On the Martingale Problem for Jumping Diffusions

M.A.H. Dempster

Centre for Financial Research

Judge Institute of Management Studies

University of Cambridge, Cambridge, England.

mahd2@cam.ac.uk

www-cfr.jims.cam.ac.uk

and

G.Ch. Gotsis

Hellenic Capital Market Commission

Syntagma Sq., Athens, Greece.

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Abstract

Jumping diffusion models for financial prices and returns are finding increasing application in the pricing of current contingent claims. Generalizing the theory for a Markov process with continuous sample paths – characterized in terms of its infinitesimal generator – we establish existence and uniqueness of solutions to jump stochastic differential equations. We adapt the Stroock and Varadhan approach, which develops a variant of the 'weak sense' solution of a stochastic differential equation by formulating it as a diffusion process solution of a *martingale problem*. Our approach to deriving the integro-partial differential equation for the value of a contingent claim through the corresponding Kolmogorov forward equation is illustrated by a generalization of the recent work of Babbs and Webber describing fixed income derivative valuation in the presence of central bank rate changes.

Keywords: jumping diffusions, Markov processes, martingale problem, semimartingales, fixed income derivatives.

1 Introduction

The concept of *jump-diffusion* Markov process models of financial prices and returns is finding increasing application in pricing and hedging ever more complex contingent claims. Some of these applications include equity index jumps [24, 18], central bank rate changes [3, 19], credit rating changes [7], catastrophe risk [1], etc. On the other hand, usually either formal statements are simply assumed correct or practically restrictive assumptions are made in setting out the corresponding *integro-partial differential equation* (PDE) for claim valuation and hedging. In this paper, we marshall appropriate results from the mathematical literature to establish the existence and uniqueness of a weak solution to a general *jump stochastic differential equation* (SDE) in terms of the corresponding *martingale problem* stated in terms of the infinitesimal generator of the Markov solution process. Although, for example, non-Markovian versions of Heath-Jarrow-Morton compatible short term interest rate models are possible [5], Markov models are almost universally used in practice. We illustrate the use of the solution to a martingale problem for a Markov process in a typical financial application under a risk neutral measure to derive rigorously the integro-PDE satisfied by the claim value. The machinery treated in this paper can be applied *ceteris paribus* to all the financial situations mentioned above – and many more involving point process events.

A number of authors have studied the existence and uniqueness of solutions to stochastic differential equations whose solutions are termed *diffusion* processes. Loosely speaking the term diffusion is attributed to a Markov process which has continuous sample paths and can be characterized in terms of its *infinitesimal generator*. The latter is specified in terms of drift and diffusion coefficients which have natural interpretations in financial modelling.

There are several approaches to the study of diffusions ranging from the purely analytical to the purely probabilistic. The methodology of *stochastic differential equations* (SDEs) was suggested by P. Levy [17] as an 'alternative' probabilistic approach to the deterministic theory of heat diffusion and was carried out by K. Ito. [14]. His *strong solution* is constructed on a given probability space, with respect to a given filtration and a given Brownian motion $\mathbf{W}$. The idea of *weak solution* is a notion in which the probability space, the filtration and the driving Brownian motion are all part of the solution, rather than the statement, of the problem. Stroock and Varadhan [22, 23] developed a variation of the solution of a stochastic differential equation in the 'weak sense' by formulating the search for a diffusion process with given drift and dispersion coefficients in terms of the martingale problem. Uniqueness of the solution process is required in the sense of its finite-dimensional distributions and the continuity of its sample paths. Although finding this process is equivalent to solving the related stochastic differential equation in the weak sense, the SDE is not explicitly involved. Rather, the infinitesimal evolution of suitable functionals of the solution process – specified by the infinitesimal characteristics of the process through its *infinitesimal generator* – are determined up to an additive martingale error term.

In this paper we consider the existence and uniqueness of the solution to the Stroock-Varadhan [23] martingale problem for Markov *semimartingales*, i.e. jumping diffusions. The next section contains notation and preliminaries, after which the statements and proofs of the main results are given in Section 3. Loosely speaking, these theorems imply that the *infinitesimal characteristics* of a jumping diffusion – drift, dispersion, jump rate and post-jump measure – together with the requirement that its sample paths be right continuous and left limited, uniquely determine the solution process of the martingale problem. In Section 4, an illustrative application of this practically important result is given to a contemporary problem in mathematical finance. However the techniques developed in this paper are much more broadly applicable, as is noted in the conclusions of Section 5.

2 Notation and Preliminaries

In this section we review some basic definitions and results from Markov process theory [8, 11, 16, 23].

Throughout the paper $(\Omega, \mathcal{F}, P)$ denotes a *probability space*, E is a metric space and $\mathcal{B}(E)$ is the σ-algebra of Borel subsets of E. A collection $\{\mathcal{F}_t\} := \{\mathcal{F}_t,\ t \in [0,\infty)\}$ of σ-algebras of sets of $\mathcal{F}$ is a *filtration* iff $\mathcal{F}_t \subset \mathcal{F}_{t+s}$ for $t, s \in [0,\infty)$. For a *stochastic process* $\mathbf{X}$ we define $\mathcal{F}_t^{\mathbf{X}} := \sigma\{\mathbf{X}_s : 0 \leq s \leq t\}$ to be the $\sigma-$algebra which corresponds to the information known to an observer watching $\mathbf{X}$ up to time t. A filtration $\{\mathcal{F}_t\}$ is said to satisfy the *usual conditions* if it is increasing, right-continuous and $\mathcal{F}_0$ contains all the

P-negligible events in $\mathcal{F}$. The process $\mathbf{X}$ is *Markov* iff

$$P\{\mathbf{X}(t+s) \in B | \mathcal{F}_t^{\mathbf{X}}\} = P\{\mathbf{X}(t+s) \in B | \mathbf{X}(t)\} \tag{1}$$

for all $s, t \geq 0$ and $B \in \mathcal{B}(E)$.

