Bond options and swaptions pricing: a computational investigation of volatility inference

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Abstract

Derivative pricing is especially challenging in novel and illiquid markets where pricing relies greatly on assumptions and models rather than on known flow of market prices. In the novel market of shekel bond options the estimate of implied volatility for different strikes could be based on the information about other - sometimes more liquid - financial instruments in the market. Here we show relevance but not equivalence of the information from the market of swaps (volatility of swap rates) to the market of bonds (volatility of bond prices). In particular we show why the proxy to bond's yield to maturity from the swap market should be based on the swap rate maturing simultaneously with the bond. Numerical simulations and analysis of historical data are applied to examine the approximation and assumptions which, in the presence of swaption market, can be applicable for inferring information about bond price volatility smile. Hypothetically the procedure is invertible - inferring information about swap rate volatility smile based on the data from the bond options market. Our analysis is implemented for the Israeli market while the rationale is relevant for similar instruments elsewhere.

1 Introduction

Pricing of vanilla contingent claims (our case consists of bond options and swaptions) in the market is based on Black-Scholes formula. Black-Scholes formula assumes that the fluctuations of the logarithm of the asset price (bond price or swap rate in our case) are normally distributed. Traders correct that generally vague assumption by using non-constant implied volatility represented as either a function of strike or as a function of option's delta in the Black-Scholes formula. Therefore the notion of implied volatility is an alternative way to cite an option price.

In illiquid market (the market of shekel bond options[1] belongs to this category) the flow of bid and ask prices is not readily available or the spreads between bid and ask prices at various strikes are large. However, bond price is not a standalone asset as its price is defined by the corresponding interest rate curve. Therefore we expect interplay between options in two interest rate markets: 1) bonds and 2) swaps. Although the 1) bond yields and 2) swap rates *are derived from different interest rate curves*, the values

[1] Shekel options are traded on specific bonds listed in Table 1. An alternative practice of trading options on cheapest to deliver bond futures is adopted, for example, in the US market.

and fluctuations over time of these two notions possess similarities which is supported by our empirical analysis of the shekel market.

Here we analyze the plausible connection between the contingent claims on bonds and on swaps. Assuming that sometimes the swaption market could be more liquid than the bond options market, our goal is to estimate how a gist about the volatility smile for bond options can be practically inferred from the volatility smile for swaptions. Of course in cases when bond options market is more liquid than the swaptions market the procedure is reversible.

A simple approximate relationship between bond price volatility (σ_B) and yield volatility (σ_y) is provided by a well known approximation formula $\sigma_B \approx D\, y\, \sigma_y$ (D is bond's duration, y is yield to maturity, more details about the formula are provided further in text). Here we analyze the validity of this relationship for pricing purposes in the framework of inferring implied volatility from one market to the other as described above.

The notation and methodology are introduced in "Background and methods". The findings are presented in the "Results" sections which is followed by "Discussion". Derivations and more details are provided in Appendices A, B.

1.1 Background and methods

Given a zero coupon interest rate curve derived from the bonds' market, the price of a bond is related to the interest rates as follows:

$$\text{Bond's dirty price} = C\sum_{i=1}^{N}\frac{T_i - T_{i-1}}{(1+r_i)^{T_i}} + \text{Par}\cdot\frac{1}{(1+r_N)^{T_N}}$$

with

C	Fixed coupon
i	i-th future cashflow
r_i	Annually compounded zero coupon interest rate corresponding to time T_i
N	Number of future annual coupon payments
T_i	Time from now to the i-th cash flow measured in years, $T_0 = 0$
Par	Bond's price at maturity without interest; the amount of coupon payment corresponds to the percent of Par

We will further assume that the term $T_i - T_{i-1}$ which is very close to 1 for bonds with annual coupon[2], is equal to 1:

$$\text{Bond's dirty price} = C\sum_{i=1}^{N}\frac{1}{(1+r_i)^{T_i}} + \text{Par}\cdot\frac{1}{(1+r_N)^{T_N}} \quad (1.1)$$

1.1.1 Bond price volatility versus yield volatility

A more convenient representation of the bond price in terms of interest rates is based on the notion of a *single* interest rate called yield to maturity (y) which defines a discounting of the future cashflows leading to the correct bond price instead of discounting with a number of different interest rates from the curve in formula (1.1):

$$\text{Bond price} = C\sum_{i=1}^{N}\frac{1}{(1+y)^{T_i}} + \text{Par}\cdot\frac{1}{(1+y)^{T_N}} \quad (1.2)$$

Following ample literature, eg [1], [3], the linear sensitivity of the bond price (B) to small fluctuations in continuously compounded yield to maturity is related via a notion called duration (D) for the case of continuous compounding. It is called modified dura-

[2]Bond options in Tel Aviv stock exchange are traded on Shahar bonds which bear fixed annual coupon.

tion for the case of non-continuous compounding[3]:

$$\frac{dB}{dy} = -DB \ . \tag{1.3}$$

Therefore the relative change in bond price can be approximated to the first order with the relative change in yield to maturity as follows:

$$\frac{\Delta B}{B} \approx (-Dy)\,\frac{\Delta y}{y}$$

implying an approximate relationship between the volatilities of bond price and yield (eg. [1]).

$$\sigma_B \approx D y \sigma_y \ . \tag{1.4}$$

However we could not find information concerning the validity of approximation (1.4) for bond option's pricing, in case yield volatility is available, in the literature and *our study intends to fill at least part of this gap*.

