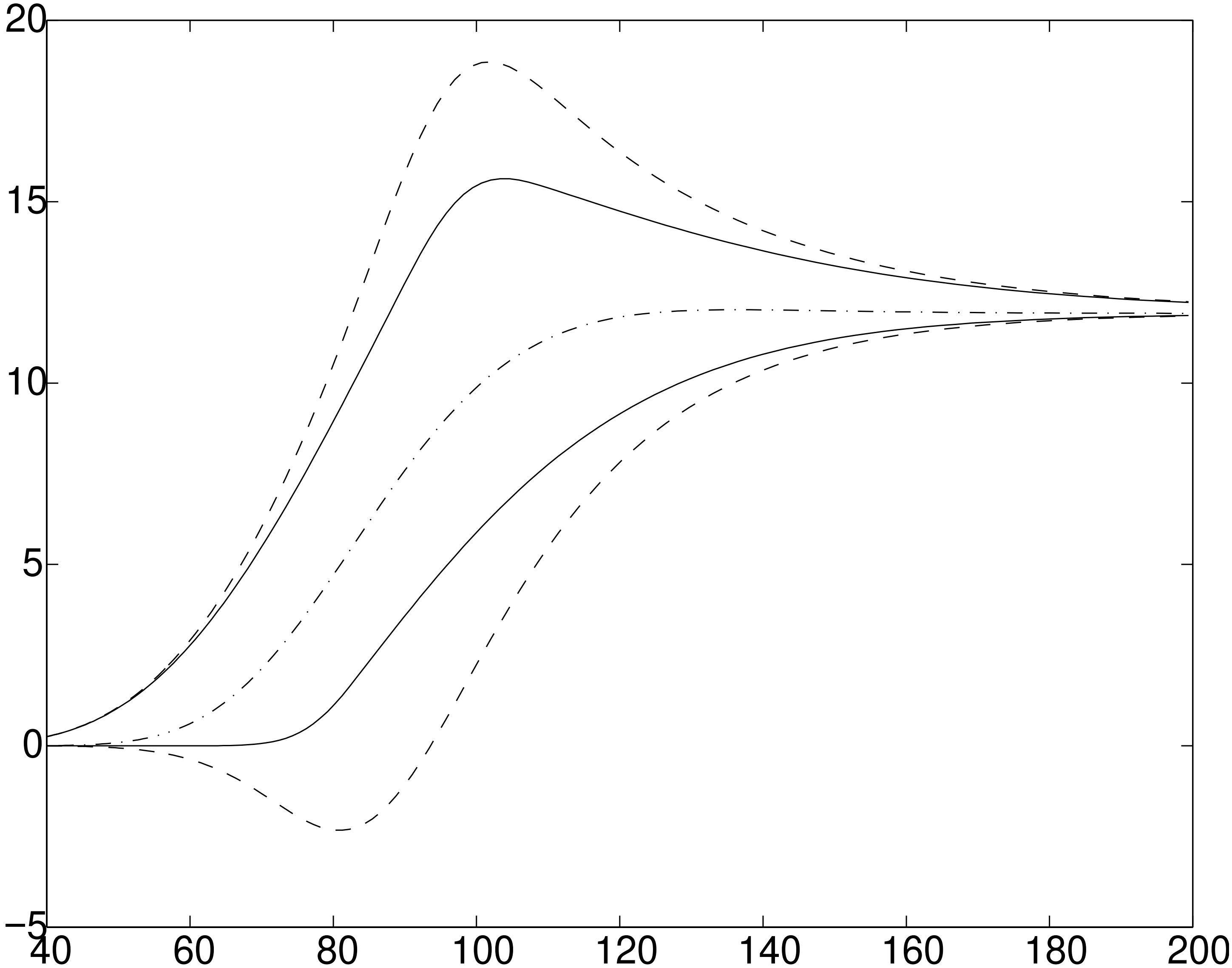


Dashed lines = upper/lower envelopes of BS with constant vol

Calendar spread: pooled vs. separate pricing

Short pos. : short 1 90 call with 6 mo. to maturity
long 1 100 call with 1 yr to maturity