Let L denote a *real Banach space* with *norm* $\| \cdot \|$ and let $M(E)$ denote the collection of all *real-valued, Borel measurable functions* on E. $B(E) \subset M(E)$ denotes the Banach space of *bounded functions* with norm $\| f \| := \sup_{x \in E} | f(x) |$. $C(E)$ denotes the subspace of $B(E)$ of *bounded continuous functions* and $C_0(E)$ denotes the Banach space of *continuous functions which vanish at infinity* (in terms of the one point compactification of E) equipped with this norm. $\mathcal{P}(E)$ denotes the family of *Borel probability measures* on E.

Taking into consideration that most of the stochastic processes which arise in financial applications have the property that they have right and left limits at each time point for almost every sample path, we denote by $D_E[0, \infty)$ the *Skorohod space* of right continuous functions $x : [0, \infty) \to E$ with left limits. We take $D_E[0, \infty)$ to be the path space of all processes considered here for a suitable space E (usually $I\!R^n$) and $C_E[0, \infty)$ will denote the subspace of $D_E[0, \infty)$ containing the continuous functions $x : [0, \infty) \to E$.

Defining a *metric* d on E we can induce a topology on the space $D_E[0, \infty)$. It is proven in Ethier and Kurtz [11], Theorem 5.6, p.121, that with the topology induced by the *Skorohod metric* $D_E[0, \infty)$ is a separable space if E is separable and is complete if (E, d) is complete. The processes of interest to us here will take values in a complete, separable metric space E and will have sample paths in $D_E[0, \infty)$ equipped with the *Skorohod topology*.

A one-parameter family $\{T(t), t \geq 0\}$ of bounded linear operators on a Banach space L with norm $\| \cdot \|$ is called a *semigroup* iff $T(0) = I$ and $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$. A semigroup $T(t)$ on L is said to be strongly *continuous* iff $\lim_{t \to 0} \| T(t)f \| = \| f \|$ for every $f \in L$; it is said to be a *contraction* semigroup iff $\| T(t) \| < 1$ for all $t \geq 0$. We define a semigroup $\{T(t)\}$ on L to be *measurable* iff $T(\cdot)f$ is a measurable function on $([0, \infty), \mathcal{B}[0, \infty))$ for each $f \in L$.

Let A be a linear operator on L, which is a linear mapping whose domain $\mathcal{D}$(A) is a subspace of L and whose range $\mathcal{R}(\mathrm{A})$, another subspace, lies in L.

A linear operator A on L is said to be *dissipative* iff $\| \mu f - Af \| \geq \mu \| f \|$ for every $f \in \mathcal{D}(\mathrm{A})$ and $\mu > 0$.

We define for $f \in L$

$$Af := \lim_{t \to 0} \frac{1}{t} [T(t)f - f] \tag{2}$$

to be the *infinitesimal generator* of a semigroup $\{T(t)\}$ on L. We also define:

$$gr\hat{A} := \{(f, g) \in L \times L : \ T(t)f - f = \int_0^t g(s)ds, \ \ t \geq 0\} \tag{3}$$

to be the graph of the *full generator* $\hat{A}$ of a measurable contraction semigroup $\{T(t)\}$ on L with extended domain $\mathcal{D}(\hat{A}) \supset \mathcal{D}(A)$.

If, for some real μ', $\mu' - A := \mu' I - A$ is one-to-one, $\mathcal{R}(\mu' - A) = L$ and $(\mu' - A)^{-1}$ is a bounded linear operator on L, then μ' is said to belong to the *resolvent set* of A, and $\mathcal{R}_{\mu'} = (\mu' - A)^{-1}$ is called the *resolvent* at μ' of A.

A set $M \subset B(E)$ is called *separating* iff whenever $P, Q \in \mathcal{P}(E)$ and $\int f dP = \int f dQ$, for all $f \in M$ we have $P = Q$. Conversely, if $P \neq Q$, then there exists $f \in \mathrm{M}$ such that $\int f dP \neq \int f dQ$, i.e. M *separates* points of $\mathcal{P}(E)$.

An operator A on $C(E)$ is said to satisfy the *positive maximum principle* iff whenever $f \in \mathcal{D}(\mathcal{A})$, $x_0 \in E$ and $\sup_{x \in E} f(x) = f(x_0) \geq 0$ we have $Af(x_0) < 0$.

A process $\mathbf{X}$ is a *martingale* with respect to the filtration $\{\mathcal{F}_t\}$ iff $\mathbb{E}[\mathbf{X}_t | \mathcal{F}_s] = \mathbf{X}_s$ *a.s.* and a *sub (super) martingale* according as the equality becomes $\leq$ $(\geq)$. If with respect to $\{\mathcal{F}_t\}$ there exists a sequence of stopping times $\boldsymbol{\tau}_n \nearrow \infty$ such that $\mathbf{X}_{(\cdot) \wedge \boldsymbol{\tau}_n}$ is a martingale for all $n = 1, 2, \ldots$, we say that $\mathbf{X}$ is a *local martingale.* All martingales are local martingales, but not conversely. An adapted process $\mathbf{X}$ is a *semimartingale* iff it has a decomposition of the form

$$\mathbf{X}_t = \mathbf{X}_0 + \mathbf{M}_t + \mathbf{A}_t,$$

where $\mathbf{X}_0$ is a random variable with distribution ρ (we write $\mathbf{X}_0 \sim \rho$), $\mathbf{M}$ is a local martingale and $\mathbf{A}$ has sample paths in $D_E[0, \infty)$ of finite variation, i.e. of bounded variation on compact time sets. Local martingales, finite variation processes, sub- and supermartingales and (for $E := I\!R_n$) n-dimensional standard Brownian motion $\mathbf{W}$, with uncorrelated Wiener coordinate processes, are all semimartingales.

