# Bounds on European Option Prices under Stochastic Volatility *

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**Abstract**

In this paper we consider the range of prices consistent with no arbitrage for European options in a general stochastic volatility model. We give conditions under which the infimum and the supremum of the possible option prices are equal to the intrinsic value of the option and to the current price of the stock respectively and show that these conditions are satisfied in most of the stochastic volatility models from the financial literature. We also discuss properties of Black-Scholes hedging strategies in stochastic volatility models where the volatility is bounded.

**Keywords:** Stochastic Volatility, Option Pricing, Incomplete Markets, Superreplication.

## 1 Introduction

A significant part of the recent research in finance has concentrated on building models for asset price fluctuations that are flexible enough to cope with the known empirical deficiencies of the geometric Brownian motion model of Black and Scholes. In particular, there is a growing literature on stochastic volatility models (SV-models) including Hull and White (1987), Hofmann, Platen, and Schweizer (1992), Heston (1993) or the survey articles Ball and Roma (1994) or Frey (1997). In this class of models the volatility is

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modelled by a stochastic process which is not adapted to the filtration generated by the Brownian motion driving the asset price process. SV-models are able to capture the succession of periods with high and low activity we observe in most financial time series. However, this increase in realism leads to new conceptual problems in the pricing and hedging of derivatives. It is well-known that SV-models are incomplete, i.e. one can no longer perfectly replicate the payoff of a typical derivative by a dynamic trading strategy in the stock and some riskless asset. By the second fundamental theorem of asset pricing this is equivalent to the fact that the model admits many equivalent martingale measures. Hence for typical derivatives such as options there are many prices consistent with absence of arbitrage.[1]

In this paper we determine for a large class of SV-models the supremum and the infimum of the set of possible option prices, i.e. the supremum and the infimum of the expected value of the terminal payoff of an option under all equivalent martingale measures. Obviously in an arbitrage-free market a call option is always worth less than the underlying security. On the other hand we know from Merton's theorem on the equivalence of European and American call options that the price of a European call option on some non-dividend paying asset must exceed the intrinsic value. For a general SV-model we give conditions on the distribution of the average volatility which are equivalent to supremum and infimum of the set of option prices being equal to these extreme values. Here our arguments rely strongly on the observation that in a SV-model the asset price process can be represented as a time-changed Brownian motion. In the second part of the paper we show that these conditions are satisfied for a large class of SV-models in which volatility is modelled as a one-dimensional diffusion. This class contains most of the models that have been considered in the financial literature including Wiggins (1987), Hull and White (1987), Heston (1993) and Renault and Touzi (1996).

These findings are of importance for the hedging of options in the context of SV-models. It is easily seen that the minimum initial value of a self-financing strategy that super-replicates the payoff of a derivative is no smaller than the supremum of the expected value of the terminal payoff under all equivalent martingale measures[2]; our

[1] Of course the market may still be complete, if there are other options on the underlying asset with different strike traded in the market. A detailed analysis of market completion by the introduction of options is carried out in Bajeuz and Rochet (1996).

[2] A deep result from mathematical finance shows that for derivatives with payoff bounded below these two quantities are actually equal, see e.g. Delbaen (1992), El Karoui and Quenez (1995), or Kramkov (1996).

results therefore show that for a large class of SV-models with unbounded volatility there is no nontrivial super-replicating strategy for options. Similarly, our results on the infimum of the set of feasible option prices show that for most SV-models there is no nontrivial sub-replicating strategy either. Hence in most popular SV-models the concept of super- or sub-replication does not lead to satisfactory answers for the pricing and hedging of derivative securities. Consequently interesting approaches to the risk management of derivatives in these models must involve some sort of risk-sharing between buyer and seller; in particular the seller must necessarily bear some of the "unhedgeable" volatility risk.

The picture changes, if we study models where the volatility is assumed to be bounded from above by some constant $\sigma_{\max}$. As shown by El Karoui, Jeanblanc-Picqué, and Shreve (1998), in all such models the use of a Black-Scholes strategy corresponding to the constant volatility $\sigma_{\max}$ induces a superreplicating strategy; see also Avellaneda, Levy, and Paras (1995) or Lyons (1995) for related results. From an economic viewpoint this approach appears to be somewhat problematic as this "universal" superhedging strategy neglects the particular form of the volatility dynamics in a given SV-model. Adding to the existing literature we provide some justification for the use of this strategy and show that for a wide class of SV-models where the volatility follows a bounded diffusion process this strategy is actually the cheapest superhedging strategy for European options.

Related results have been obtained by a number of authors in various contexts. Eberlein and Jacod (1997) showed the absence of non-trivial bounds on option prices in a model where the logarithm of the asset price process is a purely discontinuous Levy process with unbounded jumps. Frey (1997) observed that nontrivial bounds on option prices do not exist in the well-known SV-model of Hull and White (1987). Finally, Cvitanic, Pham, and Touzi (1997) have independently obtained results which are very similar to ours. They study the supremum of the set of all arbitrage prices for non-path-dependent derivatives whose payoff satisfies certain regularity conditions. They are working in a SV-model where the stock price and the volatility are given by a two-dimensional diffusion process. Under regularity assumptions they are able to characterize the supremum of the set of all arbitrage prices as a viscosity super-solution of the Bellman equation associated to the infinitesimal generator of this two-dimensional diffusion process. From this characterization they deduce that this supremum is independent of the current volatility level, decreasing over time and concave in the current stock price. They conclude that the supremum is given by the smallest concave majorant of the terminal payoff. If the payoff is convex this is precisely our result.

Cvitanic, Pham and Touzi are able to handle non-convex payoff functions which are not considered in the present paper, and they also deal with the problem of super-replication under convex portfolio constraints. On the other hand there analysis is restricted to models where asset price and volatility follow a two-dimensional diffusion process whereas our general results cover also models with more general volatility dynamics such as the model proposed by Naik (1993). Moreover, in order to obtain their viscosity super-solution characterization they have to impose relatively strong regularity conditions on the terminal payoff and on the coefficients of the SDE for the asset price process. This excludes for instance the popular square root model of Heston (1993) which is covered by our results.

The remainder of the paper is organized as follows: Section 2 gives a general characterization of the case when the supremum and infimum of the range of option prices are equal to their extreme values. In Section 3 we verify this criterion in the special case where the volatility follows a one-dimensional diffusion process. Section 4 deals with the case of bounded volatility. Section 5 is the conclusion.

## 2 The General Criterion

Throughout our analysis we consider a frictionless financial market where securities are traded continuously, including a risky asset called the stock and a riskless money market account. We use the money market account as a numeraire thereby making interest rates implicit to our model. The stock price process is given by a locally bounded nonnegative semimartingale $S$ defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with $(\mathcal{F}_t)$ satisfying the usual conditions. In this paper we are mainly interested in the case where the evolution of the stock price is described by some *stochastic volatility model* (SV-model). In this class of models it is assumed that $(\mathcal{F}_t)$ is rich enough to support a Wiener process $B_t$ and that $S_t$ is a solution to the equation

$$dS_t = S_t \sigma_t dB_t + S_t \mu_t dt \,, \qquad S_0 = x \tag{2.1}$$

for suitably integrable adapted processes $\sigma_t$ and $\mu_t$.