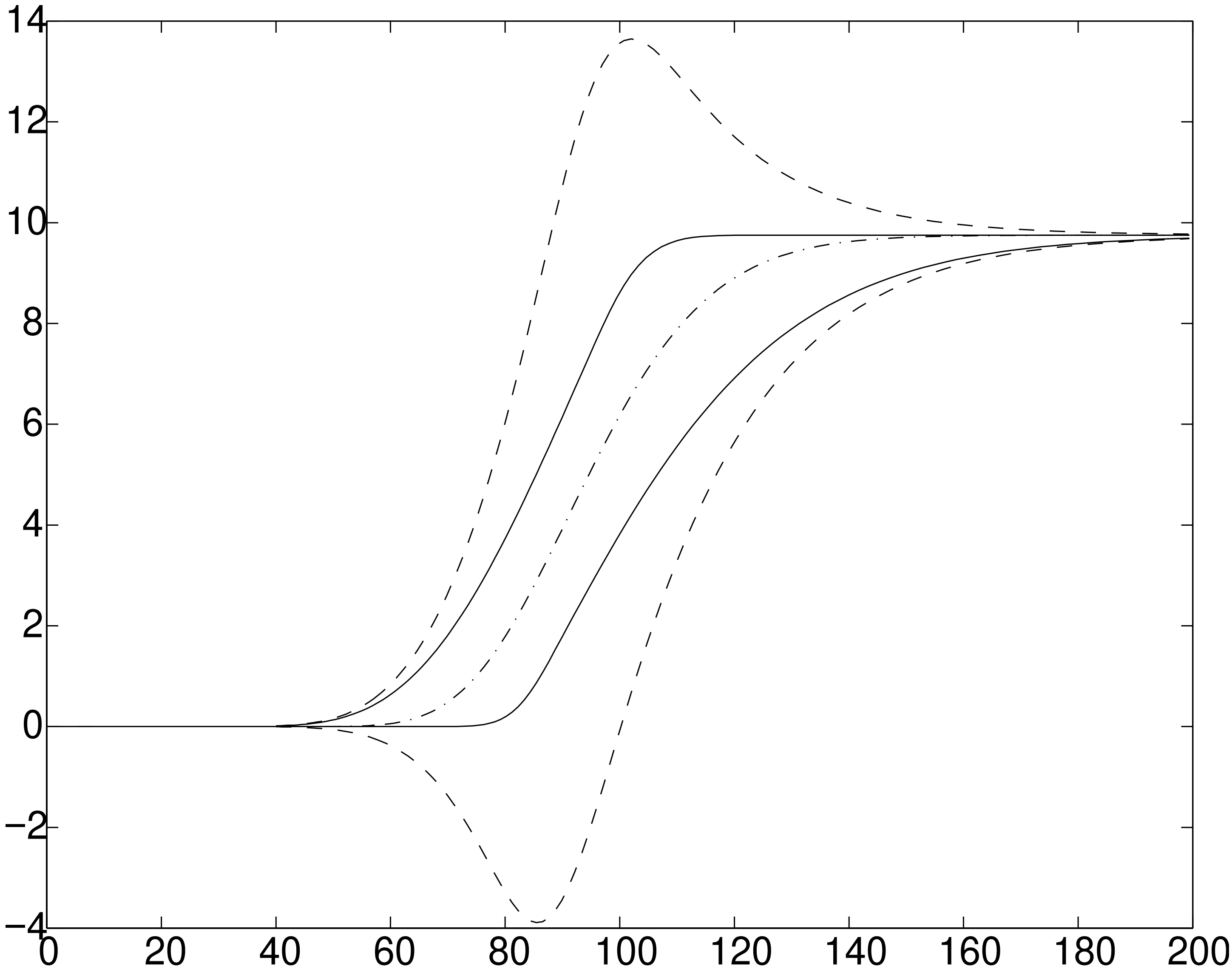
$\sigma_{min} = 0.1$, $\sigma_{max} = 0.4$



Call-spread: "pooled" pricing (BSB) vs. separate pricing (BS) with extreme vols

Short pos. : short 1 90 Call, long 1 100 call , 6 mo. to maturity

$\sigma_{min} = 0.1$, $\sigma_{max} = 0.4$



A basic property of UVM: diversification of volatility risk

Let

$$F \;=\; \sum_{j=1}^{N} \; e^{-r(t-t_j)} \, F_j(S_{t_j}),$$

$$G \;=\; \sum_{k=1}^{N'} \; e^{-r(t-t_k)} \, G_k(S_{t_k})$$

represent the discounted payoffs of two derivatives. Then,

$$\underset{P}{Sup} \; \mathrm{E}_t^P \left[\, F \; + \; G \, \right] \;\leq\; \underset{P}{Sup} \; \mathrm{E}_t^P \left[\, F \, \right] \; + \; \underset{P}{Sup} \; \mathrm{E}_t^P \left[\, G \, \right]$$

$$\underset{P}{Inf} \; \mathrm{E}_t^P \left[\, F \; + \; G \, \right] \;\geq\; \underset{P}{Inf} \; \mathrm{E}_t^P \left[\, F \, \right] \; + \; \underset{P}{Inf} \; \mathrm{E}_t^P \left[\, G \, \right]$$