The implied volatility of the option is derived from the option price. Option price is equal to the discounted expectation of option's payout at maturity [1]:

$$\text{Call(Bond price)} = e^{-rT}\int_{\text{Strike B}}^{\infty} (B - \text{Strike}B)\,\pi_B(B)\,dB$$

$$\text{Call(yield)} = e^{-rT}\int_{\text{Strike y}}^{\infty} (y - \text{Strike}y)\,\pi_y(y)\,dy \ .$$

Here $\pi(\cdot)$ denotes the risk neutral probability density[4] of the corresponding asset value at option's maturity, T is the time to delivery of option's payout if the option expired in the money and r is continuously compounded interest rate corresponding to T.

To see how the implied volatilities for yield and bond price can be related, we notice from formula (1.2) that whenever the yield grows the bond price decreases and vice versa - when the yield decreases the bond price grows. That is the relationship between the bond yield and bond price is monotonous. Therefore whenever we know the risk neutral probability of the yield (at option expiry) to be below some number, say $\Pi_y(y < a) = A$, we also know that the risk neutral probability for the bond price (at option expiry) to be above $B(a)$ is the same[5]:

$$\Pi_B(B > B(a)) = \Pi_y(y < a) = A \ . \tag{1.5}$$

In this case, whenever we know the probability distribution of bond yield (at option expiry), we will know the probability distribution of bond price (at option expiry). So the price of a vanilla option on bond price can be computed given risk neutral probability distribution of bond's yield to maturity as presented in Appendix B. Therefore, whenever we are provided with yield volatility smile and thus with the option prices on yield, we can estimate the prices of the options on bond and to further derive volatility smile for the bond price.

The above mentioned approach is based on the numerical differentiation of option prices and further integration with respect to risk neutral probabilities. Therefore one needs to know the yield volatility smile for a wide range of strikes covering probability almost 1 of yield's forward value being in the range. So the approach is purely theoretical in case of poorly liquid market and is called "theoretical" later on in text. Finally, in simulations, the volatility smile for the bond price obtained based on the *theoretical approach* can be compared with the volatility smile obtained based on the approximation in formula (1.4). In this way the feasibility of the approximation formula (1.4) is tested. Figure 1 depicts the schematic representation of the above mentioned approach.

[3]For the case of annual compounding the value of duration is computed as follows, based on formulae from Chapter 4 of [1]: $D = \dfrac{C\sum_{i=1}^{N}\frac{1}{(1+r_i)^{T_i}} + BT_N\cdot\frac{1}{(1+r_N)^{T_N}}}{B\cdot(1+y)}$ with y being yield to maturity.

[4]Using the terminology of [1], in the world which is risk neutral with respect to zero-coupon bond maturing at time T.

[5]Formula (1.5) is correct as payouts of digital calls on bond price are simultaneous with the payouts of digital puts on yield with corresponding strikes.