Fix a time horizon $T < \infty$. The following two sets of probability measures $Q$ equivalent to $P$ on $(\Omega, \mathcal{F}_T)$ will be very important:

$$\mathbb{M}^e := \{Q \,|\, Q \sim P \text{ and } S \text{ is a } Q\text{-local martingale } \}$$
$$\mathbb{Q} := \{Q \,|\, Q \sim P \text{ and } S \text{ is a } Q\text{-martingale } \}.$$

It is well known that our model precludes arbitrage if and only if the set $\mathbb{M}^e$ is nonempty. We make the slightly stronger assumption that also the smaller set $\mathbb{Q}$ is nonempty.[3] As shown in Theorem 13 of Delbaen and Schachermayer (1995), the set $\mathbb{Q}$ is nonempty if and only if the claim $S_T - S_0$ is maximal, i.e. if and only if there is no admissible trading strategy that requires an initial investment of $S_0$ and that yields a terminal value $V_T \geq S_T$ with $P(V_T > S_T) > 0$. As nonmaximality of the claim $S_T - S_0$ is an undesirable feature of any model used for pricing derivative securities on $S$, our assumption that $\mathbb{Q}$ is nonempty makes economic sense. According to Theorem 5.2 of Delbaen and Schachermayer (1997), if the set $\mathbb{Q}$ is nonempty it is dense in $\mathbb{M}^e$ in the following sense.[4]

**Proposition 2.1.** *Suppose that $S$ is a locally bounded nonnegative semimartingale and that $\mathbb{Q}$ is nonempty. Then for every $Q \in \mathbb{M}^e$ there is a sequence $Q_n \in \mathbb{Q}$ such that*

$$\left\| \frac{dQ_n}{dP} - \frac{dQ}{dP} \right\|_{\mathcal{L}^1(\Omega, \mathcal{F}_T, P)} \to 0 \,. \tag{2.2}$$

We consider a European call option on the stock with strike $K$ and maturity $T$, and denote by $C_K^Q$ its expected payoff under the measure $Q \in \mathbb{M}^e$, i.e. $C_K^Q = E^Q((S_T - K)^+)$. Now let

$$\overline{C_K} = \sup\{C_K^Q : Q \in \mathbb{Q}\} \text{ and } \underline{C_K} = \inf\{C_K^Q : Q \in \mathbb{Q}\}\,.$$

Using Jensen's inequality and the martingale property of $S$ we obtain for all $Q \in \mathbb{Q}$ the bounds

$$(S_0 - K)^+ = (E^Q(S_T) - K)^+ \leq C_K^Q \leq S_0. \tag{2.3}$$

Hence we have the estimate $(S_0 - K)^+ \leq \underline{C_K} \leq \overline{C_K} \leq S_0$.

**Remark 2.2.** It follows easily from Proposition 2.1 and Fatou's Lemma that we may replace $\mathbb{Q}$ by $\mathbb{M}^e$ in the definition of $\overline{C_K}$, i.e. $\overline{C_K} = \sup\{C_K^Q : Q \in \mathbb{M}^e\}$. This is not true for the lower bound $\underline{C_K}$, as $E^Q(S_T) < S_0$ for $Q \in \mathbb{M}^e - \mathbb{Q}$. It may even happen that

[3] Sin (1988) gives an example of an SV-model where both $\mathbb{Q}$ and $\mathbb{M}^e - \mathbb{Q}$ are nonempty.

[4] We are very grateful to Freddy Delbaen for bringing this result and its implications for our analysis to our attention.

$\inf\{C_K^Q : Q \in \mathbb{M}^e\} < (S_0 - K)^+$; see e.g. Emanuel and Macbeth (1982) or Sin (1996) for examples in the context of the constant elasticity of variance models. This shows that the distinction between the sets $\mathbb{Q}$ and $\mathbb{M}^e$ is of economic significance.

As mentioned already in the introduction, the quantities $\overline{C_K}$ and $\underline{C_K}$ are of importance for the hedging of options. Results from El Karoui and Quenez (1995) and Kramkov (1996) imply that $\overline{C_K}$ is the minimum initial value of an admissible self-financing strategy that super-replicates the payoff of the call option. Similarly, by imposing extra integrability conditions on the admissible strategies, it is possible to show that $\underline{C_K}$ is the maximum initial value of a trading strategy that sub-replicates the call; see El Karoui and Quenez (1995) for details. The following two results provide conditions under which the quantities $\overline{C_K}$ and $\underline{C_K}$ are equal to their extreme values $S_0$ and $(S_0 - K)^+$ respectively, in which case there is no non-trivial super- or sub-replicating strategy for the option.

**Proposition 2.3.** *Let $S$ be a locally bounded nonnegative semimartingale such that the set $\mathbb{Q}$ of equivalent martingale measures is nonempty. Then the following four conditions are equivalent:*

*i) There exists some $K^* > 0$ such that $\overline{C_{K^*}} = S_0$.*

*ii) There exist $Q_1, Q_2, \ldots \in \mathbb{M}^e$ such that the law of $S_T$ under $Q_n$ converges weakly to $\delta_0$ (the Dirac-measure in $0$).*

*iii) There exist $Q_1, Q_2, \ldots \in \mathbb{Q}$ such that the law of $S_T$ under $Q_n$ converges weakly to $\delta_0$.*

*iv) $\overline{C_K} = S_0$ for all $0 < K < \infty$ .*

*If $S$ follows a SV-model, i.e. if the dynamics of $S$ are of the special form (2.1), each of the above statements is equivalent to*

*v) There exists a sequence of probability measures $Q_1, Q_2, \ldots \in \mathbb{M}^e$ such that for all constants $L \in \mathbb{R}^+$*

$$Q_n\left(\int_0^T \sigma_t^2 dt > L\right) \to 1 \qquad \text{as } n \to \infty . \tag{2.4}$$

**Proposition 2.4.** *Let $S$ be a locally bounded nonnegative semimartingale such that the set $\mathbb{Q}$ of equivalent martingale measures is nonempty. Then the following four statements are equivalent:*

*i)* $\underline{C}_{S_0} = 0$

*ii) There exist* $Q_1, Q_2, \ldots \in \mathbb{M}^e$ *such that the law of* $S_T$ *under* $Q_n$ *converges weakly to* $\delta_{S_0}$.

*iii) There exist* $Q_1, Q_2, \ldots \in \mathbb{Q}$ *such that the law of* $S_T$ *under* $Q_n$ *converges weakly to* $\delta_{S_0}$.

*iv)* $\underline{C}_K = (S_0 - K)^+$ *for all* $0 < K < \infty$ .

*If the dynamics of* $S$ *are given by form* (2.1) *each of the above statements is equivalent to*

*v) There exists a sequence of probability measures* $Q_1, Q_2, \ldots \in \mathbb{M}^e$ *such that for all constants* $L \in \mathbb{R}^+$

$$Q_n \left( \int_0^T \sigma_t^2 dt > L \right) \to 0 \qquad \text{as } n \to \infty \,. \tag{2.5}$$

**Remark 2.5.** The most important result in Propositions 2.3 and 2.4 is of course the implication v) $\Longrightarrow$ iv). These implications are very intuitive: as stated by the referee, "we would expect the supremum of the option price over all equivalent martingale measures to be as high as possible, i.e. equal to the stock price, if we can find a sequence of martingale measures under which the price becomes more and more volatile, thus making the option more "valuable." Similarly, the infimum of all possible option prices will be as low as possible, if we can find a sequence of martingale measures under which the price becomes less and less volatile and, in the limit, "freezes" at $S_0$."

An easy proof of the implication v) $\Longrightarrow$ iv) that draws directly on this intuition can be given for SV-models where, conditional on the realization of the volatility path $(\sigma_t)_{0 \le t \le T}$, the asset price is lognormally distributed under a sequence $Q^n$ of martingale measures satisfying v). In that case the expected values $C_K^{Q_n}$ can be represented as mixture of Black-Scholes prices, and the result follows immediately from v); see e.g. Frey (1997) for details. However, in many SV-models the asset price is not lognormally distributed conditionally on the volatility, which is why our proof is based on a different idea; see Remark 2.6. As an empirically relevant example we mention the SV-models treated in Section 3, where the volatility follows a one-dimensional Brownian motion and where the Brownian motion driving the volatility is correlated to the Brownian motion driving the asset price process.