The UVM captures risk-diversification effects in derivatives portfolios.

Pricing with the UVM

- **Options** are priced with the Black-Scholes equation with σ_{min} ("bid") and σ_{max} ("offer")

- Option portfolios and other derivative securities are priced with the "worst-case volatility paths"

$$\sigma_t \;=\; \sigma\left[\;\frac{\partial^2\; W(S_t\;,\;t)}{\partial S^2}\right]$$

with vol oscillating randomly between σ_{min} and σ_{max}

- Agent short the derivative security will use σ_{max} if he is "short Gamma" and σ_{min} if he is "long Gamma " — after solving BSB (not BS).

Finding extremal prices: the Black-Scholes-Barenblatt equation (BSB)

$$\frac{\partial W(S,t)}{\partial t} \;+\; \frac{1}{2} S^2 \; \sigma^2 \Big[\; \frac{\partial^2 W(S,t)}{\partial S^2} \; \Big] \cdot \frac{\partial^2 W(S,t)}{\partial S^2} \;+$$

$$r \left(\, S \, \frac{\partial W(S,t)}{\partial S} \; - W(S,t) \; \right) \;=\; \sum_{t_k \;>\; t}^{N-1} \; F_k \,(S) \cdot \delta(t \;-\; t_k)$$

$$W(S, t_N) \;=\; F_N(S) \;.$$

- For W^+, $\sigma\,[\,\Gamma\,] \;=\; \sigma_{min}$ if $\Gamma \;< 0$

  $\;=\; \sigma_{max}$ if $\Gamma \;\geq\; 0$

- For W^-, $\sigma\,[\,\Gamma\,] \;=\; \sigma_{min}$ if $\Gamma \;\geq 0$

  $\;=\; \sigma_{max}$ if $\Gamma \;<\; 0$

Extremal non-arbitrageable prices

For a derivative security with cash-flows $F_j(\, S_{t_j}\,)$,
$j \; = \; 1, 2, ..., N$

$$W^{+}(\, S_t \, , \, t) \; = \; \underset{P}{Sup} \;\; \mathrm{E}_t^P \, [\, \sum_{j=1}^{N} \, e^{- \, r \, (t_j \, - \, t)} \;\; F_j(\, S_{t_j} \,) \,]$$

$$W^{-}(\, S_t \, , \, t) \; = \; \underset{P}{Inf} \;\; \mathrm{E}_t^P \, [\, \sum_{j=1}^{N} \, e^{- \, r \, (t_j \, - \, t)} \;\; F_j(\, S_{t_j} \,) \,]$$

- Determined by worst-case scenarios for forward vol paths(short, long positions).

The Uncertain Volatility Model (UVM)

(Avellaneda, Levy, Parás, 1994, 1995)

Risk-averse approach for hedging volatility risk (large vol changes)

- Agent assumes that the forward volatility **path** $\sigma_\tau,\ t \le \tau \le T$ remains between two bounds:

$$\sigma_{min} \le \sigma_\tau \le \sigma_{max}$$

and that, under any admissible martingale measure P,

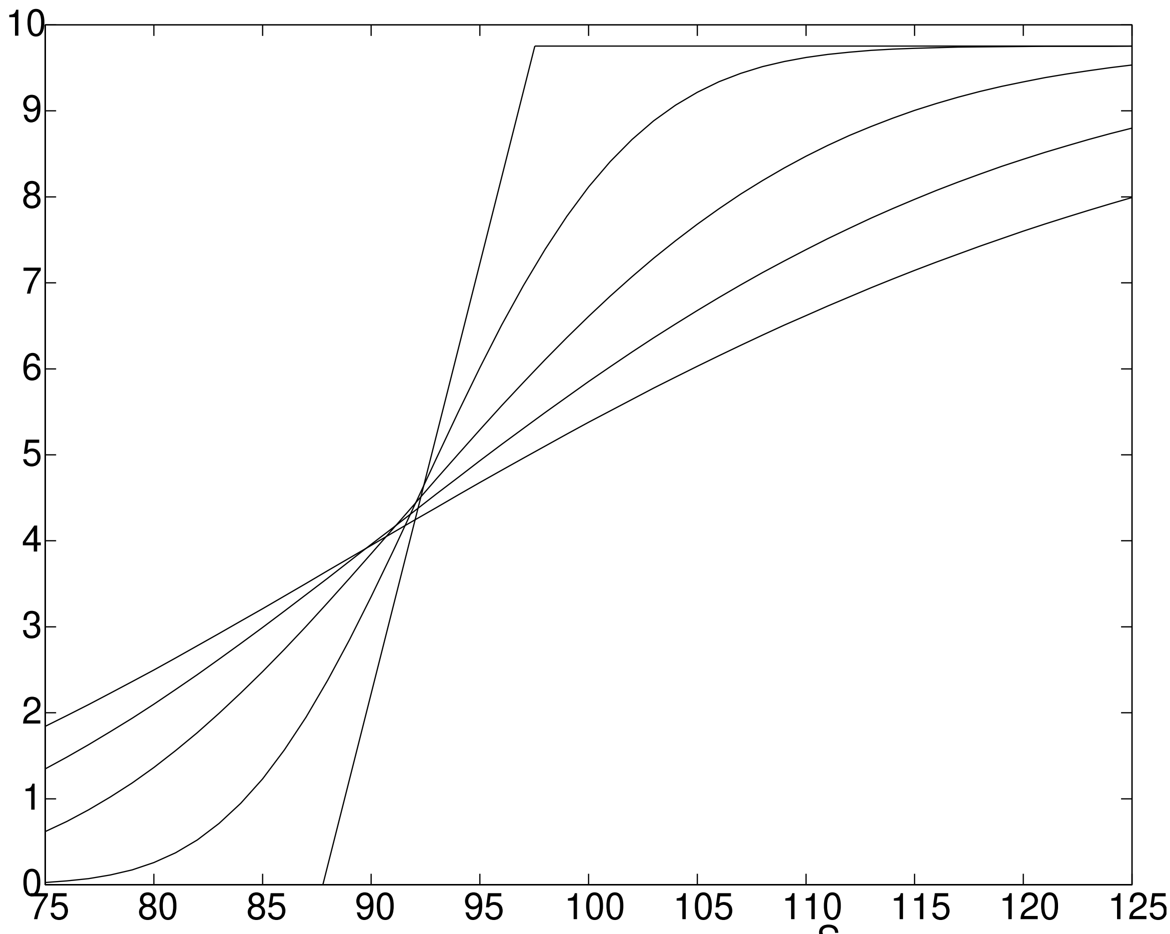
$$dS_t = S_t \left(\sigma_t \, dZ_t + r dt \right) .$$

- Bounds are determined from extreme values of implied vols, historical or seasonal info., agent's aversion to volatility risk. (Not necess. constant in time .)
- Jumps in σ_t are allowed

Pricing a Call-Spread using Black-Scholes with different vols

Short position = short 1 call with strike K_1, long 1 call with strike $K_2 > K_1$.

