Principal Component Analysis of Volatility Smiles and Skews

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This paper develops a model for volatility sensitivity to the underlying asset price $\partial\sigma/\partial S$. It has applications to option pricing and dynamic delta hedging under stochastic volatility. The model, which corresponds to a quadratic approximation to the volatility surface, extends the linear parameterizations of Derman (1999) by allowing ATM volatility sensitivity to change continuously with S. The extension to fixed strike volatility sensitivities is achieved using a principal component analysis on the *deviation* of fixed strike volatilities from ATM volatility.

1. Introduction

Many financial markets are characterized by a high degree of collinearity. It occurs when there are only a few important sources of information in the data, which are common to many variables. This paper is about a standard method for extracting the most important uncorrelated sources of variation in a multivariate system, which is called principal component analysis (PCA).

PCA is not just about term structures of interest rates or futures, although most readers will be familiar with the method in this context. The standard interpretation of the first component as the trend, the second component as the tilt and the third component as the curvature holds for any highly correlated ordered system, not just a term structure. So when implied volatilities are ordered by strike or moneyness, an application of PCA will reveal the standard trend-tilt-curvature interpretation of the first three principal components.

Several principal component models of volatility smiles and skews have been based on daily changes in implied volatilities, by strike and/or by moneyness. Derman and Kamal (1997) analyze S&P500 and Nikkei 225 index options where the daily change in the volatility surface is specified by delta and maturity. Skiadopoulos, Hodges and Clewlow (1998) apply PCA to first differences of implied volatilities for fixed maturity buckets, across both strike and moneyness metrics. And Fengler et. al. (2000) employ a common PCA that allows options on equities in the DAX of different maturities to be analyzed simultaneously.

There is an important difference between the research just cited and the approach taken in this paper. Instead of applying PCA to daily changes in implied volatilities, a PCA is applied to daily changes in the deviations of fixed strike volatilities from at-the-money volatility. The advantages of this approach are both empirical and theoretical.

On the empirical front, time series data on fixed strike or fixed delta volatilities often display much negative autocorrelation, possibly because markets over-react. But the daily variations in fixed strike deviations from ATM volatility are much less noisy than the daily changes in fixed strike (or fixed delta) volatilities. Consequently the application of PCA to fixed strike deviations from ATM volatility, denoted $\Delta(\sigma_K - \sigma_{ATM})$, yields more robust results.

There is also a theoretical model that supports this. It will be shown below that the models of the skew in equity markets that were introduced by Derman (1999) can be expressed in a form where fixed strike volatility deviations from ATM volatility always have the same relationship with the underlying index. The particular market regime is determined only by a different behaviour in ATM volatility. Thus the stability of PCA on $\Delta(\sigma_K - \sigma_{ATM})$ is implied by Derman's models.

Derman (1999) asked 'how should implied volatilities be changed as an equity index moves?'. Derman's models are described below and in each model there will be a parallel shift in all volatilities as the index moves, where the size of this shift is determined by the current market regime. The model presented in this paper extends Derman's models to allow non-parallel shifts in the skew as the index moves. It uses PCA to actually quantify the sensitivities of implied volatilities to changes in the underlying price.

The results in this paper have two important applications: to dynamic delta hedging and the quantification of price-volatility scenarios in different market regimes. The delta of an option with value $f(S, \sigma)$, written as a function of the underlying asset price S and its volatility σ is given by

$$\Delta = \Delta_{BS} + (\partial f/\partial\sigma)(\partial\sigma/\partial S)$$

where Δ_{BS} is the Black-Scholes delta. Assuming constant volatility the delta of a European call option is $\Delta = \Delta_{BS} = \Phi(x)$ where $x = \ln\left(S/Ke^{-r\tau}\right)/\sigma\sqrt{\tau} + \sigma\sqrt{\tau}/2$ measures the 'moneyness' of the option.[1] However if volatility is not constant the additional term 'vega $(\partial\sigma/\partial S)$' needs to be included in the delta. Vega = $\partial f/\partial\sigma$ is the volatility sensitivity of the option value and it is normally approximated with finite differences. However $\partial\sigma/\partial S$ is more difficult to quantify. Many traders assume that $\partial\sigma/\partial S = \partial\sigma/\partial K$; that is, the volatility sensitivity to movements in the underlying price is taken from the slope of the skew (or smile) by strike.

This paper explains how principal component analysis can be used to obtain more precise measures of the volatility sensitivity to movements in the underlying price. The method used to measure these sensitivities is shown to be equivalent to a quadratic parameterization of the volatility surface; as such it extends Derman's linear skew parameterization. This paper will show how to generate scenarios for non-parallel movements in the smile surface that are appropriate for given movements in the underlying price.

The model has applications to all types of implied volatility surfaces, including currency option smiles and swaption skews. The present paper focuses on its application to the skew in the FTSE 100 between 4th January 1998 and 31st March 1999. It is found that the sensitivity of a fixed strike volatility to movements in the index will change according to market conditions and that the range of the skew (the difference between low strike volatility and high strike volatility) will normally fluctuate over time. However in jumpy markets the range of the skew is quite static and shifts in fixed strike volatilities are more likely to be parallel, as predicted by Derman's models.

2. Volatility Regimes in Equity Markets

Figure 1 shows the 1-month implied volatilities for European options of all strikes on the FTSE100 index for the period 4th January 1998 to 31st March 1999.[2] The bold red line