By a solution of the *martingale problem* for A we mean a measurable stochastic process $\mathbf{X}$ with values in E and sample paths in $\Omega := D_E[0, \infty)$ defined in a probability space $(\Omega, \mathcal{F}, P)$, where is the Borel σ-field relative to the Skorohod topology, such that for each $(f, g) \in A$, i.e. $g := Af$,

$$f(\mathbf{X}(t)) - \int_0^t g(\mathbf{X}(s))ds \tag{4}$$

is a martingale with respect to the filtration $\mathcal{F}_t^{\mathbf{X}}$ for all $f \in B(E)$. Equivalently, we say that P solves the martingale problem for A.

When an initial distribution $\rho \in \mathcal{P}(E)$ is specified for the solution process $\mathbf{X}$ and (4) holds, we say that $\mathbf{X}$ – equivalently P – is a *solution* of the martingale problem for (A, ρ) and without loss of generality write (4) in the form

$$f(\mathbf{X}(t)) - f(\mathbf{X}(0)) - \int_0^t g(\mathbf{X}(s))ds. \tag{5}$$

If a solution of the martingale problem for (A, ρ) exists and is unique, we say that the martingale problem for (A, ρ) is *well-posed.*

Denoting by Δ the martingale process $\mathbf{X}$ with $\mathbf{X}(0) \sim \rho$ given by (5), we have for all $f \in B(E)$

$$f(\mathbf{X}(t)) = f(\mathbf{X}(0)) + \int_0^t Af(\mathbf{X}(s))ds + \Delta(t), \tag{6}$$

which is a stochastic version of the fundamental theorem of calculus due to Dynkin [8] involving the 0-mean martingale *error* Δ. Taking expectations conditional on $\mathbf{X}(0) = x$, using Lebesgue's dominated convergence theorem and differentiating with respect to time yields the *Kolmogorov forward equation* for $\mathbf{E}_x f(\mathbf{X}_t)$ as

$$\frac{\partial}{\partial t}\mathbf{E}_x f(\mathbf{X}(t)) - A\mathbf{E}_x f(\mathbf{X}(t)) = 0. \tag{7}$$

Finally, consider the Borel measurable *drift* vector function $b : [0,\infty)\times I\!R^n \to I\!R^n$, $(t,x) \to b(t,x)$ and *volatility* matrix function $\sigma : [0,\infty)\times I\!R^n \to I\!R^{n^2}$, $(t,x) \to \sigma(t,x) := (\sigma_{ij}(t,x))$. The *diffusion* matrix function $a : [0,\infty)\times I\!R^n \to I\!R^{n^2}$, $(t,x) \to a(t,x) := (a_{ij}(t,x))$ is defined by the *diffusion* (covariance) coefficients

$$a_{ij}(t,x) := \sum_{k=1}^{n} \sigma_{ik}(t,x)\sigma_{kj}(t,x).$$

We define a *weak solution* of the *stochastic differential equation* (SDE)

$$d\mathbf{X}(t) = b(t,\mathbf{X}(t))dt + \sigma(t,\mathbf{X}(t))d\mathbf{W}(t) \tag{8}$$

to be a triple $\{(\mathbf{X},\mathbf{W}),(\Omega,\mathcal{F},P),\{\mathcal{F}_t\}\}$ where:-

(i) $(\Omega,\mathcal{F},P)$ is a probability space.

(ii) $\{\mathcal{F}_t\}$ is a filtration of sub-σ-fields of $\mathcal{F}$ satisfying the usual conditions.

(iii) $\mathbf{X} = \mathbf{X}(t)$, $0 \le t < \infty$, is a continuous adapted $I\!R^n$-valued process and $\mathbf{X}$ has sample paths in $C_{I\!R^n}[0,\infty)$,

(iv) $\mathbf{W} := \{\mathbf{W}(t) : 0 \le t < \infty\}$ is an n-dimensional standard Brownian motion and the following conditions are satisfied:

  (a) $P\{\int_0^t [\,|b_i(s,\mathbf{X}(s))| + \sigma_{ij}^2(s,\mathbf{X}(s))\,]\,ds \le \infty\} = 1$ holds for every $1 \le i,j \le n$ and $0 \le t < \infty$, and

  (b) the integral version of (8)

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t b(s,\mathbf{X}(s))ds + \int_0^t \sigma(s,\mathbf{X}(s))d\mathbf{W}(s) \quad 0 \le t < \infty \tag{9}$$

  holds almost surely.

2.1 The martingale problem and weak solution of SDEs

The following propositions demonstrate the existence of a unique solution to a martingale problem for diffusions in $I\!R^n$ and the equivalence of the martingale problem solution and a weak solution to the corresponding stochastic differential equation. A weak solution to the SDE (8) induces on $C_{I\!R^n}[0,\infty) \subset D_{I\!R^n}[0,\infty)$ a probability measure P which solves the martingale problem (A,ρ), for A defined for all $f \in B(I\!R^n)$ by

$$Af(t,x) := \frac{\partial f(t,x)}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 f(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x)\frac{\partial f(t,x)}{\partial x}, \tag{10}$$

and conversely.