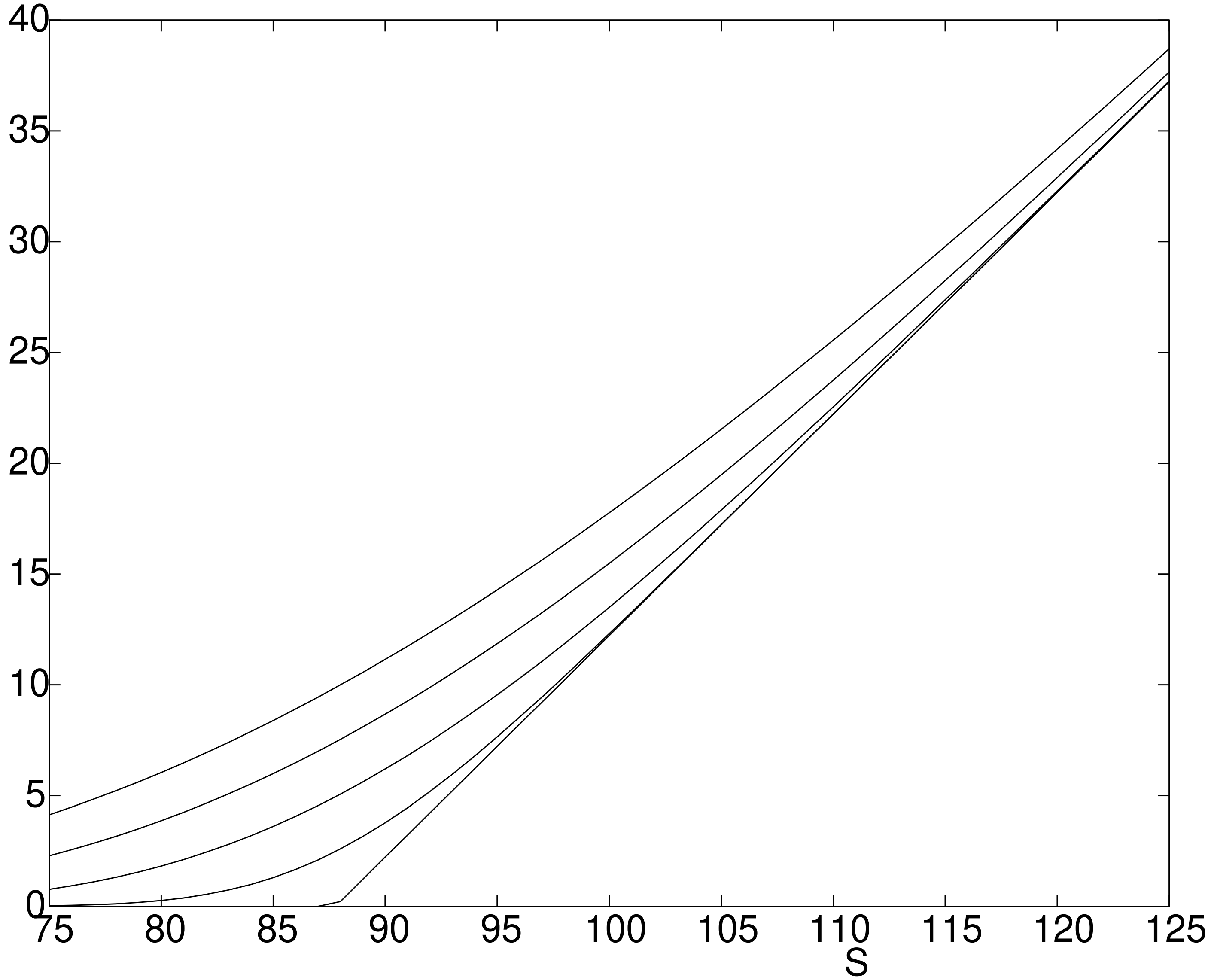
Short position exposed to **rise** in vol out-of-the-money and **drop** in vol in-the-money



short 90 Call, long 100 Call, 6 mo. to maturity, int 5%
Vol=0.1, 0.2, 0.3, 0.4

Black-Scholes option prices with different vols.

Short position exposed to **rise** in vol



90 Call, 6 mo. to maturity, interest rate 5%

Vol=0.1, 0.2, 0.3, 0.4

Option pricing, implied volatility
& forward volatility

Implied vol is not constant: imperfect correlation between underlying index and option prices.

- Implied volatility depends on moneyness, $S_t - K$
- depends on time to maturity
- is affected by **new information**

Vega = commonly used measure of volatility risk =

$$\frac{S_t \sqrt{T-t}}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2\sigma^2 (T-t)} \left(\ln \frac{S_t \, e^{r(T-t)}}{K} + \frac{\sigma^2 \, (T-t)}{2} \right)^2 \right\}$$

Sensitivity to **infinitesimal** changes in vol.

Arbitrage Pricing Theory

Fair Value of a Security = Discounted expected divs.

- There exists a *pricing measure* or *martingale measure* such that every traded security satisfies

$$S_t = \mathrm{E}_t^P \left\{ e^{-\int_t^T (r_s - q_s)\, ds} S_T \right\}$$

(Harrison and Kreps, 1979; Duffie, 1992)

In practice P is not unique.

NON-LINEAR PDES AND DIVERSIFICATION OF VOLATILITY RISK IN DERIVATIVES MARKETS

Institute for Advanced Study, March 16, 1995

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