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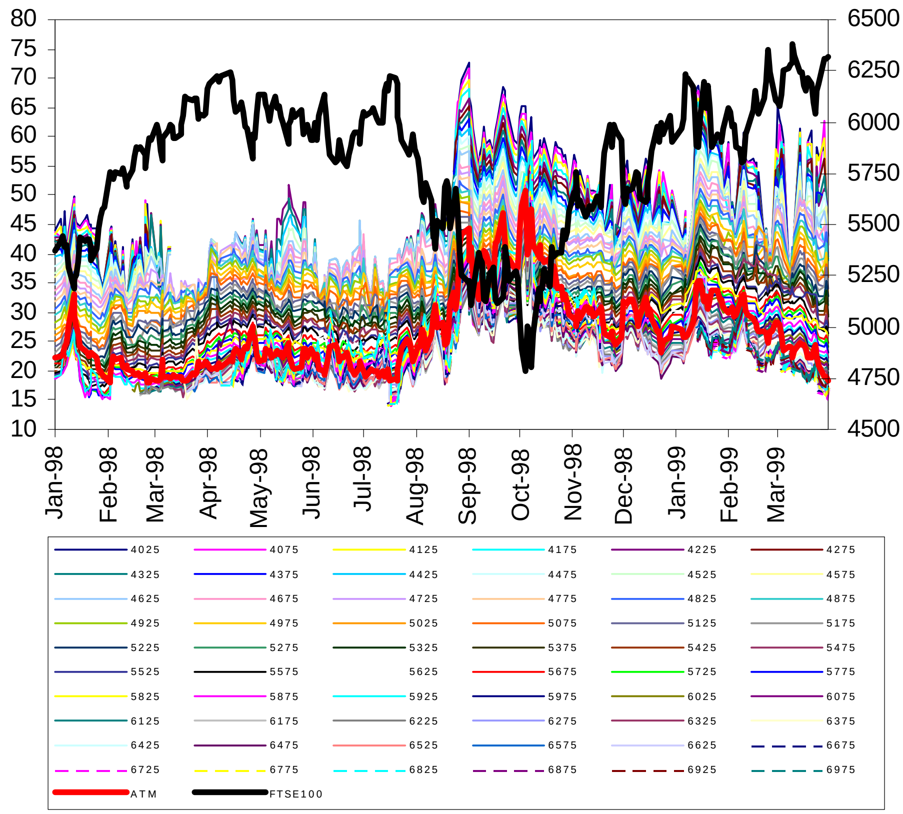
[1] With the standard notation, K denotes the strike, τ the time to expiry and r the interest rate.

[2] The fixed maturity implied volatility data used in this section have been obtained by linear interpolation between the two adjacent maturity option implied volatilities. However this presents a problem for the 1 month volatility series because often during the last few working days before expiry

indicates the ATM volatility and the bold black line the FTSE100 index price (on the right-hand scale).

Observation of data similar to these, but on the S&P500 index option 3 month volatilities, has motivated Derman (1999) to formulate three different types of market regime and to define a different linear parameterization of the volatility skew in each regime. These are known as 'sticky' models, because each parameterization implies a different type of 'stickiness' for the local volatility in a binomial tree. Denote by $\sigma_K(\tau)$ the implied volatility of an option with maturity τ and strike K, $\sigma_{ATM}(\tau)$ the volatility of the τ-maturity ATM option, S the current value of the index and σ_0 and S_0 the initial implied volatility and price used to calibrate the tree:

Figure 1: 1mth Fixed-Strike Volatilities, At-the-Money Volatility and the FTSE 100 Index Level



(a) In a range bounded market skews should be parameterized as

$$\sigma_K(\tau) = \sigma_0 - b(\tau)\,(K - S_0)$$

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data on the near maturity option volatilities are totally unreliable. So the 1 month series rolls over to the next maturity, until the expiry date of the near-term option, and thereafter continues to be interpolated linearly between the two option volatilities of less than and greater than 1 month.

If the index changes, fixed strike volatilities $\sigma_K(\tau)$ will not change but σ_{ATM} will decrease as the index increases: this can be seen by substituting in S = K above, giving

$$\sigma_{ATM}(\tau) = \sigma_0 - b(\tau)\ (S-S_0)$$

(b) In a stable trending market skews should be parameterized as:

$$\sigma_K(\tau) = \sigma_0 - b(\tau)\ (K-S)$$

In this model fixed strike volatility $\sigma_K(\tau)$ will increase with the index level but $\sigma_{ATM}(\tau) = \sigma_0$ so it will be independent of the index.

(c) In jumpy markets skews should be parameterized as:

$$\sigma_K(\tau) = \sigma_0 - b(\tau)\ (K+S) + 2b(\tau)S_0$$

Fixed strike volatility $\sigma_K(\tau)$ will decreases when the index goes up, and increase when the index falls. Since

$$\sigma_{ATM}(\tau) = \sigma_0 - 2b(\tau)\ (S-S_0)$$

the ATM volatility will also decreases as the index goes up and increases as the index falls, and twice as fast as the fixed strike volatilities do.

The range-bounded model (a) is called the 'sticky strike' model because local volatilities will be constant with respect to strike. That is, each option has its own binomial tree, with a constant volatility that is determined by the strike of the option. As the index moves all that happens is that the root of the tree is moved to the current level of the index. The same tree is still used to price the option.

The trending markets model (b) is called the 'sticky delta' model because local volatilities are constant with respect to the moneyness (or equivalently the delta) of the option. That is, it is the moneyness of the option that determines the (still constant) local volatility in the tree. As the index moves the delta of the option changes and we consequently move to a different tree, the one corresponding to the current option delta.

In the 'sticky tree' model (c) the local volatilities are no longer constant. There is, however one unique tree that can be used to price all options, that is determined by the current skew. This is the implied tree described in Derman and Kani (1994).

3. Fixed Strike Deviations from ATM Volatility

The relationship between fixed strike deviations from at-the-money volatility and the underlying price is the same in all of Derman's 'sticky' models. In fact for any maturity τ there will be a linear relationship between the deviation of a fixed strike volatility from ATM volatility and the underlying price that is given by:

$$\sigma_K(\tau) - \sigma_{ATM}(\tau) = -b(\tau)\ (K-S) \qquad (1)$$

For any given maturity, the deviations of all fixed strike volatilities from at-the-money volatility will change by the same amount $b(\tau)$ as the index level changes, as shown in figure 2a. Four strikes are marked on this figure: a low strike K_L, the initial at-the-money strike K_1, the new at-the-money strike after the index level moves up K_2, and a high strike K_H. The volatilities at each of these strikes are shown in figure 2b, before and after a unit rise in the index level. In each of the three market regimes the range of the skew between K_L and K_H, that is $\sigma_L - \sigma_H$, will be the same after the index move. Thus as the underlying price moves, the fixed strike volatilities will shift parallel, and the range of the skew will remain constant. The direction of the movement in fixed strike volatilities depends on the relationship between the original ATM volatility σ_1 and the new ATM volatility σ_2:

- In a range bounded market $\sigma_2 = \sigma_1 - b(\tau)$, but fixed-strike volatilities have all increased by the same amount $b(\tau)$, so a static scenario for the skew by strike should be applied;
- When the market is stable and trending, $\sigma_2 = \sigma_1$ and there is an upward shift of $b(\tau)$ in all fixed-strike volatilities;
- In a jumpy market $\sigma_2 = \sigma_1 - 2b(\tau)$, so a parallel shift downward of $b(\tau)$ in the skew by strike should be applied.