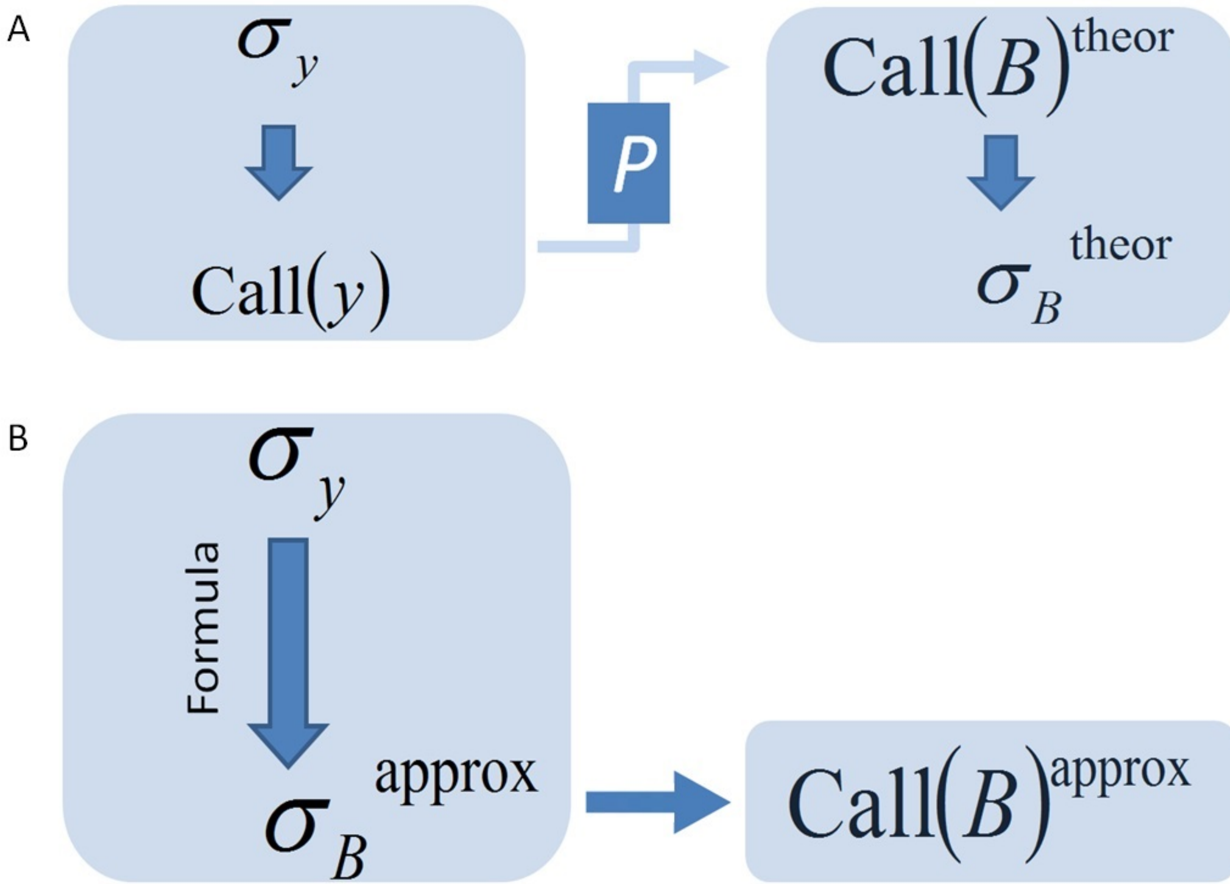

Figure 1: Schematic representation of the "theoretical" (A) and approximation based on formula (1.4) (B) approaches to computing bond price implied volatility smile when the volatility smile for yield to maturity is known. **(A)** Given volatility smile for bond's yield, corresponding option prices are computed. Bond option prices are derived from the prices of the yield options using equivalence of probabilities mentioned above. Volatility smile for bond price is computed directly from the option prices. **(B)** An approximated volatility smile for bond price is computed directly from the yield volatility smile using formula (1.4). Yield option prices and bond option prices can be directly computed given corresponding volatility smiles. The simulations aimed at analyzing the accuracy of approach based on formula (1.4) provide us with the volatilities and bond option prices in (A) which are further compared with the volatilities and bond option prices in (B). Yield volatilities in (A) and (B) are the same. Bond price volatilities and bond option prices in (A) and (B) are different.

Pricing of bond options by traders should satisfy high standard of accuracy. Here we consider the requirement for the price to be within the bid-ask spread (which is usually limited by several tens basis points[6] of the principle in the shekel swaption market). Nevertheless, the validity of using the approximation (1.4) for bond option pricing is unknown. To this end, the "theoretical" approach is used to assess the accuracy of formula (1.4) in *simulations*: bond option prices are derived based on two methods: 1) "theoretical", 2) approximation based on formula (1.4) and the results of the two approaches are compared.

We attempt to use the implied volatility of swap rates (from swaption market) as a proxy for pricing bond options. Here we rely on a possible closeness between the fluctuations of the swap rate and bond's yield to maturity[7]. The closeness of fluctuations is tested for empirical data. An example of interest rate curves from the swap and government bonds markets is depicted in figure 2A. The spread between the two curves is depicted in Figure 2B. In this example the curves are usually very close one to the other. The magnitude of the difference between the interest rates corresponding to the same periods is smaller than the interest rates by tens times, which is typical. However the spread between the swap and bond curves is large enough to consider these two markets separately.

Options on swap rate (swaptions) and bond options are both priced based on Black Scholes formula. The Black-Scholes framework assumes normal distribution of asset's relative fluctuations. However relative fluctuations of bond price and yield (assuming that swap rate is a proxy for the bond yield) cannot be normal simultaneously[8]. Therefore simplistic assumption of constant implied volatility for both parameters simultaneously necessarily leads to arbitrage opportunities. So the volatility smile for at least one of the two parameters (1. bond price, 2. bond yield) must be non flat. Finally, if one of the two assets, say yield, has non-flat volatility smile, still the non-flat shape of yield smile does not necessarily lead to the flat bond price smile. Therefore we apriori use the entire volatility smile (instead of a single value of the at the money volatility) for the swap rates in order to infer information about presumably non-flat volatility smile for the bond price.

1.2 Swap rate versus bond yield to maturity

From year 2011 bond options in Tel Aviv stock exchange are traded on two Shahar bonds paying fixed annual coupon. The two bonds are listed in Table 1. Options' time to expiry is up to 3 months.

Here we consider the bond and the fixed leg of the swap[9] both paying fixed annual coupon. Bond and swap both pay annually a constant amount, coupon and swap rate respectively. However, only bond pays Par amount at maturity, which constitutes the conceptual difference between the bond price and the net present value (NPV) of the swap (fixed) rate payments[10]. Still, in the simplistic world where bond and swap are priced based on the same interest rate curve,

[6] One basis point of a quantity is equal to 0.0001 or 0.01% of that quantity. If the principle of the option is 10 million shekel, then 1 basis point would be equal to 1,000 shekel.

[7] Interest rate curves in the swap and bond markets are different due to different credit properties of swaps and bonds. The differences between zero coupon interest rates in those two markets vary along the interest rate period and along registration time. Anyhow, the difference is tens times smaller than the rates themselves.

[8] Assume that σ_y is constant in formula (1.4). Different strikes y would then imply non-constant values of σ_B. Modifications of D due to changing y do not compensate for the effect of changing y. Our argument is applied to approximation formula and is not a rigorous proof. However it clearly demonstrates a clear phenomenon. A more rigorous proof can be constructed based on the definition of the yield to maturity notion.

[9] We consider swap whose fixed leg provides annual payments. The market swap rate then is equal to the fixed annual payments whose net present value (NPV: the sum of future cashflows discounted to today) is equal to the NPV of the floating leg payments. Standard shekel interest rate swaps (IRS) provide quarterly floating rate payments and annual fixed rate payments.