**Remark 2.6.** The key idea behind the proof of the equivalence of ii) and v) is the use of a stochastic time change

$$S_t = S_0 \exp(B_{\langle M\rangle_t} - 1/2\langle M\rangle_t) =: S_0 U_t\,,$$

where $\langle M\rangle_t := \int_0^t \sigma_s^2 ds$ is the quadratic variation of the local martingale $M_t := \int_0^t \sigma_s dW_s$, and where $B$ is a $Q_n$-Brownian motion w.r.t. a new filtration $(\mathcal{G}_t)$; see also (2.8) below. $U$ is therefore a $Q_n$-geometric Brownian Motion adapted to $(\mathcal{G}_t)$ and $C_K^{Q_n}$ equals the Black-Scholes price of a call with strike price $K$ and random maturity date $\langle M\rangle_T$. As this maturity date tends to $+\infty$ or 0 the Black-Scholes price tends to $S_0$ or $(S_0 - K)^+$ respectively.

**Remark 2.7.** Our results allow us to draw conclusions on the cheapest superhedging strategy for certain other derivatives as well. Consider for instance any bounded convex payoff function $g$. Obviously $g(x) \le g(0)$ for all $x > 0$. Now Proposition 2.3 implies that under condition v) $\sup\{E^Q(g(S_T))\,,\ Q \in \mathbb{Q}\}$ is equal to $g(0)$. Hence the cheapest superhedging strategy is trivial and consists of $g(0)$ units of the money market account and a zero position in the stock. More generally Propositions 2.3 and 2.4 allow us to draw inference on the minimal super-replicating strategy and maximal sub-replicating strategy for any convex function by the representation theorem for convex functions.

PROOF OF PROPOSITION 2.3:

**i) $\Longrightarrow$ ii).** By assumption there is a sequence $Q_n \in \mathbb{Q}$ such that $E^{Q_n}((S_T - K^*)^+) \to S_0$. Hence we get from the put-call-parity and the fact that $S$ is a $Q_n$-martingale

$$E^{Q_n}((K^* - S_T)^+) = E^{Q_n}(K^* - S_T + (S_T - K^*)^+) = K^* - S_0 + E^{Q_n}((S_T - K^*)^+) \to K^*\,,$$

as $n \to \infty$. This immediately implies that for all $\delta > 0$ we have that $Q_n(S_T > \delta) \to 0$ as $n \to \infty$ and hence iii) and therefore also ii).

**ii) $\Longrightarrow$ iii).** It suffices to show that for every $\epsilon > 0$ there is some $Q \in \mathbb{Q}$ such that $Q(S_T > \delta) < \epsilon$. Let $\epsilon$, $\delta$ be given. By ii) there is some $\tilde{Q} \in \mathbb{M}^e$ with $\tilde{Q}(S_T > \delta) < \epsilon/2$. By Proposition 2.1 we can find some $Q \in \mathbb{Q}$ so that $\left\|\frac{d\tilde{Q}}{dP} - \frac{dQ}{dP}\right\|_{\mathcal{L}^1(\Omega,\mathcal{F}_T,P)} < \frac{\epsilon}{2}$. Hence

$$Q(S_T > \delta) \le \tilde{Q}(S_T > \delta) + \int\limits_{\{S_T > \delta\}} \left|\frac{d\tilde{Q}}{dP} - \frac{dQ}{dP}\right| dP \ \le \frac{\epsilon}{2} + \left\|\frac{d\tilde{Q}}{dP} - \frac{dQ}{dP}\right\|_{\mathcal{L}^1(\Omega,\mathcal{F}_T,P)} \le \epsilon\,.$$

**iii) $\Longrightarrow$ iv).** This implication follows again from the put-call-parity, as iii) implies that for every put option $\lim_{n\to\infty} E^{Q_n}((K - S_T)^+) = K$.

**iv) $\Longrightarrow$ i)** trivial.

Now assume that under $P$ the stock price process $S$ is given by a SV-model (a solution to the SDE (2.1)). By Girsanov's theorem there exists for every $Q \in \mathbb{M}^e$ a new $Q$-Wiener process $W_t$ such that $S_t$ satisfies

$$dS_t = S_t \sigma_t dW_t \,, \qquad S_0 = x. \tag{2.6}$$

We define $M_t = \int_0^t \sigma_s dW_s$, a $Q$-local martingale with quadratic variation process $\langle M \rangle_t = \int_0^t \sigma_s^2 ds$. Itô's formula yields that $S$ is given by $S_t = S_0 \exp(M_t - \frac{1}{2}\langle M \rangle_t)$.

**v) $\Longrightarrow$ ii).** By definition of $M_T$ we have for every $\delta > 0$

$$Q_n(S_T > \delta) = Q_n(\log(S_T) > \log(\delta)) = Q_n\left(M_T - \frac{1}{2}\langle M \rangle_T > \log\left(\frac{\delta}{S_0}\right)\right). \tag{2.7}$$

Levy's characterization of continuous local martingales implies that the process

$$B_t = M_{f_t}, \qquad \text{where } f_t = \inf\{s > 0 : \int_0^s \sigma_u^2 du \geq t\} \tag{2.8}$$

is a Brownian motion relative to the new filtration $(\mathcal{G}_t) = (\mathcal{F}_{f_t})$, and $M_t = B_{\langle M \rangle_t}$; see e.g. Section 3.4.B of Karatzas and Shreve (1988). By the law of large numbers for the Brownian motion we can find for every $\epsilon > 0$ some $L$ large enough such that

$$Q\left(\forall t \geq L, \quad \frac{B_t}{t} < \frac{\log(\delta/S_0)}{t} + \frac{1}{2}\right) > 1 - \frac{\epsilon}{2} \text{ for all } n\,,$$

and by assumption we find $n$ large so that $Q_n(\langle M \rangle_T > L) > 1 - \epsilon/2$. Hence

$$\begin{aligned}
Q_n(S_T > \delta) &\leq Q_n(S_T > \delta,\ \langle M \rangle_T > L) + Q_n(\langle M \rangle_T \leq L) \\
&\leq Q_n(B_{\langle M \rangle_T} - \frac{1}{2}\langle M \rangle_T > \log\left(\frac{\delta}{S_0}\right),\ \langle M \rangle_T > L) + \frac{\epsilon}{2} \\
&\leq Q_n(\frac{B_t}{t} > \frac{1}{2} + \frac{\log(\delta/S_0)}{t} \quad \text{for some } t > L) + \frac{\epsilon}{2} < \epsilon\,.
\end{aligned}$$

Therefore, for any $\delta > 0$ we have $Q_n(S_T > \delta) \to 0$ as $n \to \infty$ which is equivalent to ii).

**ii) $\Longrightarrow$ v).** We prove this claim by contradiction. Suppose there are $L$ and $\epsilon > 0$ such that $Q_n(\langle M \rangle_T > L) < 1 - \epsilon$ for infinitely many $n$. We have for these $n$

$$Q_n(S_T \leq K) \leq Q_n(S_T \leq K,\ \langle M \rangle_T \leq L) + Q_n(\langle M \rangle_T > L)\,.$$

Now we have the following estimate for the first term on the right:

$$\begin{aligned}
Q_n(S_T \leq K,\ \langle M \rangle_T \leq L) &= Q_n\left(B_{\langle M \rangle_T} \leq \log(\frac{K}{S_0}) + \frac{1}{2}\langle M \rangle_T,\ \langle M \rangle_T \leq L\right) \\
&\leq Q_n\left(\inf_{0 \leq t \leq L} B_t \leq \log(\frac{K}{S_0}) + \frac{1}{2}L\right) \\
&< \epsilon/2 \quad \text{for } K \text{ small enough,}
\end{aligned}$$

so $Q_n(S_T > K) > \epsilon/2$ for infinitely many $n$ which contradicts ii). $\square$.