Proposition 1. *Let $a(t,x)$ and $b(t,x)$ be bounded and Borel measurable. Suppose that for every $x \in I\!R^n$ and for all ϕ from a given separating set $M \subset C_{I\!R^n}[0,\infty)$*

$$E^{P_1}[\phi(\mathbf{X}(t))] = E^{P_2}[\phi(\mathbf{X}(t))], \tag{11}$$

where the probability measures P_1 and P_2 both solve the martingale problem for (A,ρ) starting from (s,x). Then $P_1 = P_2$, i.e. starting from (s,x) there is at most one solution to the martingale problem (A,ρ) with sample paths in $C_E[0,\infty)$.

Proof : See Stroock and Varadhan [23], Theorem 6.2.3, p.147. ■

The next general proposition shows existence of solution processes $\mathbf{X}$ to the martingale problem in locally compact spaces E such as $I\!R^n$.

Proposition 2. *Let E be locally compact and separable and let A be a linear operator on $C_0(E)$. Suppose that $\mathcal{D}(A)$ is dense in $C_0(E)$ and that A satisfies the positive maximum principle. Define the linear operator A^Δ on $C(E^\Delta)$, where $E^\Delta := E \cup \{\Delta\}$ is the one-point compactification of E, by*

$$(A^\Delta f)|_E := A(f - f(\Delta))|_E \qquad A^\Delta f(\Delta) := 0 \tag{12}$$

for all $f \in C(E^\Delta)$, so that $(f - f(\Delta))|_E \in \mathcal{D}(A)$. Then for each $\rho \in \mathcal{P}(E^\Delta)$ there exists a *solution of the martingale problem for* (A^{Δ}, ρ) *with sample paths in* $D_E[0, \infty)$.

Proof : See Ethier and Kurtz [11], Theorem 5.4, p.199. ■

Proposition 3. *The existence of a solution* P *on* $C_{I\!R^n}[0, \infty)$ *to the martingale problem* (A, ρ) *for* A *defined by (10) is equivalent to the existence of a weak solution* $\{(\mathbf{X}, \mathbf{W}), (\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}\}$ *to the stochastic integral equation (9) or, formally, the stochastic differential equation (8) with initial condition* $\boldsymbol{X}(0) \sim \rho$.

Proof : Since a martingale is a local martingale, we may apply Karatzas and Shreve [16], Proposition 4.6, p.315, to show the existence of the required weak solution. The converse follows from Ito's lemma and will be established more generally in Theorem 2 below. ■

3 Main Results

We define a *jumping diffusion* to be a solution process $\mathbf{X}$ in $I\!R^n$ of the *jump stochastic differential equation*

$$d\mathbf{X}(t) = b(t, \mathbf{X}(t_-))dt + \sigma(t, \mathbf{X}(t_-))d\mathbf{W}(t) + \Delta\mathbf{X}(t_-), \tag{13}$$

where the jump saltus $\Delta\mathbf{X}(\boldsymbol{\tau}_-)$ of $\mathbf{X}$ at the jump epoch $\boldsymbol{\tau}$ has infinitesimal characteristics at $(t, x) \in [0, \infty) \times I\!R^n$ given by $\lambda(t, x)$ and $Q(t, x, \cdot)$. Here $\lambda \in B([0, \infty) \times I\!R^n)$ is the nonnegative *jump rate* function and $Q : [0, \infty) \times I\!R^n \to \mathcal{P}(I\!R^n)$ is the *post-jump (transition probability) measure.* The term $\Delta\mathbf{X}(t_-)$ is zero at non-jump epochs $t \in [0, \infty)$. The remaining infinitesimal characteristics of $\mathbf{X}$ are the Borel measurable *drift* $b : [0, \infty) \times I\!R^n \to I\!R^n$ and *volatility matrix* $\sigma : [0, \infty) \times I\!R^n \to I\!R^{n^2}$ as before.

First we state the generalisation of *Ito's rule* for semimartingales.

Proposition 4. *Suppose* $\mathbf{X}$ *is a semimartingale in* $I\!R^n$ *with decomposition of the form*

$$\mathbf{X}_t = \mathbf{X}_0 + \mathbf{B}_t + \mathbf{M}_t, \tag{14}$$

for $\boldsymbol{B}$ an adapted process of finite variation and $\boldsymbol{M}$ a local martingale, and let $f: \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable with bounded first and second derivatives. Then

$$\begin{aligned} f(\mathbf{X}_t) &= f(\mathbf{X}_0) + \sum_{i=1}^{n} \int_0^t \frac{\partial}{\partial x_i} f(\mathbf{X}_{s-}) d\mathbf{X}_{is} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^t \frac{\partial^2}{\partial x_i x_j} f(\mathbf{X}_{s-}) d\langle {\mathbf{X}_i}^c, {\mathbf{X}_j}^c \rangle_s \\ &+ \sum_{\boldsymbol{\tau} \in (0,t]} (f(\mathbf{X}_s) - f(\mathbf{X}_{s-})) \quad a.s., \end{aligned} \tag{15}$$

where $\mathbf{X}^c$ is the continuous part of $\mathbf{X}$ defined for $t \in [0, \infty)$ as

$$\mathbf{X}^c(t) := \mathbf{X}(t) - \sum_{\boldsymbol{\tau} \in [0,t]} \Delta \mathbf{X}(\boldsymbol{\tau}_-) \tag{16}$$

in terms of the jump times $\boldsymbol{\tau}$ and $\langle \mathbf{X}_i^c, \mathbf{X}_j^c \rangle_t$ is the quadratic covariation, i.e. the instantaneous covariance, of ${\mathbf{X}_i}^c$ and ${\mathbf{X}_j}^c$ at time t.