[10] Formally the Par value might be paid at the maturity of the swap however the payments would identical at the floating and the fixed legs and therefore practically this formal payment

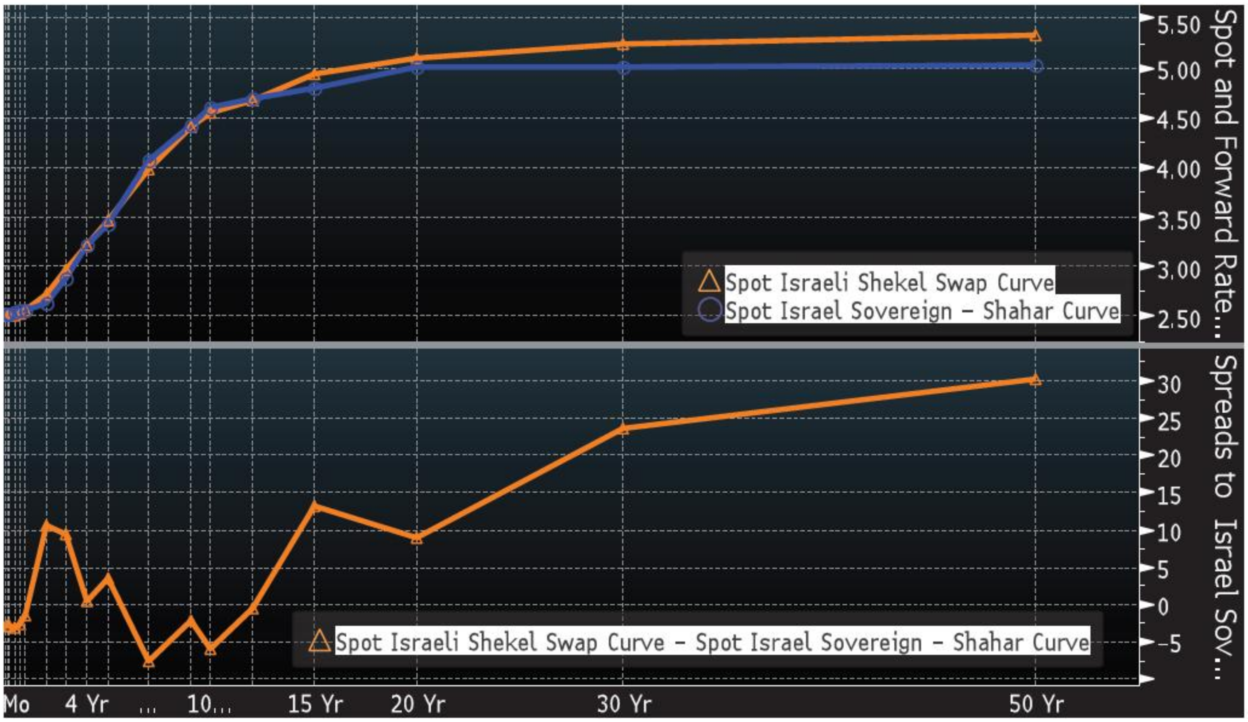

Figure 2: **Upper plot.** Interest rate curves from the bond (blue) and swap (orange) markets measured in percents. **Below plot.** Difference between the interest rates of the two curves (spread) measured in basis points; 1bp = 0.01%. The spread is tens times smaller than the interest rates themselves. For most frequently used time periods (up to 20 years) the spread is bounded by 15 basis points. The curves correspond to April 11, 2012. In this typical example the two curves are close to each other both in terms of their values and in terms of how the interest rates change over the tenor. Chart courtesy of Bloomberg LP.

Bond	Maturity	Years to maturity	Coupon
0120	31 Jan 2020	~ 6.5	5%
0217	28 Feb 2017	~4.75	5.5%

Table 1: Shahar bonds on which options with 3 types of time to maturity (0-1 month, 1-2 months, 2-3 months) were traded in 2011-2012. The table corresponds to August 2012.

bond coupon and swap rate are equal when the bond is at par[11] (see Appendix A). Moreover bond coupon and swap rate are equal to annually compounded yield to maturity of a bond in such a case (Appendix A). However bonds are rarely traded at par. For example, bond may pay 5% coupon while bond's yield to maturity is only 3.5%. Nevertheless, as we show in Appendix A, the value of bond's yield to maturity is rather conservative compared to "fluctuations"[12] of bond's coupon from the yield to maturity. Therefore, when bond and swap are priced with the same interest rate curve[13] (a seemingly vague simplification is discussed based on the results of the analysis of empirical data further in text), bond's yield to maturity and swap rate will be close[14] even if the difference between bond's yield to maturity and coupon is large, say 3% (see Table 3 and Figure 13 in Appendix A). In such cases bond's yield to maturity and swap rate correspond to nearly equivalent notions but in different markets, assuming the schedule for swap's fixed payments and bond's coupon payments is the same, say annum.

Similarity of the two notions (1. bond yield to maturity; 2. swap rate) when both bond and swap's fixed leg have the same schedule of payments justifies our motivation to relate information about the volatility of swap rates to volatility of bond's yield to maturity or vice versa. Note that bond's duration used in formula (1.4) is usually smaller than bond's time to maturity. The relationship between the bond and swap markets is analyzed for empirical data in the Results section.

A practical tool to computing the bond price volatility smile based on yield volatility smile would be approximation formula (1.4). However approximation accuracy is a critical issue for correct trading. To this end we apply the simulations illustrated in Figure 1 in order to test the accuracy of the approximation in formula (1.4) for the whole range of strikes. We are not aware of other studies which have analyzed the applicability of the accuracy of formula (1.4) for pricing derivatives.

1.3 Data analysis

Similarity between the swap rates and bond yields is the core assumption in our analysis. In order to compare the swap rates and yield to maturity from the market we analyze the historic data for synthetic bond yield to maturity (5 years to maturity, 5% coupon; bond yields were defined via historic sovereign interest rate curves, data source: HedgeTech)[15] and 5 years swap rate (data and chart in Figure 2 courtesy of Bloomberg LP). Both sets of data correspond to end of day measurement.