PROOF OF PROPOSITION 2.4:

**i) $\Longrightarrow$ ii).** For $Q \in \mathbb{Q}$ the put-call-parity yields that $E^Q((S_T - S_0)^+) = E^Q((S_0 - S_T)^+)$. Now let $Q_n \in \mathbb{Q}$ be a sequence with

$$\lim_{n\to\infty} E^{Q_n}((S_T - S_0)^+) = \lim_{n\to\infty} E^{Q_n}((S_0 - S_T)^+) = 0\,.$$

Then for any $\delta > 0$ we get from the first submartingale inequality

$$Q_n\left(\sup_{0\le t\le T}(S_t - S_0) > \delta\right) \le \frac{E^{Q_n}((S_T - S_0)^+)}{\delta} \to 0 \tag{2.9}$$

as $n \to \infty$ and also

$$Q_n\left(\inf_{0\le t\le T}(S_t - S_0) < -\delta\right) \le \frac{E^{Q_n}((S_0 - S_T)^+)}{\delta} \to 0 \tag{2.10}$$

as $n \to \infty$ which implies iii) and hence ii).

**ii) $\Longrightarrow$ iii).** This follows as in the proof of Proposition 2.4 from the fact that $\mathbb{Q}$ is dense in $\mathbb{M}^e$.

**iii) $\Longrightarrow$ iv).** This implication follows again from the put-call-parity, as iii) implies that for every put option $\lim_{n\to\infty} E^{Q_n}((K - S_T)^+) = (K - S_0)^+$.

Now consider the case where $S$ is given by the SV-model (2.1). We first show that v) implies iii). Suppose there exist $Q_1, Q_2, \ldots \in \mathbb{M}^e$ that satisfy (2.5). Then, for any $\delta > 0$ and every $\epsilon > 0$ we have

$$\begin{aligned}
Q_n(S_T < S_0 - \delta) &\le Q_n(S_T < S_0 - \delta,\ \langle M\rangle_T \le L) + Q_n(\langle M\rangle_T > L)\\
&\le Q_n\left(B_{\langle M\rangle_T} - \frac{1}{2}\langle M\rangle_T < \log\left(\frac{S_0 - \delta}{S_0}\right),\ \langle M\rangle_T \le L\right) + Q_n(\langle M\rangle_T > L)\\
&\le Q\left(\inf_{0\le t\le L} B_t - \frac{1}{2}L < \log\left(\frac{S_0 - \delta}{S_0}\right)\right) + Q_n(\langle M\rangle_T > L) < \epsilon\,,
\end{aligned}$$

provided $L > 0$ small enough such that the first probability is less than $\epsilon/2$ and then $n$ large enough such that the second probability is less than $\epsilon/2$. Similarly

$$Q_n(S_T > S_0 + \delta) \le Q\left(\sup_{0\le t\le L} B_t > \log\left(\frac{S_0 + \delta}{S_0}\right)\right) + Q_n(\langle M\rangle_T > L) < \epsilon\,,$$

which together imply (ii).

Conversely assume that i) is satisfied and let $Q_n$ be a sequence with $\lim_{n\to\infty} E^{Q_n}((S_T - S_0)^+) = 0$. Define $\tau_\delta = \inf\{t > 0 : |S_t - S_0| \ge \delta\}$. By (2.9) and (2.10) for arbitrary $\epsilon > 0$

we can find $n$ large such that $Q_n(\tau_\delta < T) < \epsilon$. Hence we get

$$
\begin{aligned}
Q_n(\langle M\rangle_T > L) &\le Q_n(\langle M\rangle_T > L,\ \forall t \le T\ |S_t - S_0| < \delta) + \epsilon \\
&= Q_n(\langle M\rangle_{T\wedge\tau_\delta} > L) + \epsilon \\
&\le Q_n(\langle S\rangle_{T\wedge\tau_\delta} > L(S_0-\delta)^2) + \epsilon \\
&\le \frac{E^{Q_n}(\langle S\rangle_{T\wedge\tau_\delta})}{L(S_0-\delta)^2} + \epsilon \\
&= \frac{E^{Q_n}((S_{T\wedge\tau_\delta} - S_0)^2)}{L(S_0-\delta)^2} + \epsilon \\
&\le \frac{\delta^2}{L(S_0-\delta)^2} + \epsilon
\end{aligned}
$$

for arbitrarily small $\delta, \epsilon > 0$ and hence $Q_n(\langle M\rangle_T > L) \to 0$ as $n \to \infty$. $\square$

# 3 The Case of Diffusion Volatility

We now consider a large class of stochastic volatility models where the instantaneous variance follows itself a diffusion process. As we are only interested in the range of possible arbitrage prices for options it is legitimate to model the asset price dynamics directly under a local martingale measure $Q$. We assume that — possibly after an equivalent change of measure — $S$ satisfies the equations

$$
dS_t = S_t|v_t|^{1/2}dW_t^{(1)}, \qquad S_0 = 1 \tag{3.1}
$$

$$
dv_t = a(v_t)dt + \eta_1(v_t)dW_t^{(1)} + \eta_2(v_t)dW_t^{(2)}, \qquad v_0 = \sigma_0^2 \tag{3.2}
$$

for $W_t = (W_t^{(1)}, W_t^{(2)})$ a Wiener process under $Q$. This implies that $S_t$ is a positive local martingale under $Q$ (the semimartingale exponential of $\int |v_t|^{1/2}dW_t^{(1)}$) and $v_t$ is a one-dimensional diffusion.

We will impose a further set of conditions on the coefficients:

A0) The SDE (3.2) has a strictly positive, non-exploding solution.

A1) The real functions $a, \eta_1, \eta_2$ are locally Lipschitz in $\mathbb{R}^+$, and $b(x) = \sqrt{\eta_1^2(x) + \eta_2^2(x)}$ belongs to $\mathcal{C}^1(\mathbb{R}^+)$.

A2) $\eta_2(v) > 0$ for all $v \in \mathbb{R}^+$. This condition ensures that volatility innovations and asset returns are not perfectly correlated which in turn implies that the market is incomplete.

**Remark 3.1.** The above class of volatility models contains the models considered by Wiggins (1987), Hull and White (1987), Heston (1993) and Renault and Touzi (1996) as special cases. Note that in contrast to most of these papers we allow for nonzero $\eta_1$ and hence for nonzero correlation between volatility innovations and asset returns.

We will moreover assume that the set $\mathbb{Q}$ of equivalent martingale measures for $S$ is nonempty. The following Proposition from Sin (1988) is very helpful when it comes to verifying this condition for a particular model. Sin (1988) shows that in general the solution to (3.1), (3.2) can be a strictly local martingale, so checking the martingale property of $S$ is not just a purely technical exercise.

**Proposition 3.2.** *Suppose that weak uniqueness holds for the following SDE*

$$d\widetilde{v}_t = a(\widetilde{v}_t)dt + \eta_1(\widetilde{v}_t)|\widetilde{v}_t|^{1/2}dt + \eta_1(\widetilde{v}_t)dW_t^{(1)} + \eta_2(\widetilde{v}_t)dW_t^{(2)}\,, \qquad \widetilde{v}_0 = \sigma_0^2\,. \tag{3.3}$$

*Then $S_t$ defined in (3.1) and (3.2) is a $Q$-martingale if and only if the SDE $\widetilde{v}_t$ admits a non-exploding solution.*

The proof of this result follows exactly the one that appears in Lemma 4.2 of Sin (1998) for the case where $v$ is given by a linear diffusion.