Proof : See Elliott [10], p.132, where the result is obtained for $\mathbb{R}$. The extension to $\mathbb{R}^n$ is straightforward. ■

3.1 The martingale problem for time homogeneous jumping diffusions

We consider first the time homogeneous case of the martingale problem and define the *bounded* post-jump measure $Q : E \to \mathcal{P}(E)$ to have the property $Q(\cdot, B) \in B(E)$ for all $B \in \mathcal{B}(E)$.

Proposition 5. *Let (E,d) be complete and separable and let $grA_1 \subset B(E) \times B(E)$, the jump rate $\lambda \in B(E)$, Q be a bounded jump measure and A_2 be given by*

$$A_2 f(x) = \lambda(x) \int_E (f(y) - f(x)) Q(x, dy). \tag{17}$$

Suppose that for every $\rho \in \mathcal{P}(E)$ there exists a solution of the $C_E[0, \infty)$ martingale problem for (A_1, ρ). Then for every $\rho \in \mathcal{P}(E)$ there exists a solution of the $D_E[0, \infty)$ martingale problem for $(A_1 + A_2, \rho)$.

Proof : See Ethier and Kurtz [11], Proposition 10.2, p.256. ∎

Proposition 6. *Let L be a family of functions which is defined on a Borel subset of $\mathbb{R}^n$ and is closed under addition, scalar multiplication and weak-convergence. If L contains all twice continuously differentiable functions with compact support, then L contains all bounded Borel functions and is separating.*

Proof : See Dynkin [8], Lemma 5.12, p.160. ∎

Proposition 6 enables us to make use of the following theorem of Ethier and Kurtz to show the uniqueness of the solution to the martingale problem for jumping diffusions by setting the separating set $L := B(\mathbb{R}^n) \supset C_0^2(\mathbb{R}^n)$, the space of *twice continuously differentiable functions on $\mathbb{R}^n$ which vanish at infinity.*

Proposition 7. *Let E be separable and let $grA \subset B(E) \times B(E)$ be linear and dissipative. Suppose there exists a linear operator A' such that $grA' \subset grA$, $\overline{\mathcal{R}(\mu' - A')}$=$\overline{\mathcal{D}(A')}$=$L$ for some μ' and L is separating. Let $\rho \in \mathcal{P}(E)$ and suppose $\mathbf{X}$ is a solution of the martingale problem for (A, ρ). Then $\mathbf{X}$ is the Markov process corresponding to the semigroup on L generated by the closure of A' and uniqueness holds for the solution of the martingale problem for (A, ρ).*

Proof : See Ethier and Kurtz [11], Theorem 4.1, p.182. ∎

We now state our first essentially original theorem.

Theorem 1. *Consider a diffusion process generator A_1 on $B(\mathbb{R}^n)$, i.e. the weak closure of*

$$A_1'f(x) := \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial f(x)}{\partial x_i} \tag{18}$$

defined on $C_0(\mathbb{R}^n)$ with locally bounded and Borel measurable drift coefficients b_i and diffusion coefficients a_{ik}, and a jump process generator A_2 with jump rate $\lambda \in B(\mathbb{R}^n)$ and *bounded post-jump measure Q, i.e.*

$$A_2 f(x) = \lambda(x) \int_{I\!R^n} (f(y) - f(x)) Q(x, dy) \tag{19}$$

for $f \in B(I\!R^n)$. Then there exists a Markov process $\mathbf{X}$ which is the unique solution of the martingale problem for $(A{=}A_1 + A_2, \rho)$.

Proof : In order to find a solution to the martingale problem using Proposition 7 we must have a dissipative operator, so we need to prove that $A := A_1 + A_2$ is dissipative. Indeed for $x_0 := argmax f(x)$ we have that

$$\| \mu' f - Af \| = \| \mu' f - A_1 f - A_2 f \| \geq \mu' f(x_0) - A_1 f(x_0) - A_2 f(x_0) \geq \mu' f(x_0) = \mu' \| f \|$$

because A_1 and A_2 both satisfy the positive maximum principle.

The proof now follows directly from Propositions 5, 6 and 7. ■

So the martingale problem (A, ρ) in the time homogeneous case is *well-posed.*

Of course – analogous to the $C_E[0, \infty)$ case – the solution of the martingale problem $(A := A_1 + A_2, \rho)$ in $D_E[0, \infty)$ means the (weak) solution of the jump SDE corresponding to (σ, b, ρ) with an added jump term with infinitesimal characteristics λ and Q.

Theorem 2. *A Markov process $\mathbf{X}$ in $I\!R^n$ is the unique solution of the martingale problem for $(A := A_1 + A_2, \rho)$ of Theorem 1 if, and only if, $\mathbf{X}$ is the unique weak solution of the time homogeneous form of the jump stochastic differential equation (13) given in integral form by*

$$\mathbf{X}_t = \int_0^t b(\mathbf{X}_{s-}) ds + \int_0^t \sigma(\mathbf{X}_{s-}) d\mathbf{W}_s + \int_0^t d(\mathbf{X}_s - \mathbf{X}^c_{s-}). \tag{20}$$

with initial condition $\mathbf{X}(0) \sim \rho$.

Remark : The last integral in (20) is a generalized Ito integral with semimartingale integrator defined by (16). See, for example, Rogers and Williams [20], Chapter 6. ■

Proof : Let $\mathbf{X}$ be the unique solution of the martingale problem (A, ρ) and consider an arbitrary stopping time $\boldsymbol{\tau} < \infty$ a.s.