At each analyzed date the historic volatility was computed for 66 previous measurements (approximately 3 months) as follows:

$$\sigma^{hist}(i) = \sqrt{\frac{1}{65}\sum_{k=i-65}^{i}\left(\ln\frac{x(k)}{x(k-1)} - \frac{1}{66}\sum_{k=i-65}^{i}\ln\frac{x(k)}{x(k-1)}\right)^2},$$

where x stands either for bond yield to maturity or swap rate, the computed parameter corresponds to day i.

Closeness of the data dynamics (simultaneous decrease or increase) was measured with the correlation coefficient. Positive correlation (over time) between, for example, volatility measurements in bond and swap markets means that on average both parameters increase or decrease simultaneously. Correla-

is irrelevant when both legs are of the same currency.

[11] That is (clean) bond price is equal to bond's par value (value for which interest is paid).

[12] The value of the coupon is fixed during bond's life and only yield to maturity may fluctuate according to the fluctuations of the interest rate curve. However mathematically we can model the situation when the coupon changes and see that the consequences for the yield to maturity (which depends on both the interest rate curve and the coupon) are significantly weaker than coupon's "change" (see Appendix A).

[13] An empirical comparison of yield to maturity and swap rate is presented in Results section.

[14] Even when some bond is not at par, its yield to maturity will usually be close to that of another bond at par, both bonds having the same payments schedule, see Appendix A for more details.

[15] Shekel bond options are traded for the bonds whose maturity is either below 5 years or above it, see Table 1.

tion being 1, maximal possible value, also means that the ratio between the fluctuations of both parameters is constant: ChangeBonds(i) / ChangeSwaps(i) = ChangeBonds(k) / ChangeSwaps(k) for any two dates i, k. However positive correlation does not imply that the changes are identical.

Most of time over the analyzed period the two parameters 1) 5Y swap rates and 2) yield to maturity of 5 years bond paying 5% annual coupon are close and decrease or increase simultaneously, see Figure 3. The correlation between the two parameters is always positive and most of time is above 0.8. The correlation between the daily relative fluctuations of the two parameters is depicted in Figure 4; it is always positive and most of time is above 0.5.

2 Results

Formula (1.4) and historic observations may serve as a basis of a practical tool for inferring bond price implied volatility smile from a swap rate implied volatility smile. Firstly the "theoretical" approach (see Methods) is applied to test the accuracy of the formula (1.4) for bond yield to maturity instead of the swap rate. Secondly we analyze the closeness between the volatility of bond yield to maturity and swap rates for historic data. Finally the findings are discussed.

2.1 Testing with the "theoretical" approach: accuracy of approximating bond price vol based on bond yield vol according to formula (1.4)

The tests were conducted for the options having the properties similar to those of the traded bond options (Table 1). Tested options had 3 months to maturity[16] and the underlying bonds had 5 years to maturity. The tests were implemented for two kinds of yield volatility smile: 1) artificial flat smile with volatility 0.35; 2) volatility skew observed once in USD swaption market. Detailed demonstrations of the exemplar results for both kinds of volatility smile are depicted in Figures 5-8. Upper parts of Figures 5-8 demonstrate volatility smile for the yield and two volatility smiles of the bond price, one obtained via formula (1.4) and the other resulting from "theoretical" reconstruction. The lower parts of figures 5-8 demonstrate option prices obtained based on the two volatility smiles for the bond price and difference between the option prices. The graphs in Figures 5, 7 are depicted versus strikes while in Figures 6, 8 the graphs are depicted versus delta. Visualization of the results versus delta provides the possibility to see how "probable" are certain parameters, eg. whether the strike corresponding to the maximal difference between the two graphs of option prices is close to the at the money value or not.

Maximal values of the difference between option prices in the examples depicted in Figures 5 - 8 reach 7 and 2 basis points for the artificial flat and realistic smile respectively as can be seen in the lower part of these figures. The difference for the flat smile is noticeably greater than the difference for the example with the realistic shape of the smile. The graphs versus delta (Figures 6, 8) show that the probability of achieving the asset value with the greatest difference is not negligible however such event is not that certain as it corresponds to a delta close to 0.1. We conclude that existence of approximation errors should be accounted for when formula (1.4) is used to approximately construct volatility smile.

The examples in Figures 5, 7 correspond to specific values of the bond coupon and yield to maturity: the coupon and the at the money (ATM) yield are equal to 3.5%[17]. Therefore the bond is almost at par (Ap-

[16] The "theoretical" approach compares option prices (Figure 1). The difference between option prices decreases when option's time to maturity decreases. We consider the maximal possible (for shekel bond options) time to maturity to account for maximal possible errors.