Figure 2a: Parallel Shift in Skew Deviations as Price Moves Up

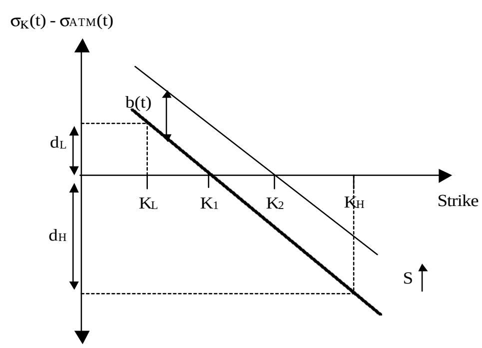
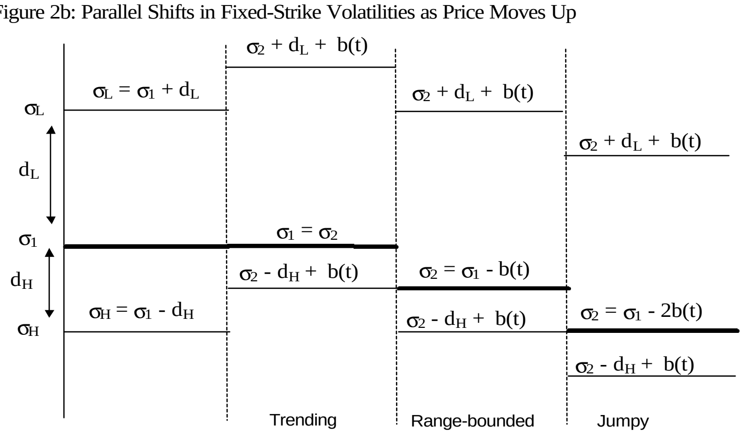


Figure 2b: Parallel Shifts in Fixed-Strike Volatilities as Price Moves Up



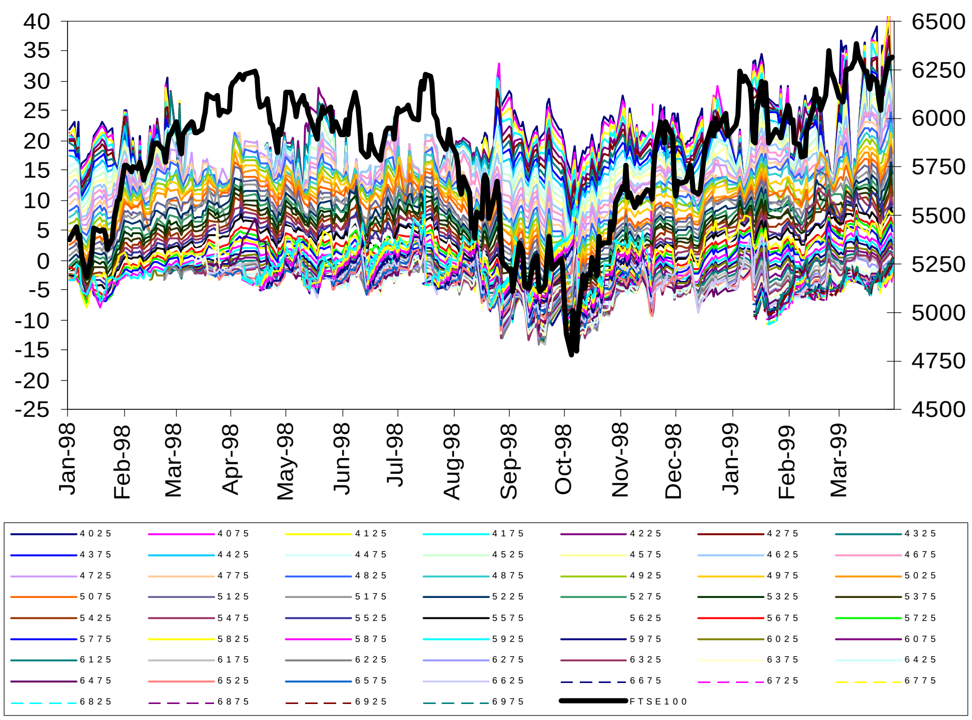
These observations suggest that a method for testing whether Derman's 'sticky' models are supported by empirical evidence is to perform a PCA of $\Delta(\sigma_K - \sigma_{ATM})$. Equation (1) implies that only the first principal component should be significant, but if it is found that the second or higher principal components are significant factors for determining movements in $\Delta(\sigma_K - \sigma_{ATM})$, then the parallel shifts in the skew that are implied by the 'sticky' models will not apply.

4. Principal Component Analysis of the Skew Deviations

There are around 60 different strikes represented in figure 1, and their implied volatilities form a correlated, ordered system that is similar to a term structure. It is therefore natural to consider using principal component analysis to identify the key uncorrelated sources of information, and there will only be a few.

A principal components analysis of daily changes in the fixed-strike volatilities shown in figure 1 may not give very good results, because the data will be rather noisy as mentioned above. But look at the deviations of the same fixed strike volatilities from the at-the-money volatility, shown in figure 3. The fixed strike deviations display less negative autocorrelation and are even more highly correlated and ordered than the fixed strike volatilities themselves. A strong positive correlation with the index is evident during the whole period.

Figure 3: Deviations of 1mth Fixed-Strike Volatility from At-the-Money Volatility



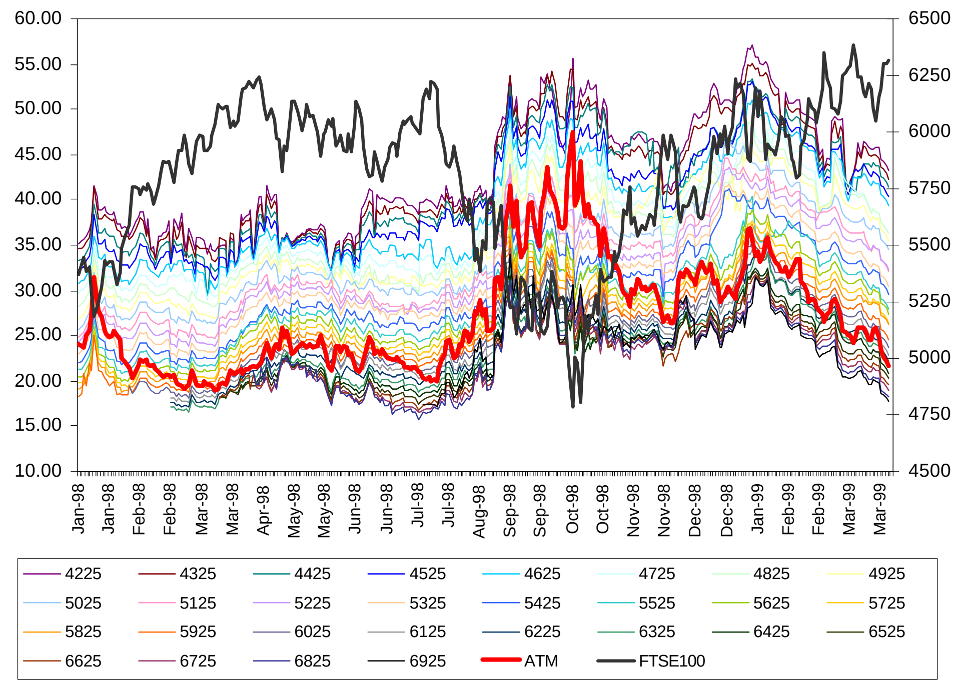
The PCA of fixed strike deviations $\Delta(\sigma_K - \sigma_{ATM})$ of a fixed volatility maturity τ is based on the model[3]