Suppose first that $\mathbf{X}$ is the corresponding unique solution of the *local* martingale problem for (A, ρ) stopped at $\boldsymbol{\tau}$. Then applying Proposition 4 for the stopped version of the semimartingale

$$\mathbf{X} = \mathbf{X}^c + (\mathbf{X} - \mathbf{X}^c_-) := \mathbf{W} + (\mathbf{X} - \mathbf{W}_-) \tag{21}$$

with $f(X) := X_i,\ i = 1, \ldots, r$, we obtain (20) for the interval $[0, \boldsymbol{\tau}]$. Taking an increasing sequence of stopping times $\boldsymbol{\tau} \nearrow \infty$ *a.s.* yields the result.

Conversely, if $\mathbf{X}$ is a weak solution of (20) with $\mathbf{X}(0) \sim \rho$, then using Proposition 4 we may conclude for $f \in C^2(\mathbb{R}^n) \cap \bar{C}(\mathbb{R}^n)$ that

$$\begin{aligned}
& f(\mathbf{X}_t) - f(\mathbf{X}_0) - \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i} f(\mathbf{X}_{s-}) d\mathbf{X}_{is} - \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{X}_{s-}) d\langle \mathbf{X}^c_i, \mathbf{X}^c_j \rangle_s \\
& \qquad - \int_0^t \lambda(\mathbf{X}_{s-}) \int_{\mathbb{R}} [f(y) - f(\mathbf{X}_{s-})] Q(\mathbf{X}_{s-}, dy) ds \\
&= f(\mathbf{X}_t) - f(\mathbf{X}_0) - \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i} f(\mathbf{X}_{s-}) d\mathbf{X}_{is} - \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{X}_{s-}) d\langle \mathbf{X}^c_i, \mathbf{X}^c_j \rangle_s \\
& \qquad - \int_0^t d[f(\mathbf{X}_s) - f(\mathbf{X}_{s-})] \\
& \qquad + \int_0^t d[f(\mathbf{X}_s) - f(\mathbf{X}_{s-})] - \int_0^t \lambda(\mathbf{X}_{s-}) \int_{\mathbb{R}^n} [f(y) - f(\mathbf{X}_{s-})] Q(\mathbf{X}_{s-}, dy) ds \\
&= 0 + \int_0^t d[f(\mathbf{X}_{s-}) - \lambda(\mathbf{X}_{s-}) \int_{\mathbb{R}^n} f(y) Q(\mathbf{X}_{s-}, dy) - [1 - \lambda(\mathbf{X}_{s-})] f(\mathbf{X}_{s-})] \\
&= \int_0^t d[f(\mathbf{X}_s) - \mathbb{E} f(\mathbf{X}_s)].
\end{aligned} \tag{22}$$

Thus the semimartingale increments of the last generalized Ito integral have expectation 0 and hence the integral represents a 0-mean martingale process Δ, as required. Applying Proposition 6 yields the result for $f \in B(\mathbb{R}^n)$ and we may conclude from Proposition 7 that $\mathbf{X}$ uniquely solves the martingale problem for (A, ρ). ∎

3.2 Time-dependent case

Let us now consider processes whose parameters vary in time – the common situation in financial applications.

In general let $grA \subset B([0,\infty) \times E) \times B([0,\infty) \times E)$. Then a measurable E-valued process $\mathbf{X}$ with $\mathbf{X}(0) \sim \rho$ is the solution of the martingale problem for (A, ρ) iff for each $(f, g) \in A$

$$f(t, \mathbf{X}(t)) - f(0, \mathbf{X}(0)) - \int_0^t g(s, \mathbf{X}(s)) ds \tag{23}$$

is a 0-mean martingale with respect to the filtration $\mathcal{F}_t^{\mathbf{X}}$. Most of the basic results concerning martingale problems can be extended to the time-dependent case by considering the *space-time process* $\mathbf{X}^0(t) := (t, \mathbf{X}(t))$. It follows that in the time-dependent case the linear operator A_2 for a jumping process with jump rate $\lambda(\cdot,\cdot) \in B([0,\infty) \times E)$ and bounded post-jump measure $Q : [0,\infty) \times E \to \mathcal{P}(E)$ is given by

$$A_2 f(t,x) = \lambda(t,x) \int (f(t,y) - f(t,x)) Q(t,x,dy). \tag{24}$$

Specializing to the case $E := \mathbb{R}^n$ of the practical interest, the drift and diffusion coefficients arising from the SDE (8) now also depend on the time t and define the second-order differential operator (10). This leads to the following result.

Theorem 3. *Consider the time-dependent diffusion process generator A_1 on $B(\mathbb{R}^n)$ which is the weak closure of*

$$A_1' f(t,x) = \frac{\partial f(t,x)}{\partial t} + \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^n a_{ik}(t,x) \frac{\partial^2 f(t,x)}{\partial x_i \partial x_k} + \sum_{i=1}^n b_i(t,x) \frac{\partial f(t,x)}{\partial x_i} \tag{25}$$

defined on $C_0^2(\mathbb{R}^n)$ with locally bounded and Borel measurable drift coefficients b_i and diffusion coefficients a_{ik} and the jump process generator

$$A_2 f(t,x) = \lambda(t,x) \int (f(t,y) - f(t,x)) Q(t,x,dy) \tag{26}$$

with jump rate $\lambda \in B([0,\infty) \times \mathbb{R}^n)$ *and bounded jump transition probability measure* $Q : [0,\infty) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$.

Then there exists a Markov process $\mathbf{X}$ *in* $\mathbb{R}^n$ *which is the unique solution of the martingale problem for* $(A{=}A_1 + A_2, \rho)$.

Proof : The proof follows directly from Theorem 1 using the space-time process $\mathbf{X}^0(t)$ defined above. ■

Finally, we state without proof the straightforward generalization of Theorem 2.