[17] Here bond price corresponding to the yield equal to ATM yield (B(ATM yield) = 100 as here ATM yield is equal to the coupon) is not identical to the ATM bond price (ATM Bond price = 100.0481) due to convexity adjustment [1]. For more clarity, ATM bond is not necessarily at par when the ATM yield = coupon, that is its price is different from 100, as appears as example in column 6 of Table 2. ATM bond is at par when (forward yield) = coupon. Forward yield is the yield corresponding to the forward bond price

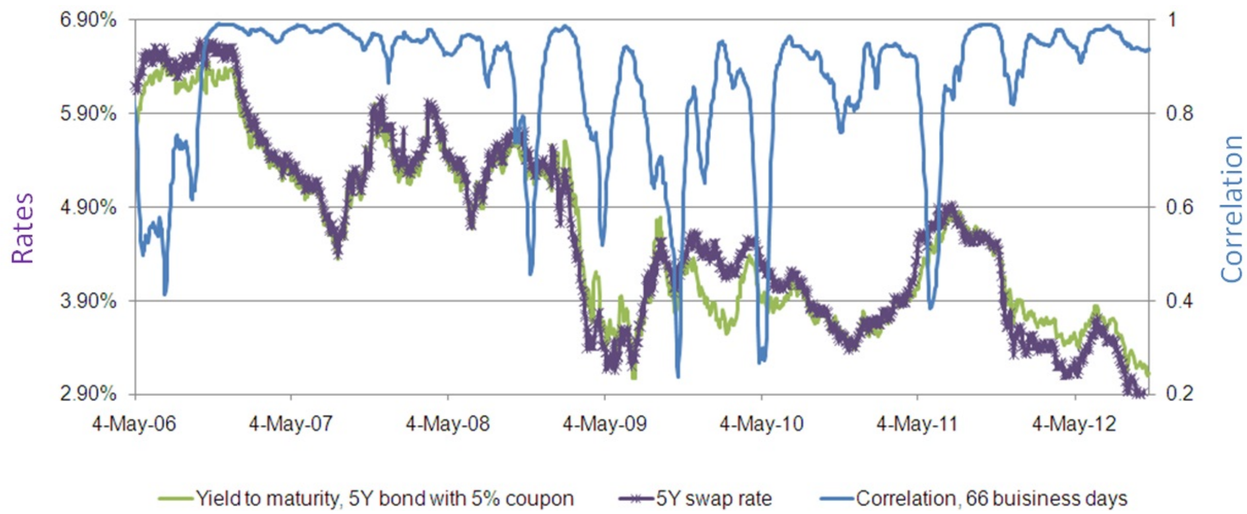

Figure 3: Yield to maturity for 5 year bond and 5 year swap rate for the period of more than 6 years. The values of the two parameters are usually close and usually increase or decrease simultaneously. Their correlation coefficient (right axis) is always positive and is above 0.8 most of time.

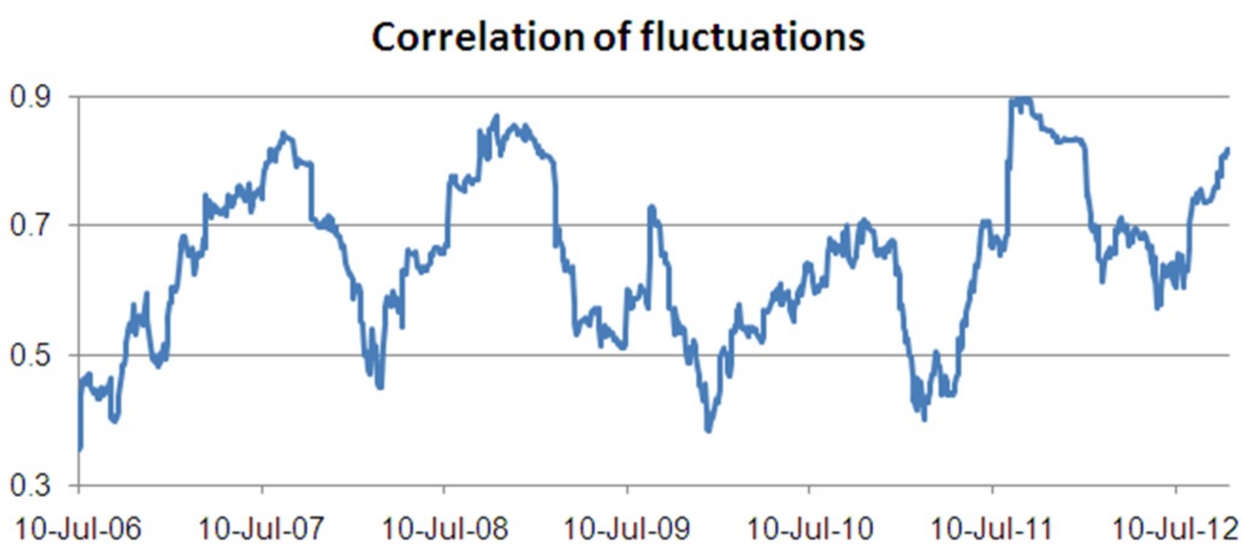

Figure 4: Correlation between daily relative fluctuations of 5Y swap rates and yield to maturity of 5 years bond with 5% annual coupon. Each correlation coefficient was computed for 3 months of measurements. The correlation coefficients are always positive and most of time above 0.5.

pendix A). We expanded the list of synthetic bonds under consideration and implemented the same analysis for the bonds with coupon and yield ranging from 1% to 10%. The results for the cases when coupon and ATM yield are the same are presented in Table 2. We see that the maximal difference (column 2 of Table 2) increases with increase of the rates. However what happens when coupon and yield to maturity undergo unequal changes? Such cases are demonstrated in Figures 9, 10. Figure 9 shows the differences when the value of the (flat) yield volatility is equal to 0.15, and the differences for the case of the flat volatility being equal to 0.35 are shown in Figure 10. The differences between the option prices increase with increase of volatility (Figure 9 vs Figure 10) and with increase of the interest rates. The differences are rather small in case of volatility being 0.15. However in the case of volatility level equal to 0.35 and when the rates approach 10% the differences in turn approach the level of the bid-ask spread for option prices in the market. Anyhow for the current level of 5Y interest rates, coupons below 6%, and yield to maturity below 4%, the differences are tolerable being below 10 basis points.