$$\Delta(\sigma_K - \sigma_{ATM}) \approx \omega_{K,1}\ P_1\ + \omega_{K,2}\ P_2\ + \omega_{K,3}\ P_3 \qquad (2)$$

Daily data on $\Delta(\sigma_K - \sigma_{ATM})$ is used to estimate the first three principal components (these are the daily time series P_1, P_2 and P_3) and the constant factor weights $\omega_{K,1}$, $\omega_{K,2}$ and $\omega_{K,3}$.

A PCA for 3 month implied volatility skew deviations based on the data shown in figure 4b gives the output in table 1. It is clear from table 1a that the first principal component is only explaining 74% of the movement in the volatility surface and that the second principal component is rather important as it explains an additional 12% of the variation over the period. It is interesting that the factor weights shown in table 1b indicate the standard interpretation of the first three principal components in a term structure, as parallel shift, tilt and convexity components. Note that sparse trading in very out-of-the money options implies that the extreme low strike volatilities show less correlation with the rest of the system, and this is reflected by their lower factor weights on the first component.

Figure 4a: 3mth Fixed Strike Volatilties, ATM Volatility and the FTSE



[3] Now that a time series analysis will be employed, to avoid confusing notation the time variable τ which indicates the volatility maturity has been dropped. The exposition in sections 4 and 5 takes it as given that a volatility maturity has been fixed (at either 1 month, 2 months or 3 months in this paper).

Figure 4b: 3mth Fixed Strike Volatility Deviations from ATM Volatility

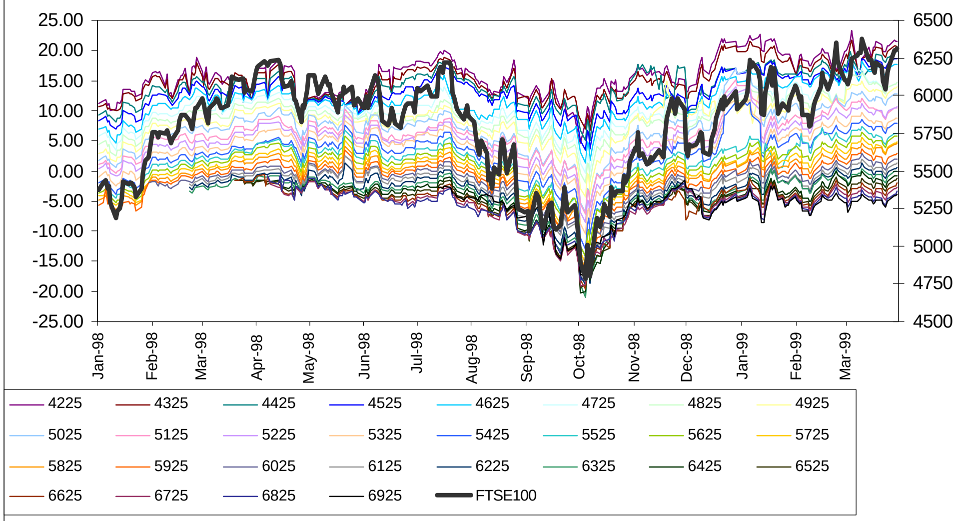


Figure 4c: 2mth Fixed Strike Deviations from ATM Volatility

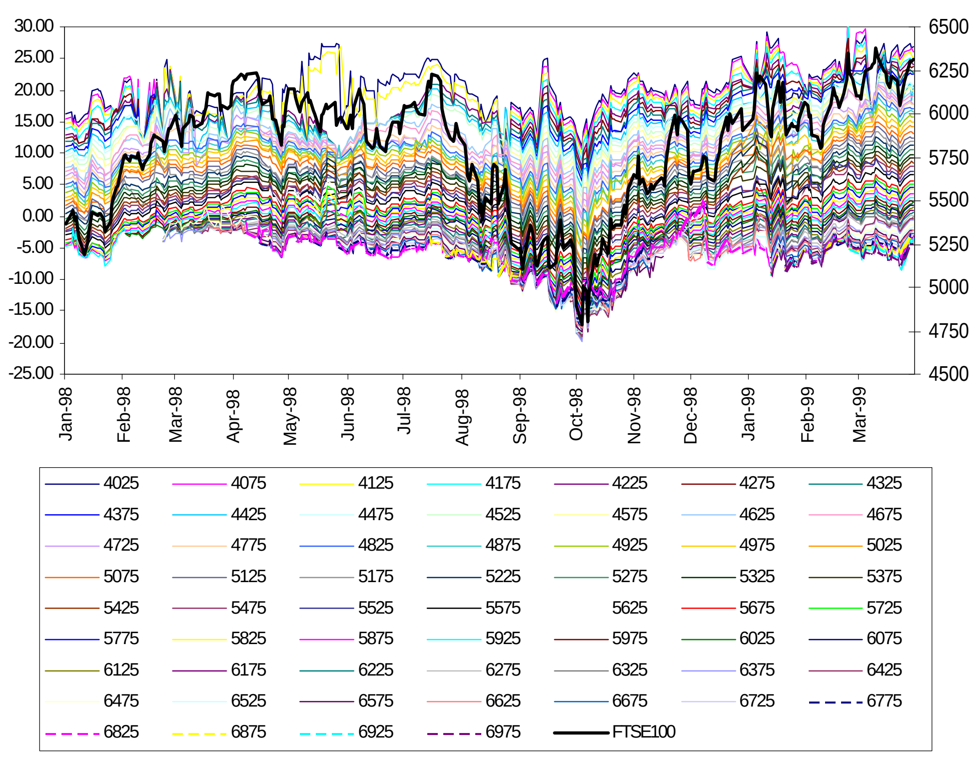


Table 1a: Eigenvalues of Correlation Matrix

Component	Eigenvalue	Cumulative R^2
P1	13.3574	0.742078
P2	2.257596	0.8675
P3	0.691317	0.905906