Theorem 4. *A Markov process* $\mathbf{X}$ *in* $\mathbb{R}^n$ *is the unique solution of the martingale problem for* $(A := A_1 + A_2, \rho)$ *of Theorem 3 if, and only if,* $\mathbf{X}$ *is the unique weak solution of the jump stochastic differential equation*

$$d\mathbf{X}(t) = b(t, \mathbf{X}(t_-))dt + \sigma(t, \mathbf{X}(t_-))d\mathbf{W}(t) + \Delta\mathbf{X}(t_-) \tag{27}$$

given in integral form by

$$\mathbf{X}_t = \int_0^t b(s, \mathbf{X}_s)ds + \int_0^t \sigma(s, \mathbf{X}_s)d\mathbf{W}_s + \int_0^t d(\mathbf{X}_s - \mathbf{X}^c_{s-}) \tag{28}$$

with initial condition $\mathbf{X}(0) \sim \rho$. ■

4 Application to Fixed Income Derivatives

As a typical application of the above results to contingent claims analysis we consider the Babbs and Webber [3] model for fixed income derivative valuation in the presence of central bank rate changes. They modelled the term structure of interest rates as involving two correlated state variables : the *official short rate* $\mathbf{r}$ and the *market short rate* or *state of the economy* $\mathbf{X}$.

We suppose given a probability space $(\Omega, \mathcal{F}, P)$ and the filtration $\mathcal{F}_t$ generated by a real-valued Wiener process $\mathbf{W}$. The σ-algebra $\mathcal{F}_t$ represents the information available to the financial markets at time t and the filtration satisfies the usual conditions. Note that the authorities may possess greater information at time t than is embodied in $\mathcal{F}_t$. Trade takes place in a fixed finite time interval $[0,T]$.

Our version of the model involves two processes: the *official short rate* $\mathbf{r}$ and a diffusion process $\mathbf{X}$ in $I\!R^{n-1}$. The vector process $\mathbf{X}$ represents the *state of economy* – including the market short rate – which, together with the official short rate $\mathbf{r}$, drives the market's assessment of the likelihood of the authorities changing the level of $\mathbf{r}$.

We have a *savings account* whose initial value is continuously compounded at interest rate $\mathbf{r}$ by the factor

$$\mathbf{P}_0(t) = \exp\left[\int_0^t \mathbf{r}(u)du\right]. \tag{29}$$

Babbs and Webber characterise the spot (instantaneous) official interest rate $\mathbf{r}$ as a pure jump process with jumps of fixed sizes. At each point in time the probability of a jump of size c_j occurring in the next infinitesimal time interval δt is given by a generalized Poisson process $\mathbf{N}_j$ with state-dependent intensity $\boldsymbol{\lambda}_j := \lambda_j(\mathbf{X})$ as $\boldsymbol{\lambda}_j \delta t + 0(\delta t)$.

The process $\mathbf{r}$ satisfies

$$d\mathbf{r} = \sum_{j=1}^{J} c_j d\mathbf{N}_j, \tag{30}$$

where $c_1, c_2, \ldots, c_J$ represent the jump sizes and $\mathbf{N}_j$ is a point process counting the number of jumps of size c_j since time zero and possessing a predictable intensity $\boldsymbol{\lambda}_j$.

Bounds are needed for both $\mathbf{r}$ and the $\boldsymbol{\lambda}_j$'s. Namely, there exists a lower bound r_{lower} and an upper bound r_{upper} such that each $\boldsymbol{\lambda}_j$ vanishes when $\mathbf{r} + c_j \notin [r_{lower}, r_{upper}]$ and there exists a constant k such that

$$P[\sup_{0\le t\le T} \max_{j=1,2,\ldots,J} \boldsymbol{\lambda}_j(t) \le k] = 1.$$

In our version of the Babbs-Webber model the official change of the short rate is assumed to be driven both by its own level $\mathbf{r}$ and the state of the economy $\mathbf{X}$ – including the market rate – and the joint evolution of $\mathbf{r}$ and $\mathbf{X}$ is assumed to be Markovian. The dynamics of $\mathbf{X}$ are given by

$$d\mathbf{X}(t) = b(t, \mathbf{r}(t-), \mathbf{X}(t_-))dt + \sigma(t, \mathbf{r}(t-), \mathbf{X}(t_-))d\mathbf{W}(t), \tag{31}$$

where $\mathbf{W}$ is vector standard Brownian motion in $I\!R^{n-1}$ and $b(\cdot,\cdot,\cdot)$ and $\sigma(\cdot,\cdot,\cdot)$ are locally bounded and Borel measurable. The evolution of $\mathbf{r}$ is given by a pure jump stochastic differential equation

$$d\mathbf{r}(t) = \Delta\mathbf{r}(t_-) \tag{32}$$

with suitable infinitesimal characteristics $\lambda(t, r, x)$ and $Q(t, r, q, x)$ allowing arbitrary jump saltae.

A unique weak solution of the vector jump stochastic differential equations (31)-(32) exists.

Theorem 5. *Let b and σ be locally bounded and Borel measurable and let $\rho \in \mathcal{P}(I\!R^n)$. Then there is a unique weak solution of the vector jump stochastic differential equation (31)-(32) corresponding to $(\sigma, b, \lambda, Q, \rho)$ if, and only if, there exists a unique solution of the $D_{I\!R^n}[0,\infty)$ martingale problem for (A, ρ), where A is given by the weak closure of*

$$\begin{aligned} A'f(t,r,x) &= \frac{\partial f(t,r,x)}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}(t,r,x)\frac{\partial^2 f(t,r,x)}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(t,r,x)\frac{\partial f(t,r,x)}{\partial x_i} \\ &+ \lambda(t,r,x)\int (f(t,q,x) - f(t,r,x))Q(t,r,q,x). \end{aligned} \tag{33}$$

Proof : This result is a direct application of Theorem 4. ∎

In particular the stochastic differential equation (31) has a weak solution $\mathbf{X}$ in the probability space $(C_{I\!R^n}[0,\infty), \mathcal{B}(C_{I\!R^n}[0,\infty)), P)$ where P is Wiener measure.