2.2 Swap rate versus bond yield to maturity, analysis of historic data

This work tests an approach suggesting to use the information about swap rate implied volatility smile to assess the bond price implied volatility smile. As the first step simulations have been used to test the accuracy of approximating the bond price volatility smile given bond yield volatility smile. However swap rates and bond yields are not identical notions. To this end as the next step of the tests the similarities and differences between the swap rate and bond yield to maturity are analyzed based on historic data. The swap and bond's fixed leg are assumed to have the same schedule of payments in our analysis. At each date yield to maturity of the bond paying 5% coupon was derived based on the corresponding interest rate curve (see Methods).

Historical volatilities for 5 years swap rates and yield to maturity of 5 years bond paying 5% annual coupon are depicted in Figure 11A. Visually both volatilities possess similar pattern along time, nevertheless there are differences between their values. Volatilities weighted with the level of the rate, the way volatilities appear in formula (1.4), are depicted in Figure 11B together with correlation between the values corresponding to the swap rates and bond yields. The correlations are mostly high. Indeed, in Figure 12 the histogram for the correlation values shows that most of time the correlation is above 0.81. However in rare cases and during short time periods historical volatilities for bond yields could increase while historical volatilities for swap rates decreased. Still, for more than 75% of data the correlation was above 0.5.

3 Discussion

3.1 General conclusions

- Approximation formula (1.4) connects volatility of the bond yield to maturity to bond price. Approximation accuracy decreases when either yield to maturity, coupon or level of volatility increase. For the current market conditions the accuracy is tolerable. The reason for the approximation error lies in the "local" validity of the Black-Scholes formula: implied volatility should only be used point-wise (for each strike) to be able to correct for the price of the option. However when the volatility smile for a bond price option is derived from the volatility of yield based on stochastic process with non-flat volatility smile, Black-Scholes formula (derived originally for stochastic processes with constant log normal volatilities) would not provide a precise price value. As explained in the Methods section, actually volatility smiles for the bond price and bond yield to maturity cannot be flat simultaneously. In simulations the accurate "theoretical" approach from the Methods section can be implemented instead of approximation according to formula (1.4). However the "theoretical" approach is impractical as it demands unrealistically accurate knowledge of volatility smile for a wide range of strikes.