Table 1b: Eigenvectors of Correlation Matrix

Factor Weights

	P1	P2	P3
4225	0.53906	0.74624	0.26712
4325	0.6436	0.7037	0.1862
4425	0.67858	0.58105	0.035155
4525	0.8194	0.48822	-0.03331
4625	0.84751	0.34675	-0.19671
4725	0.86724	0.1287	-0.41161
4825	0.86634	0.017412	-0.43254
4925	0.80957	-0.01649	-0.28777
5025	0.9408	-0.18548	0.068028
5125	0.92639	-0.22766	0.13049
5225	0.92764	-0.21065	0.12154
5325	0.93927	-0.22396	0.14343
5425	0.93046	-0.25167	0.16246
5525	0.90232	-0.20613	0.017523
5625	0.94478	-0.2214	0.073863
5725	0.94202	-0.22928	0.073997
5825	0.93583	-0.22818	0.074602
5925	0.90699	-0.22788	0.068758

Principal component analysis of $\Delta(\sigma_K - \sigma_{ATM})$ for a fixed maturity τ has given some excellent results (Box 2). Alexander (2000) shows that for fixed maturity volatility skews in the FTSE100 index option market during most of 1998, 80-90% of the total variation in skew deviations can be explained by just three key risk factors: parallel shifts, tilts and curvature changes. The parallel shift component accounted for around 65-80% of the variation, the tilt component explained a further 5 to 15% of the variation, and the curvature component another 5% or so of the variation. The precise figures depend on the maturity of the volatility (1 month, 2 month or 3 month) and the exact period in time that the principal components were measured.

The immediate conclusion must be that linear parameterizations of the skew and the consequent limitation of movements in volatility surfaces to parallel shifts alone is an over simplification of what is actually happening in the data. The next section develops a model that encompasses changes in the tilt and curvature of the volatility skew as well as a parallel shift. So the range of the skew can widen or narrow as the underlying price moves up or down, and change its curvature also.

5. The Dynamics of Fixed Strike Volatilities in Different Market Regimes

It follows from (2) that the movement in fixed-strike volatilities as the underlying moves will be determined by the movement in the principal components. Each component is assumed to have a linear relationship with daily changes ΔS in the underlying. A linear model with a time-varying parameter $\gamma_{i,\,t}$ is estimated for each component:

$$P_{i,\,t} = \gamma_{i,\,t}\ \Delta S_t + \varepsilon_{i,\,t} \qquad (3)$$

where the ε_i are independent i.i.d processes. The movement in fixed-strike volatility deviations in response to movements in the underlying will be determined by the (constant) factor weights in the principal component representation (2) and the (time-varying) gamma coefficients in (3).

Figure 5 depicts the movement in skew deviations as the index price moves up, according to the signs of γ_2 and γ_3. Note that γ_1 represents the trend component and is always assumed to be positive, an assumption that is justified by the empirical analysis below. The coefficient γ_2 determines the tilt of the fixed strike deviations and γ_3 determines the convexity, so the four combinations shown represent all stylized movements in the skew deviations.

Figure 5a: Non-Parallel Shift in Skew Deviations as Price Moves Up

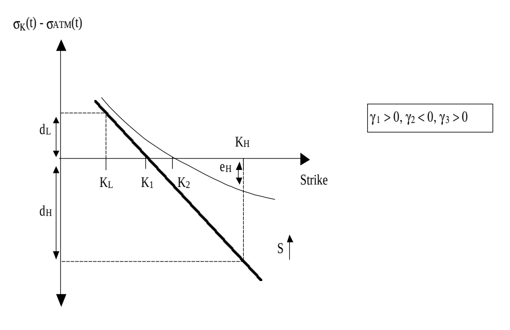


Figure 5b: Non-Parallel Shift in Skew Deviations as Price Moves Up

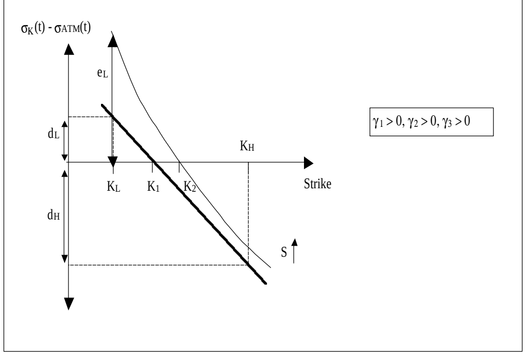


Figure 5c: Non-Parallel Shift in Skew Deviations as Price Moves Up

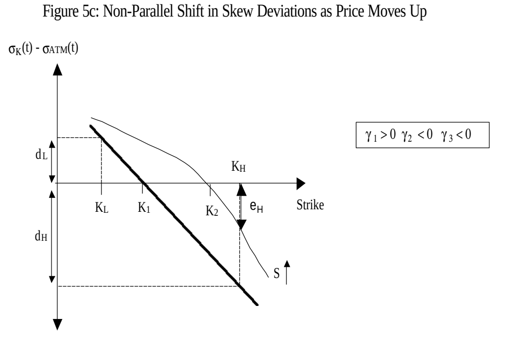
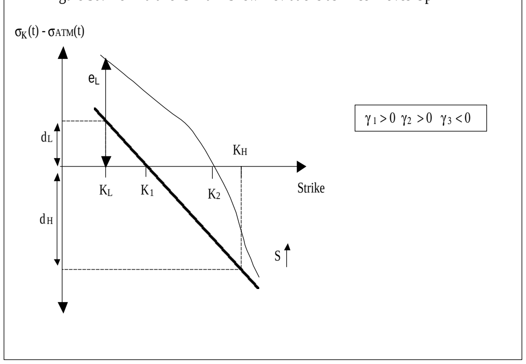