**Remark 3.3.** Under assumptions A0), A1) this condition is for instance satisfied whenever $\eta_1 \leq 0$. For a proof note first that for $\eta_1 \leq 0$ the comparison theorem for SDE's implies that $\widetilde{v} \leq v$. As $v$ has a global solution by A0) it follows that $\widetilde{v}$ cannot explode. As shown by Black (1976) and many subsequent studies a negative $\eta_1$, i.e a negative correlation between volatility innovations and asset returns is the empirically relevant case.

**Theorem 3.4.** *Suppose that assumptions A0), A1), A2) are satisfied and that the set $\mathbb{Q}$ of equivalent martingale measures is non-empty. Then the range of possible prices for the European call option with strike $K$ and maturity $T$ is given by the open interval $((S_0 - K)^+, S_0)$.*

PROOF OF THEOREM 3.4:

While the actual proof of Theorem 3.4 is rather technical, the idea underlying our approach is very intuitive. We choose a sequence of martingale measures $Q_n$, $n \in \mathbb{Z}$ such that the drift of $v$ under $Q_n$ tends to $\infty$ uniformly on compact sets as $n \to \infty$ and to $-\infty$ as $n \to -\infty$, respectively. As $n \to \infty$ the large positive drift then "drives the volatility up," whereas for $n \to -\infty$ the volatility is "driven down to zero" by the large negative drift.

For $n \in \mathbb{Z}$ we define the probability measure $Q_n$ by its Radon-Nikodym derivative with respect to $Q$,

$$\frac{dQ_n}{dQ} = \exp\left(nW_T^{(2)} - \frac{1}{2}n^2T\right).$$

Now the process $(S,v)_t$ solves the SDE

$$dS_t = S_t|v_t|^{1/2}\sigma dW_t^{(1)}, \qquad S_0 = 1 \tag{3.4}$$

$$dv_t = a(v_t)dt + n\eta_2(v_t)dt + \eta_1(v_t)dW_t^{(1)} + \eta_2(v_t)dW_t^{(n,2)} \qquad v_0 = \sigma_0^2, \tag{3.5}$$

for a $Q_n$-Wiener process $W_t^{(n)} = (W_t^{(1)}, W_t^{(n,2)})$. In particular $S$ is a $Q_n$-local martingale. Now consider the solution $v_t^{(n)}$ to the following one-dimensional SDE

$$dv_t^{(n)} = (a(v_t^{(n)}) + n\eta_2(v_t^{(n)}))dt + b(v_t^{(n)})dB_t, \qquad v_0^{(n)} = x \tag{3.6}$$

with $a$, $\eta_2$ and $b$ satisfying A1), A2) and with $B_t$ a one-dimensional Wiener process on a probability space $(\Omega, \mathcal{F}, P)$. We will explicitly use $P_x$ to denote the law of the process $v^{(n)}$ starting at $v_0^{(n)} = x$. Observe that under $Q_n$ the process $v$ satisfies this equation with an appropriate $Q_n$ Brownian motion $B^{(n)}$ and $x = \sigma_0^2$. Hence there exists a non-exploding strictly positive weak solution to (3.6) for all $n = 0, 1, 2, \ldots$, and then the locally Lipschitz property of the coefficients implies the existence and uniqueness of a strong solution. As shown below Theorem 3.4 follows from the following two lemmas.

**Lemma 3.5.** *Assume that for $n \in \mathbb{Z}$ the SDE* (3.6) *has a global solution which is strictly positive. Then the following holds:*

*i) For every $L > 0$, $T > 0$, $x > 0$ and $\epsilon > 0$ there exists $N_1 \in \mathbb{Z}^+$ such that*

$$P_x(v_t^{(n)} \geq L \text{ for some } 0 \leq t \leq T) > 1 - \epsilon \text{ for all } n \geq N_1.$$

*ii) For every $L > 0$, $T > 0$, $x > 0$ and $\epsilon > 0$ there exists $N_1 \in \mathbb{Z}^-$ such that*

$$P_x(v_t^{(n)} \leq L^{-1} \text{ for some } 0 \leq t \leq T) > 1 - \epsilon \text{ for all } n \leq N_1.$$

**Lemma 3.6.** *Assume again that for $n \in \mathbb{Z}$ the SDE* (3.6) *has a global solution which is strictly positive. Then the following holds:*

*i) For every $L > 0$, $T > 0$ and $\epsilon > 0$ there exists $N_2 \in \mathbb{Z}^+$ such that*

$$P_{2L}(v_t^{(n)} > L \text{ for all } 0 \leq t \leq T) > 1 - \epsilon \text{ for all } n \geq N_2.$$

*ii) For every $L > 0$, $T > 0$ and $\epsilon > 0$ there exist $N_2 \in \mathbb{Z}^-$ such that*

$$P_{L/2}(v_t^{(n)} < L \textit{ for all } 0 \le t \le T) > 1 - \epsilon \textit{ for all } n \le N_2 .$$

Using these Lemmas the proof of Theorem 3.4 is now relatively easy. Recall that under $Q_n$ the process $v_t$ satisfies the equation (3.6). In view of Proposition 2.3 and Proposition 2.4 we have to show that, for all $M > 0$, $\epsilon > 0$ we have

$$\exists N \in \mathbb{Z}^+, \ \forall n \ge N \quad P_{v_0}\left(\int_0^T v_t^{(n)} dt > M\right) > 1 - \epsilon \text{ and} \tag{3.7}$$

$$\exists N \in \mathbb{Z}^-, \ \forall n \le N \quad P_{v_0}\left(\int_0^T v_t^{(n)} dt < M\right) > 1 - \epsilon . \tag{3.8}$$

According to Lemma 3.5 i) by choosing $n$ large enough we may drive the process $v^{(n)}$ above any threshhold $L$ with probability arbitrarily close to one. Lemma 3.6 i) ensures that for $n$ sufficiently large we can keep $v^{(n)}$ above any threshhold $L$ with probability arbitrarily close to one, provided that we started $v^{(n)}$ at the level $2L$. The Markov property of $v^{(n)}$ now allows us to combine these two properties in order to verify Condition (3.7). For a verification of (3.8) we proceed analogously. First we "drive $v^{(n)}$ down" arbitrarily fast, using Lemma 3.5 ii). Lemma 3.6 ii) then allows us to conclude that for $n$ large enough $v^{(n)}$ will remain small with sufficiently high probability. We now give a formal proof of (3.7). Choose some $L > 0$ with $LT/2 \ge M$. Define for $L > 0$

$$\tau_L^{(n)} = \inf\{t > 0 \, : \, v_t^{(n)} \ge L\} \text{ and } \lambda_L^{(n)} = \inf\{t > 0 \, : \, v_t^{(n)} \le L\} .$$

Then

$$\begin{aligned} P_x\left(\int_0^T v_t^{(n)} dt > M\right) &> P_x\left(\tau_{2L}^{(n)} \le \frac{T}{2},\ v_t^{(n)} > L \text{ on } t \in \left[\tau_{2L}^{(n)}, \tau_{2L}^{(n)} + \frac{T}{2}\right]\right) \\ &= P_x\left(\tau_{2L}^{(n)} \le \frac{T}{2}\right) P_{2L}\left(\lambda_L^{(n)} \ge \frac{T}{2}\right) \end{aligned}$$

by the strong Markov property of $v^{(n)}$. Now Lemmas 3.5 and 3.6 imply (3.7).