Corollary 1. *A unique weak solution of the Babbs-Webber jump stochastic differential system [3] exists.*

Proof : For the Babbs-Webber model the jump rate is given by

$$\lambda(t, \mathbf{r}(t_-), \mathbf{X}(t_-)) := \sum_{j=1}^{J} \lambda_j(t, \mathbf{r}(t_-), \mathbf{X}(t_-) \tag{34}$$

and the discrete jump transition measure is given by

$$\frac{\lambda_j(t, \mathbf{r}(t_-), \mathbf{X}(t_-)}{\lambda(t, \mathbf{r}(t_-), \mathbf{X}(t_-))} \quad \text{for } \mathbf{r}(t) = c_j + \mathbf{r}(t_-), \quad j = 1, \ldots, n, \tag{35}$$

and 0 otherwise. ■

A *yield curve* can be represented in terms of the prices of zero coupon (pure discount) bonds of all maturities $\tau \in (0, T]$. Babbs and Webber [3] represent the price of a pure discount bond in terms a partial differential-difference equation – a special case of an integro-partial differential equation – which the values of the bond must satisfy. We give here a more general result – first for the case $n := 2$ for comparison with [3].

Let $B[t, \tau, \mathbf{r}, \mathbf{X}]$ to be the *price* of a pure discount bond at time t maturing at time $\tau \in (0, T]$.

Theorem 6. *For* $\mathbf{X}$ *a scalar process the pure discount bond prices* B *satisfy the integro-partial differential equation*

$$\begin{aligned} \frac{\partial B[t-, \tau, r, x]}{\partial t} &+ b^*(t, r, x)\frac{\partial B[t-, \tau, r, x]}{\partial x} + \frac{1}{2}\sigma^2(t, r, x)\frac{\partial^2 B[t-, \tau, r, x]}{\partial x^2} \\ &+ \lambda(t, r, x)\int_{I\!R} \{B[t, \tau, q, x] - B[t-, \tau, r, x]\}Q^*(t-, r, dq, x) \\ &= B[t-, \tau, r, x]r(t-) \end{aligned} \tag{36}$$

where $b^* := b - \theta_0\sigma$ *and* Q^* *is the risk-adjusted post-jump measure under any appropriate equivalent martingale measure* $\tilde{P}$.

Proof: This is a consequence of Theorem 5 and the Kolmogorov forward equation (7) applied to the present value $f(t,r,x) := e^{-r(t-)}B[t_-,\tau,q,x]$ of the price of the discount bond maturing at τ. ■

Note that θ_0 is the market price of risk and that under Q^* the original official short rate process $\mathbf{r}$ is replaced by $\mathbf{r}_t - \int_0^t \lambda(s,\mathbf{r}_s,\mathbf{x}_s)\int_{I\!R} r \ Q(s,\mathbf{r}_s,dr,\mathbf{x}_s)ds$, a martingale. $\tilde{P}$ could be taken to be the *minimal* equivalent martingale measure in the sense of [11] for the incomplete market caused by the unpredictability of official short rate jumps. In this case under $\tilde{P}$ the process $\mathbf{r}^2 - \langle \mathbf{r},\mathbf{r}\rangle$, centred by its quadratic variation *compensator* $\langle \mathbf{r},\mathbf{r}\rangle$ [10, 20], is a martingale and expected squared losses due to official short rate jumps are minimized.

Extending Theorem 6 to the vector process $\mathbf{X}$ for arbitrary n, equation (36) must be replaced by

$$\begin{aligned}
&\frac{\partial B[t-,\tau,x]}{\partial t} \\
&+ \sum_{i=1}^{n} b_i^*(t,x)\frac{\partial B[t-,\tau,x]}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sigma_{ik}^2(t,x)\sigma_{kj}^2(t,x)\frac{\partial^2}{\partial x_i \partial x_j}B[t-,\tau,x] \\
&+ \lambda(t,x)\int_{I\!R^n}\{B[t,\tau,y] - B[t-,\tau,x]\}Q^*(t-,x,dy) \\
&= B[t-,\tau,x]r(t-),
\end{aligned} \tag{37}$$

where we now consider the pure jump process $\mathbf{r}$ to be the first coordinate of $\mathbf{X}$.

Finally, we note that this generalization remains valid when applied to an appropriate version of the general vector jump stochastic differential equation (27) or (28) in the statement of Theorem 4. This allows any of the last $n-1$ coordinates of $\mathbf{X}$ to be diffusions, pure jump processes or jumping diffusions with arbitrary time and state dependent correlation structure as might be required for financial modelling purposes.

Conclusion

In this paper we have given a rigorous path to the derivation of the integro-PDE satisfied by the value of contingent claims based on semimartingale – i.e. jumping diffusion – price or rate processes. Our approach through the unique solution of the Stroock-Varadhan martingale problem is equivalent to unique weak solution of the corresponding jump stochastic differential equation which expresses the financial analysts' intuition. Applications to claim valuation and hedging are numerous for underlyings which include jumping equity indices, official short rates, credit spreads, catastrophe risks, etc. Note that the results we have presented – unlike those in the previous literature [24, 15, 9, 18, 21] – allow the correlated financial variables underlying the models to be an arbitrary combination of diffusions, pure jump processes and jumping diffusions with time and state varying correlation. In a forthcoming paper we will present some computational results for such processes.

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