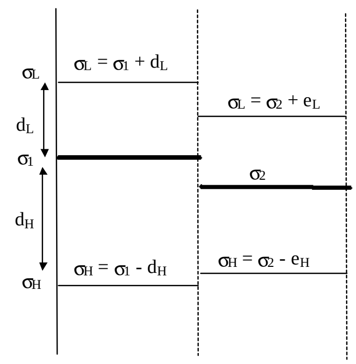
Figure 5d: Non-Parallel Shift in Skew Deviations as Price Moves Up



The deviation at the high strike K_H is denoted d_H before the move and e_H after the move, and similarly d_L and e_L denote the before and after deviations at the low strike volatility K_L. The relation between d_H and e_H and the relation between d_L and e_L will depend on the values of γ_1, γ_2 and γ_3. When γ_2 is negative it is clear from figures 5a and 5c that e_H will be less than d_H and that e_L is normally a little greater than d_L, unless γ_2 is very large and negative.[4] On the other hand when γ_2 is positive as in figures 5b and 5d, it is clear that $e_L > d_L$ but now the sign of e_H - d_H will ambiguous. But normally e_H will be a little less than d_H unless γ_2 is very large indeed.

The movements in skew deviations are translated in figure 6 to movements in the fixed strike volatilities themselves. In both cases there will be a change in the range of the skew as the index moves. When γ_2 is negative the range will narrow as the index moves up and most of the movement will be coming from low strike volatilities. But when γ_2 is positive the range will widen as the index moves up and there will be more movement in high strike volatilities.

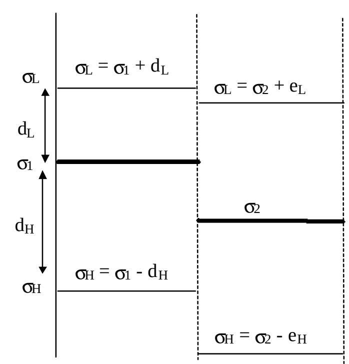
Figure 6a: Effect on Fixed-Strike Volatilities as Price Moves Up ($\gamma_2 < 0$)



Normally e_L is a little greater than d_L unless γ_2 becomes very large and negative. The range of the skew will narrow, more so when γ_2 is very large and negative.

But e_H will be less than d_H so most of the movement in the skew will come from the low strike volatilities and there may be little movement in high strike volatilities.

Figure 6b: Effect on Fixed-Strike Volatilities as Price Moves Up ($\gamma_2 > 0$)



Normally e_H is a little less than d_H and e_L is certainly greater than d_L.

Unless e_H is substantially less than d_H more than usual movement in high strike volatilities will be observed. And low strike volatilities will move less than they do when γ_2 is negative.

[4] If γ_3 were extremely large and negative then e_H would be less than d_H but this never occurs empirically.

The principal components have zero unconditional covariance; however by (3) their conditional covariance $\mathrm{Cov}_t\,(P_{i,\,t}\,,\,P_{j,\,t}\,) = \gamma_{i,\,t}\ \gamma_{j,\,t}\ \sigma_t^2$ where σ_t^2 is the conditional variance of the index, $V_t\,(\Delta S_t)$. In this paper the time-varying parameters are estimated using an exponentially weighted moving average model as an approximation to an integrated bivariate GARCH(1,1). This choice allows one to bypass the issue of parameterization of the bivariate GARCH which is a difficult issue in its own right.[5] It does of course introduce another issue, and that is which smoothing constant should be chosen for the exponentially weighted moving averages. For the sake of conformity with standard covariance calculations such as those in JP Morgan/Reuter's RiskMetrics[6] the smoothing constant $\lambda = 0.94$ has been taken.

Exponentially weighted moving average estimates of γ_1, γ_2 and γ_3 for each of the 1 month, 2 month and 3 month maturities have been calculated for each day from the beginning or March 1998 to the end of March 1999. These time series are shown in figure 7. The first point to note about all the graphs is that the estimate of γ_1 is positive throughout, and that it is generally higher and more stable than the estimates of γ_2 and γ_3. Since γ_1 captures the parallel shift component of movements in the skew, we can deduce that most of the movement in the skew at all maturities can be attributed to a parallel shift up when the index falls.

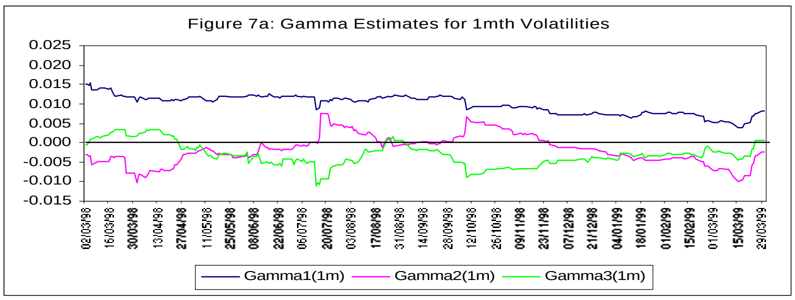


Figure 7a: Gamma Estimates for 1mth Volatilities

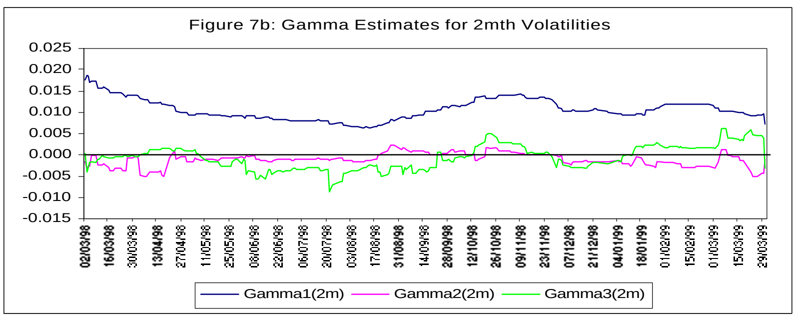


Figure 7b: Gamma Estimates for 2mth Volatilities

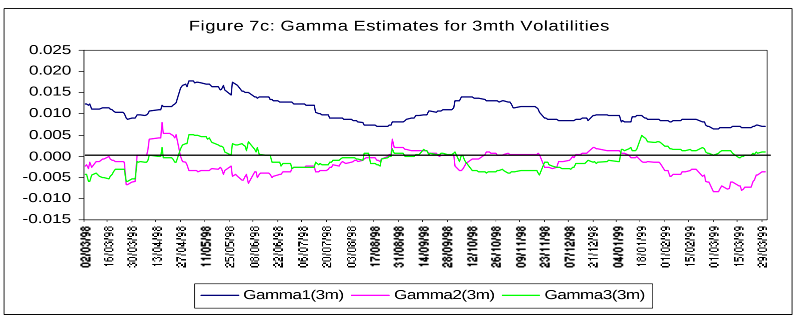


Figure 7c: Gamma Estimates for 3mth Volatilities

[5] A detailed discussion of this is given in Alexander 2001a, b or c.
[6] Available from www.riskmetrics.com.