For a proof of (3.8) we first choose some $L$ such that $P_x((v^{(0)})_T^{\sup} > L) < \epsilon/2$, where $v_T^{\sup} = \sup_{0 \le t \le T} |v_t|$. It follows from the comparison theorem for SDE's that for all $n < 0$ we have $(v^{(n)})_T^{\sup} < (v^{(0)})_T^{\sup}$ and hence also $P_x((v^{(n)})_T^{\sup} > L) < \epsilon/2$. Now

$$P\left(\int_0^T v_t^{(n)} dt < M\right) > P\left(\int_0^T (v_t^{(n)} \wedge L) dt < M\right) - P((v^{(n)})_T^{\sup} > L) .$$

Let $\overline{L} = \frac{M}{2T}$. On the set $\{v_t^{(n)} < \overline{L} \text{ for } \frac{M}{2L} \le t \le T\}$ we estimate

$$\int_0^T (v_t^{(n)} \wedge L) dt \le \left(L\frac{M}{2L}\right) + \left(T - \frac{M}{2L}\right)\overline{L} \le M ,$$

hence

$$P_x\left(\int_0^T (v_t^{(n)}\wedge L)dt < M\right) \geq P_x\left(\lambda_{\overline{L}/2}^{(n)} < \frac{M}{2L},\ v_t^{(n)} < \overline{L} \text{ on } [\lambda_{\overline{L}/2}^{(n)}, T]\right)$$
$$\geq P_x\left(\lambda_{\overline{L}/2}^{(n)} < \frac{M}{2L}\right) P_{\overline{L}/2}\left(\tau_{\overline{L}}^{(n)} > T\right),$$

again by the strong Markov property, and the result follows from Lemmas 3.5 and 3.6.
□

PROOF OF LEMMA 3.5:

**Part i):** Define the function $F(v) \triangleq \int_1^v \frac{1}{b(x)}dx$. As $b(x) > 0$ and $C^1$ on $(0,\infty)$, the function $F$ is well-defined, strictly increasing and $C^2$ on $(0,\infty)$. We get from Itô's formula

$$F(v_t^{(n)}) = F(v_0^{(n)}) + \int_0^t \frac{a(v_u^{(n)}) + n\eta_2(v_u^{(n)})}{b(v_u^{(n)})} - \frac{1}{2}b'(v_u^{(n)})du + B_t\,,$$

i.e. the process $F(v^{(n)})$ is a Brownian motion with (stochastic) drift. The proof now uses the fact that this drift tends to $\infty$ as $n\to\infty$ uniformly on compacts: Find $\delta > 0$ such that $P_x((v^{(0)})_T^{\inf} < \delta) < \epsilon/2$ (such $\delta$ exists as $v^{(0)}$ is strictly positive). By the comparison theorem for SDEs we get that for $n > 0$

$$P_x((v^{(n)})_T^{\inf} < \delta) < P_x((v^{(0)})_T^{\inf} < \delta) < \epsilon/2\,.$$

This implies that

$$P_x\left(\tau_L^{(n)} > T\right) \leq P_x\left(F(v_T^{(n)}) < F(L)\right) \leq P_x\left(F(v_T^{(n)}) < F(L),\ (v^{(n)})_T^{\inf} \geq \delta\right) + \epsilon/2\,.$$

Now by A1) and A2) we can find constants $M_0$ and $M_1$ with $M_1 > 0$ such that

$$\forall v \in [\delta, L] \quad \frac{a(v)}{b(v)} - \frac{1}{2}b'(v) > M_0 \quad \text{and} \quad \frac{\eta_2(v)}{b(v)} > M_1\,.$$

Now $P_x(-B_t > F(v_0) - F(L) + TM_0 + nTM_1) < \epsilon/2$ for $n$ sufficiently large. Hence

$$P_x(\tau_L^{(n)} > T) \leq P_x(-B_t > F(v_0) - F(L) + TM_0 + nTM_1) + \epsilon/2 < \epsilon\,.$$

Part **ii):** Essentially this part of the Lemma follows by applying i) to the process $y_t^{(n)} \triangleq 1/v_t^{(n)}$. We have

$$P_x(\lambda_L^{(n)} < T) = P_x(v_t^{(n)} < L \text{ for some } t\in[0,T]) = P_x(1/v_t^{(n)} > L^{-1} \text{ for some } t\in[0,T])\,.$$

Now, writing $v_t$ for $v_t^{(n)}$ we get by Itô's formula

$$d(v_t^{-1}) = -v_t^{-2}(a(v_t) - v_t^{-1}b^2(v_t)) - n\eta_2(v_t)v_t^{-2}dt - v_t^{-2}b(v_t)dB_t\,. \tag{3.9}$$

Defining new functions $\widetilde{a}(y) = -a(y^{-1})y^2 + b^2(y^{-1})y^3$, $\widetilde{\eta}_2(y) = \eta_2(y^{-1})y^2$ and $\widetilde{b}(y) = b(y^{-1})y^2$ we get from (3.9) that $y_t^{(n)} \triangleq 1/v_t^{(n)}$ satisfies the SDE

$$dy_t^{(n)} = (\widetilde{a}(y_t^{(n)}) - n\widetilde{\eta}_2(y_t^{(n)}))dt + \widetilde{b}(y_t)^{(n)} dB_t\,, \tag{3.10}$$

and applying the result of part i) to (3.10) yields the claim. □

PROOF OF LEMMA 3.6:

We consider only statement ii); i) follows as in the proof of Lemma 3.5 by considering the process $y_t = v_t^{-1}$. First we replace $v_t^{(n)}$ by the solution $\widetilde{v}_t^{(n)}$ of the SDE

$$d\widetilde{v}_t^{(n)} = (a(\widetilde{v}_t^{(n)}) + n\phi(\widetilde{v}_t^{(n)})\eta_2(\widetilde{v}_t^{(n)}))dt + b(\widetilde{v}_t^{(n)})dB_t\,, \qquad \widetilde{v}_0^{(n)} = L/2$$

where $\phi : [0, L] \to [0, 1]$ is an increasing Lipschitz continuous function with

$$\phi(x) = 0, \quad 0 \le x \le \frac{L}{2} \qquad \phi(x) = 1, \quad x \ge \frac{3L}{4}\,.$$

By the comparison theorem for SDEs we get for $n < 0$ that $\widetilde{v}_t^{(n)} \ge v_t^{(n)}$ for all $t$ and hence, for every $\delta \in [0, L/2]$

$$\begin{aligned} P_{L/2}((v^{(n)})_T^{\sup} > L) &\le P_{L/2}((\widetilde{v}^{(n)})_T^{\sup} > L) \\ &\le P_{L/2}((\widetilde{v}^{(n)})_T^{\sup} > L,\ (\widetilde{v}^{(n)})_T^{\inf} > \delta) + P_{L/2}((\widetilde{v}^{(n)})_T^{\inf} < \delta)\,, \end{aligned} \tag{3.11}$$

so to finish the proof we only need to show that for $n$ large enough each of these two probabilities is less than $\epsilon/2$ for arbitrary $\epsilon > 0$.