The second point to note about figure 7 is that for the 2 month and 3 month maturities the index seems to have little effect on the second and third principal components, in fact the estimates of γ_2 and γ_3 are close to zero for almost all the sample period. There are a couple of negative γ_2 periods during the springs of 1998 and 1999, when the range of the skew will have narrowed as the index moved up and widened as it moved down. But this effect is not as pronounced as it is in the 1 month skew. Therefore, and particularly during the crash period, the results show that it is reasonable to apply parallel shift scenarios for fixed strike volatilities of 2 month and 3 month skews in the strike metric.

A different picture emerges, however, for the movement of the 1 month skew (figure 7a). The estimate of γ_2 is often negative, particularly during the spring of 1998 and the spring of 1999. At these times the range of the skew was clearly decreasing when the index rose and increasing when the index fell, an effect that is very evident in figure 1. But there are two notable periods, just before the beginning of the crash and during the market recovery, when the estimate of γ_2 was strongly positive and γ_3 was strongly negative (this is the case shown in figure 5d). On 14th July 1998, several days before the FTSE 100 price started to plummet, there was a dramatic increase in γ_2 and decrease in γ_3 so that $\gamma_2 > 0$ and $\gamma_3 < 0$. During this period the range of the 1 month skew will have narrowed as the index fell. Then between 8th and 12th October 1998, the FTSE 100 jumped up 8% in 2 days trading, from 4803 to 5190. At the same time γ_2 jumped up and γ_3 jumped down, so that again $\gamma_2 > 0$ and $\gamma_3 < 0$, and the range of the 1 month skew will have widened as the index moved up. The narrowing of the range of the skew as the index fell, and the consequent widening again as the market recovered, has been driven by movements in high strike volatilities. Examination of figure 1 shows that during this unusual period the high strike volatilities did indeed move more than usual.

6. Quantification of $\partial\sigma/\partial S$ and the Volatility Surface

For a fixed volatility maturity τ we assume that

$$\Delta\sigma_{ATM,\,t} = \beta_t\ \Delta S_t + \varepsilon_t \tag{4}$$

where the error process is again i.i.d. To capture the dependence of ATM volatility changes on the current market the time-varying parameter β_t is again estimated with an exponentially weighted moving average with $\lambda = 0.94$. These estimates are shown for $\tau = 1$, 2 and 3 months in figure 8. As expected the sensitivity of ATM volatility to changes in the FTSE is greater in 1 month options than in 2 month options, which in turn have greater sensitivity than 3 month options.

There is a striking pattern in figure 8: it is very clear indeed that the sensitivity of ATM volatility moves with the level of the index. It does not jump unless the index jumps. This finding contradicts the assumptions of Derman's models that have three distinct regimes, according as $\beta = 0$ (sticky delta), $\beta < 0$ (sticky strike) and $\beta << 0$ (sticky tree); in this framework the market will jump between different regimes as the value of β jumps between different constant values and $\partial\sigma_{ATM}/\partial S = \beta$.

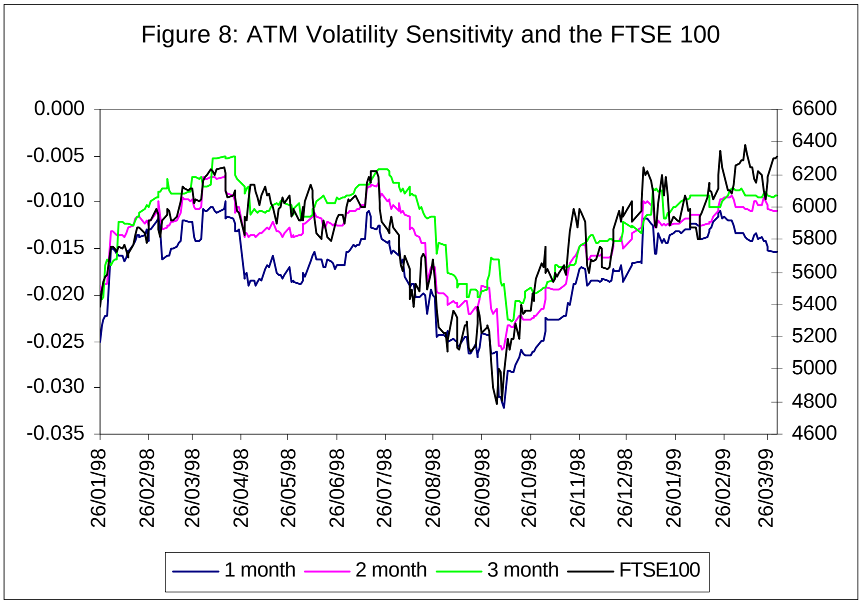
The empirical results of figure 8 suggest that that ATM volatility sensitivity β_t changes over time because the level of the index changes over time. Suppose therefore that

$$\partial\sigma_{ATM}(\tau)/\partial S = 2a(\tau)S \quad \text{where} \quad a(\tau) < 0.$$

From (1), $\partial\sigma_K(\tau)/\partial S = 2a(\tau)S + b(\tau)$, giving the quadratic parameterization of the volatility surface:

$$\sigma(S,\tau) = a(\tau)S^2 + b(\tau)S + c(\tau)$$

The model of time-varying ATM volatility sensitivity that depends on the index level is therefore equivalent to a second order Taylor approximation to the volatility surface.



In the discrete time framework, combining (2), (3) and (4) yields:

$$\Delta\sigma_{K,t} \approx \beta_{K,t}\, \Delta S_t \qquad (5)$$

where the sensitivity of the fixed strike volatility to the index is given by

$$\beta_{K,t} = \beta_t + \Sigma\, \omega_{K,i}\, \gamma_{i,t} \qquad (6)$$

Figure 9 shows the estimates of $\beta_{K,t}$ for strikes K between 4675 and 5875 and volatilities of 1 month maturity that are obtained from (6). These lowest and highest strikes are picked out in red and green. The index sensitivity of all fixed strike volatilities are negative, so they move up as the index falls but by different amounts.

During the crash period the sensitivities of all volatilities are greater and the change in the 5875 strike volatility sensitivity is very pronounced at this time. Before the crash it ranged between -0.005 and -0.01, indicating an increase of between 0.5 and 1 basis points for every FTSE point decrease. At the beginning of the crash the 5875 sensitivity increased to about 1.5 basis points, and since the FTSE fell by 1500 points during the crash, that corresponds to a 22.5% increase in 5875 volatility. Then at the height of the crash between 1st and 9th October the 5875 sensitivity became increasing large and negative as the FTSE index reached a low of 4786 on 5th October. On 9th October the 5875 sensitivity was an impressive -0.028, indicating a further 2.8 basis point increase in 5875 volatility would have occurred for every point off the FTSE at that time.

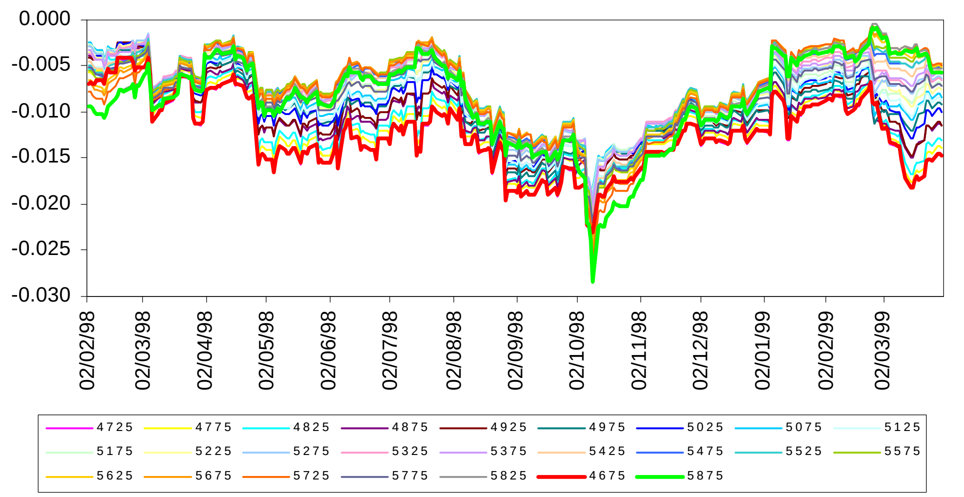


Figure 9: Change in 1mth Fixed Strike Volatility per Unit Increase in Index

Around the time of the crash an increase in low strike volatility sensitivities is much less pronounced. What is interesting about the 4675 volatility sensitivity is that it is often far greater (in absolute terms) than the high strike volatility sensitivities. So most of the movement will be coming from the low strikes as the range of the skew narrows when the index rises and widens when the index falls. Very approximately the 4675 volatility gains about 1 or 2 basis points for every point fall in the FTSE index during the period, although the sensitivity varies considerably over the period. At the end of the data period it is extraordinarily large, and it can be seen in figure 1 that range narrowing of the skew was very considerable at this time.

7. Summary and Conclusion

This paper has presented a new principal component model of fixed strike volatility deviations from ATM volatility. It has been used to quantify the change that should be made to any given fixed strike volatility per unit change in the underlying, that is $\partial\sigma/\partial S$. This quantity is an important determinant of an option delta when volatility is not constant. Market traders often approximate this sensitivity by $\partial\sigma/\partial K$ but the method outlined here explains how to calculate the volatility sensitivity to underlying price changes directly. This sensitivity has been found to depend on the current conditions in the equity market.

The methods of this paper have also been used to construct scenarios for the skew surface in equity markets that should accompany given moves in the underlying price. In many cases these scenarios should involve non-parallel shifts in the surface. Derman's 'sticky' models only allow for parallel shifts, and correspond to a local linear approximation for the volatility surface. The principal component approach that has been developed here allows for non-parallel shifts, which are shown to be particularly important for short maturity volatilities. It has also been shown that the model corresponds to a local second order approximation for the volatility surface.

Empirical application of the model to the FTSE 100 index options has shown that 2 month and 3 month skews should normally be shifted parallel as the index moves, as predicted by Derman's models. In the range-bounded markets in the spring of 1998 and 1999 there was also some narrowing in the range of the skew as the index moved up (and widening as the index moved down) but the range narrowing of the skew in a range bounded market scenario is much more pronounced in the1month skew.

The empirical analysis has revealed two distinct regimes for short-term volatility in equity markets. In stable markets the range of the 1 month skew narrows as the index moves up and widens as the index moves down. Most of the movement is in low strike volatilities and high strike volatilities remain relatively static as the underlying moves. There is a second regime in short-term volatilities that applies to the jumpy markets during a market crash and recovery period. In this regime the high strike volatilities move much more than usual and in the recovery period after the 1998 crash the 1 month skew range actually widened as the index moved up.

The model present in this paper has very general applications because it admits non-linear movements in the volatility smile as the underlying moves. Principal component analysis is shown to be a powerful analytical tool for the construction of scenarios of the implied volatility smile surface in different market regimes.

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References

Alexander, C. (2000) "Principal Component Analysis of Implied Volatility Smiles and Skews" *ISMA Centre Discussion Papers in Finance* 2000-10 University of Reading, UK.

Alexander, C. (2001a) "Orthogonal GARCH" in *Mastering Risk Volume 2* (C.Alexander, Ed.) FT-Prentice-Hall, London

Alexander, C. (2001b) "Orthogonal Methods for Generating Large Covariance Matrices" in *Wilmott Magazine Issue 2 (June)* available from www.wilmott.com

Alexander, C. (2001c) *Market Models: A Practitioners Guide to Data Analysis* John Wiley and Sons, Chichester (due for publication in July 2001).

Derman, E. (1999) "Volatility Regimes" *RISK Magazine* 12:4 pp55-59

Derman, E. and M. Kamal (1997) "The Patterns of Change in Implied Index Volatilities" *Quantitative Strategies Research Notes*, Goldman Sachs

Derman, E. and I. Kani (1994) "Riding on a Smile" *RISK Magazine* 7:2 pp32-39

Fengler, M., W. Hardle and C. Villa (2000) "The Dynamics of Implied Volatilities: A Common Principal Component Approach" *Preliminary version (September 2000) available from* fengler@wiwi.hu-berlin.de

Skiadopoulos, G., S. Hodges and L. Clewlow (1998) "The Dynamics of Implied Volatility Surfaces" *Financial Options Research Centre Preprint* 1998/86 Warwick Business School, University of Warwick, UK.