To show that the first term is less than $\epsilon/2$ observe that we can replace $\widetilde{v}_t^{(n)}$ by the stopped process $\widetilde{v}_{t\wedge\tau}^{(n)}$ where $\tau = \inf\{t > 0 : v \notin [\delta/2, 2L]\}$. Therefore we shall assume that $\widetilde{v}_t^{(n)}$ takes values in the interval $[\delta/2, 2L]$. For every $n$ we consider the socalled scale function $p^{(n)}$ of $\widetilde{v}^{(n)}$. $p^{(n)}$ is a strictly increasing function which solves the ordinary differential equation $(p^{(n)})'(a + \eta_2 n\phi) + \frac{1}{2}(p^{(n)})''b^2 = 0$. Hence, by Itô's formula the process $p^{(n)}(\widetilde{v}^{(n)})$ is a local martingale. In our case the scale function is given by

$$p^{(n)}(v) = \int_{L/2}^{v} \exp\left(-2\int_{L/2}^{x} \frac{a(u) + \eta_2(u)n\phi(u)}{b^2(u)}du\right) dx\,,$$

see also Section 5.5.B of Karatzas and Shreve (1988). As $\widetilde{v}_t^{(n)} \in [\delta/2, 2L]$ we know that $p^{(n)}(\widetilde{v}_t^{(n)})$ is bounded for every $n$ so it is in fact a real martingale. Now the large negative drift of $\widetilde{v}_t^{(n)}$ implies that the scale function is rapidly increasing on the interval $(L/2, L)$; in particular our assumptions on $\eta_2$, $a$ and $b$ imply that $\lim_{n\to-\infty} p^{(n)}(L) = \infty$. Now we obtain

$$\begin{aligned} P_{L/2}((\widetilde{v}^{(n)})_T^{\sup} > L,\ (\widetilde{v}^{(n)})_T^{\inf} > \delta) &\le P_{L/2}((p^{(n)}(\widetilde{v}^{(n)}))_T^{\sup} > p^{(n)}(L)) \\ &\le \frac{E_{L/2}((p^{(n)}(\widetilde{v}_T^{(n)}))^+)}{p^{(n)}(L)}\,, \end{aligned}$$

where the last inequality follows from the first submartingale inequality. Now the martingale property of $p^{(n)}(\widetilde{v}_t^{(n)})$ and the definition of the scale function implies that

$$E_{L/2}((p^{(n)}(\widetilde{v}_T^{(n)}))) = p^{(n)}(L/2) = 0\,.$$

Hence

$$\frac{E_{L/2}((p^{(n)}(\widetilde{v}_T^{(n)}))^+)}{p^{(n)}(L)} = \frac{E_{L/2}((p^{(n)}(\widetilde{v}_T^{(n)}))^-)}{p^{(n)}(L)} \leq \frac{|p^{(n)}(\delta/2)|}{p^{(n)}(L)}\,.$$

Note that on the set $(0, L/2)$ $p^{(n)}$ is independent of $n$ by the definition of the drift of $\widetilde{v}^{(n)}$. Hence for $n \to -\infty$ the last fraction tends to zero.

In order to show that the second term in (3.11) is small, we consider the solution $x_t$ to the following SDE with reflection at $L/2$ (for an introduction to equations with reflection see for example El Karoui and Chaleyat-Maurel (1978))

$$dx_t = a(x_t)dt + b(x_t)dB_t - dK_t \qquad x_0 = L/2$$

with $x_t \leq L/2$ for all $t$, and $K_t$ is a continuous increasing process with $\int_0^T (x_t - L/2) dK_t = 0$. The locally-Lipschitz property of the coefficients implies the existence and uniqueness of the solution, which is also a strong Markov process.

Observe that, by the definition of $\phi$, $x_t$ and $\widetilde{v}_t^{(n)}$ satisfy the same equation on the interval $(0, L/2)$ for all $n$, therefore $x_t$ must be positive with probability 1 and the uniqueness of the solution to the reflection problem implies $\widetilde{v}_t^{(n)} \geq x_t$ hence

$$P_{L/2}((\widetilde{v}^{(n)})_T^{\inf} < \delta) \leq P_{L/2}(x_T^{\inf} < \delta) \to 0$$

as $\delta \to 0$ which takes care of the second term in (3.11) and we can conclude that for $\delta$ small and $n$ large $P_{L/2}((v^{(n)})_T^{\sup} > L) < \epsilon$. $\square$

# 4 Models with Bounded Volatility

We now consider superhedging strategies for European call and put options in a SV-model of the form (2.1) where the volatility is bounded from above. In this case we are able to obtain superhedging strategies which are at least potentially of practical interest. We assume that

A3) There is a constant $\sigma_{\max} < \infty$ such that a.s. $\sigma_t < \sigma_{\max}$ for all $t$.

**Remark 4.1.** In practice it might be impossible to determine a finite upper bound on the asset price volatility which holds true with a probability of 100 percent. In that case one could choose $\sigma_{\max}$ as some upper quantile of the volatility distribution such that $\sigma_{\max}$ is exceeded by the realized volatility path only with a given small probability. The superhedging strategy will work for all volatility path $(\sigma_t)_{0\le t\le T}$ for which A3) holds; see Proposition 4.2 below for details. Superhedging of options with subjective bounds on the volatility can therefore be viewed as "pragmatic" approach to pricing and hedging of derivatives under stochastic volatility.

We now show that under A3) the value of a superhedging strategy for a European call option is given by the Black-Scholes price of this option corresponding to the volatility $\sigma_{\max}$. This result has first been obtained by El Karoui and Jeanblanc-Piqué (1990), see also El Karoui, Jeanblanc-Picqué, and Shreve (1998). Avellaneda, Levy, and Paras (1995) and Lyons (1995) have independently developed several extensions of this idea. We define the *tracking error* of a hedge strategy as the difference between the actual and the theoretical value of a self-financing portfolio for a European call with strike price $K$ and maturity $T$ calculated from the Black-Scholes formula with constant volatility $\sigma_{\max}$. The theoretical value is given by the Black-Scholes price $c(t, S_t) := C_{\mathrm{BS}}(t, S_t; \sigma_{\max}, K, T)$. The actual value $V_t$ of the self-financing portfolio defined by initially investing $V_0 = c(0, S_0)$ and holding $\frac{\partial}{\partial S}c(t, S_t)$ shares of the underlying at any time $t \le T$ is given by the cumulative gains from trade, i.e.

$$V_t = V_0 + \int_0^t \frac{\partial}{\partial S} c(s, S_s)\, dS_s.$$

The *tracking error* $e_t$ is then defined as the difference between actual and theoretical value: $e_t := V_t - c(t, S_t)$. Since $c(T, S_T) = (S_T - K)^+$, $e_T$ measures the deviation of the hedge portfolio's terminal value from the payoff it is supposed to replicate. In particular, if the tracking error is always positive, the terminal value of the hedge portfolio of an investor following the above strategy always completely covers the option's payoff.

**Proposition 4.2.** *Suppose that $S$ follows the SV-model* (2.1). *Then the tracking error for a European option is given by*

$$e_t = \frac{1}{2}\int_0^t \left(\sigma_{\max}^2 - \sigma_s^2\right) S_s^2 \frac{\partial^2 c}{\partial S^2}(s, S_s)\, ds\,. \tag{4.1}$$

*In particular, if $\sigma_{\max} \ge \sigma_t$ for all $t$, the tracking error is always positive.*

PROOF:

By Itô's formula,

$$c(t, S_t) = c(0, S_0) + \int_0^t \frac{\partial c(s, S_s)}{\partial S} dS_s + \int_0^t \left(\frac{\partial c(s, S_s)}{\partial t} + \frac{1}{2}\sigma_t^2 S_s^2 \frac{\partial^2 c(s, S_s)}{\partial S^2}\right) ds\,.$$

Now by the Black-Scholes PDE

$$\frac{\partial c(s,S_s)}{\partial t} = -\frac{1}{2}\sigma_{\max}^2 S_s^2 \frac{\partial^2 c(s,S_s)}{\partial S^2}.$$

Substituting this into the last integral on the r.h.s. yields the formula for the tracking error. Moreover, $c(s,S_s)$ being convex in $S$, its second derivative is always positive. Hence by (4.1) the sign of the tracking error is entirely determined by the sign of the volatility difference $\sigma_{\max}^2 - \sigma_t^2$. $\square$

Note that this superhedging strategy is universal in the sense that it works for all SV-models satisfying A3). It is of course interesting to know, if for a given parametric SV-model superhedging strategies can be found which are less expensive than the universal superhedging strategy based on Black-Scholes hedging with volatility $\sigma_{\max}$. We now give a criterion analogous to Proposition 2.3 v) on the average volatility that implies that $\overline{C_K} = \sup\{E^Q((S_T-K)^+)\,,\ Q\in\mathbb{Q}\}$ is equal to $C_{\mathrm{BS}}(t,S_t;\sigma_{\max},K,T)$.

**Proposition 4.3.** *Assume that there is a sequence of martingale measures $Q_n \in \mathbb{Q}$ such that for all $\delta > 0$*

$$\lim_{n\to\infty} Q_n\left(\int_0^T \sigma_t^2 dt > \sigma_{\max}^2(T-\delta)\right) = 1\,. \tag{4.2}$$

*Then $\overline{C_K} \geq C_{\mathrm{BS}}(t,S_t;\sigma_{\max},K,T)$ for all $K>0$.*

Proof:

By the put-call parity we may equivalently show that (4.2) implies that for all $K>0$

$$\sup\{E^Q((K-S_T)^+)\,,\ Q\in\mathbb{Q}\} \geq P_{\mathrm{BS}}(t,S_t;\sigma_{\max},K,T)\,,$$

where $P_{\mathrm{BS}}(t,S_t;\sigma_{\max},K,T)$ denotes the Black-Scholes price of a European put in a model with volatility $\sigma_{\max}$. Fix some arbitrary $K>0$ and $\delta>0$ and define the function $g(x)=(K-x)^+$ and the stopping time

$$\tau := \inf\{t>0, \langle M\rangle_t = \int_0^t \sigma_s^2\,ds \geq \sigma_{\max}^2(T-\delta)\}\,.$$

We get from Jensens inequality and the optional sampling theorem

$$\begin{aligned}
E^{Q_n}(g(S_T)) &= E^{Q_n}\left(E^{Q_n}(g(S_T)|\mathcal{F}_{\tau\wedge T})\right)\\
&\geq E^{Q_n}\left(g(E^{Q_n}(S_T|\mathcal{F}_{\tau\wedge T}))\right)\\
&\geq E^{Q_n}(g(S_{\tau\wedge T}))\\
&= E^{Q_n}(g(S_\tau)) - E^{Q_n}(g(S_\tau)-g(S_T)\,;\,\tau>T)\\
&\geq E^{Q_n}(g(S_\tau)) - KQ_n(\tau>T)\,.
\end{aligned}$$

Now (4.2) implies that $\lim_{n\to\infty} Q_n(\tau > T) = 0$. Recall the time change introduced in the proof of Proposition 2.3. We get by definition of $\tau$

$$\begin{aligned} E^{Q_n}(g(S_\tau)) &= E^{Q_n}\left(g\left(S_0 \exp(B_{\langle M\rangle_\tau} - 1/2\langle M\rangle_\tau)\right)\right) \\ &= E^{Q_n}\left(g\left(S_0 \exp(B_{\sigma^2_{\max}(T-\delta)} - 1/2\sigma^2_{\max}(T-\delta))\right)\right) \\ &= P_{\mathrm{BS}}(t, S_t; \sigma_{\max}, K, T-\delta)\,, \end{aligned}$$

which implies the result as $\delta \to 0$. □

**Remark 4.4.** In Proposition 4.5 we will verify (4.2) for a large class of SV-models where the volatility follows a one-dimensional diffusion process. Note however, that Proposition applies also to models with more general volatility dynamics such as the model proposed by Naik (1993), where the volatility is modelled as a finite-state Markov chain. In this model condition (4.2) is easily verified directly.

**Proposition 4.5.** *Suppose that $S$ follows a SV-model of the form* (3.1), (3.2) *and that A0), A1), A3) and the following version of A2) hold*

*A2a) There is some $0 \le \sigma_{\min} < \sigma_0 < \sigma_{\max}$ such that $\eta_2(v) > 0$ for all $v \in (\sigma^2_{\min}, \sigma^2_{\max})$.*

*Then $\overline{C_K} = C_{\mathrm{BS}}(t, S_t; \sigma_{\max}, K, T)$ for all $K > 0$, i.e. the hedging strategy described in Proposition 4.2 is the cheapest superhedging strategy for European call options.*

**Remark 4.6.** According to well-known results on one-dimensional diffusions Assumptions (A1) and (A2a) ensure that the interval $(\sigma^2_{\min}, \sigma^2_{\max})$ is contained in the range of $\sigma_t$ for all $t > 0$.

Proof:

We have to show that our SV-model satisfies condition (4.2). Note first that under A3) the sets $\mathbb{Q}$ and $\mathbb{M}^e$ coincide by the Novikov-criterion, see e.g. Section 3.5.D of Karatzas and Shreve (1988). In order to reduce our problem to the situation considered in Theorem 3.4 we use some smooth and strictly increasing function $\psi$ that maps the interval $(\sigma^2_{\min}, \sigma^2_{\max})$ onto $(0, \infty)$. By Itô's formula $y_t := \psi(v_t)$ solves the SDE

$$dy_t = \tilde{a}(y_t)dt + \tilde{\eta}_1(y_t)dW_t^{(1)} + \tilde{\eta}_2(y_t)dW_t^{(2)}$$

where the coefficients $\tilde{a}$, $\tilde{\eta}_1$, $\tilde{\eta}_2$ and $\tilde{b} := \sqrt{\tilde{\eta}_1^2 + \tilde{\eta}_2^2}$ satisfy A1) and A2). As in the proof of Theorem 3.4 we use for $n \ge 0$ equivalent martingale measures $Q_n \in \mathbb{Q}$ defined by $dQ_n/dP = \exp(nW_T^{(2)} - 1/2n^2T)$. Under these measures $y$ solves the SDE

$$dy_t^n = \tilde{a}(y_t^n) + n\tilde{\eta}_2(y_t^n)dt + \tilde{b}(y_t^n)dB_t^n\,. \tag{4.3}$$

Note that Lemma 3.5 i) and Lemma 3.6 i) apply to (4.3). Hence the Proposition follows from these Lemmas by the arguments used already in the proof of Theorem 3.4. □

## 5 Conclusion

In this paper we studied the range of prices consistent with no-arbitrage for European options in a SV-model. The supremum and infimum of this range are of financial interest as they give the initial prices of the cheapest superreplication strategy and the most expensive subreplication strategy respectively for the option. Our main result is that in most SV-models with unbounded diffusion-volatility the cheapest superreplication strategy for a European call option is to "buy the stock". Hence in these models the concept of superreplication is of little practical use, and different approaches for the risk-management of derivatives under stochastic volatility are called for. One possible approach is to introduce (subjective) bounds on the volatility. We proved that in many SV-models where the volatility is bounded above by some constant $\sigma_{\max}$ the value process of the cheapest superreplication strategy for European options is given by the Black-Scholes price corresponding to the volatility $\sigma_{\max}$. This result shows that hedging under the assumption of bounded volatility is at least potentially of practical relevance.

There are of course other approaches to the risk-management of derivatives under stochastic volatility. We refer the reader to Föllmer and Schweizer (1991) or Hofmann, Platen, and Schweizer (1992) for information about the concept of risk-minimization and applications to stochastic volatility models and to Pham and Touzi (1996) for an equilibrium analysis of option pricing in SV-models.

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