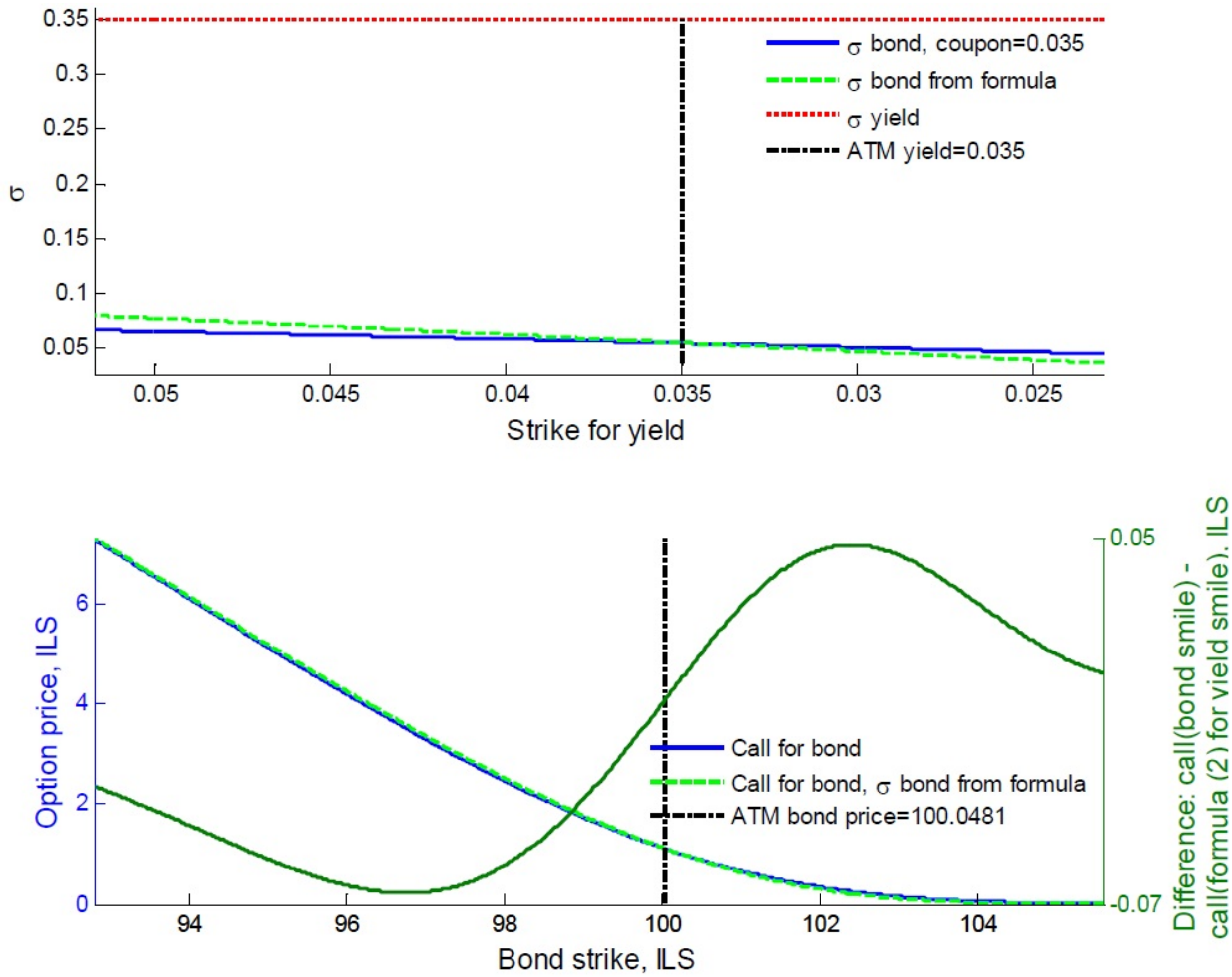

Figure 5: Example of applying the theoretical approach to specific yield volatility smile. In this example ATM yield to maturity is equal to bond's annual coupon being 3.5%, the yield has flat volatility. The options have 3 months to expiry. **Upper part.** Red: implied volatility of bond yield, green: bond volatility obtained by means of formula (1.4), blue: bond implied volatility obtained based on option prices. **Lower part.** Left axis: option prices (based on "green" and "blue" volatilities from the upper panel). Right axis: their difference ("dark green" = "green" - "blue").

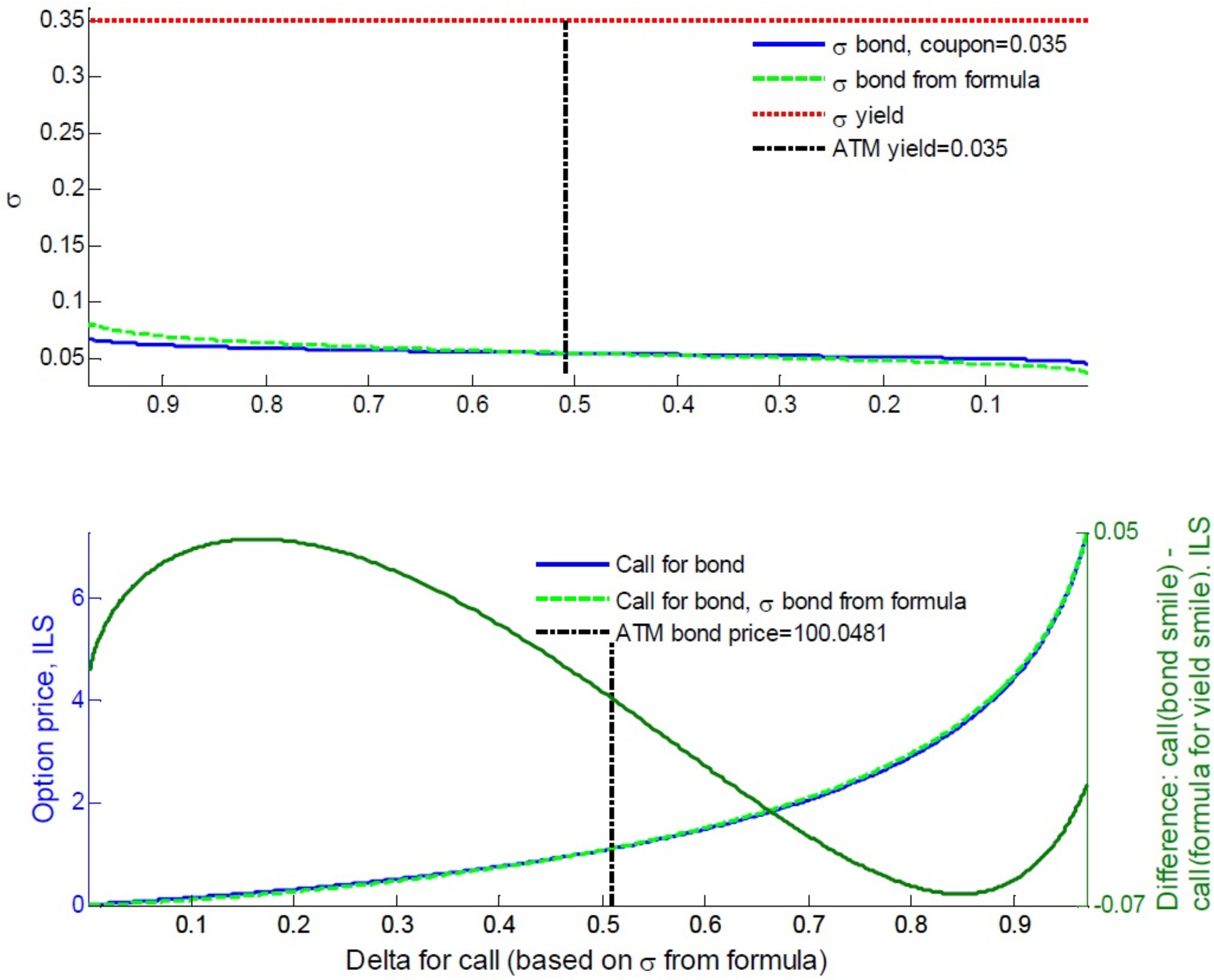

Figure 6: The same example as in Figure 5, but versus delta. In this example ATM yield to maturity is equal to bond's annual coupon being 3.5%, the yield has flat volatility. The options have 3 months to expiry. **Upper part.** Red: implied volatility of bond yield, green: bond volatility obtained by means of formula (1.4), blue: bond implied volatility obtained based on option prices. **Lower part.** Left axis: option prices (based on "green" and "blue" volatilities from the upper panel). Right axis: their difference ("dark green" = "green" - "blue").

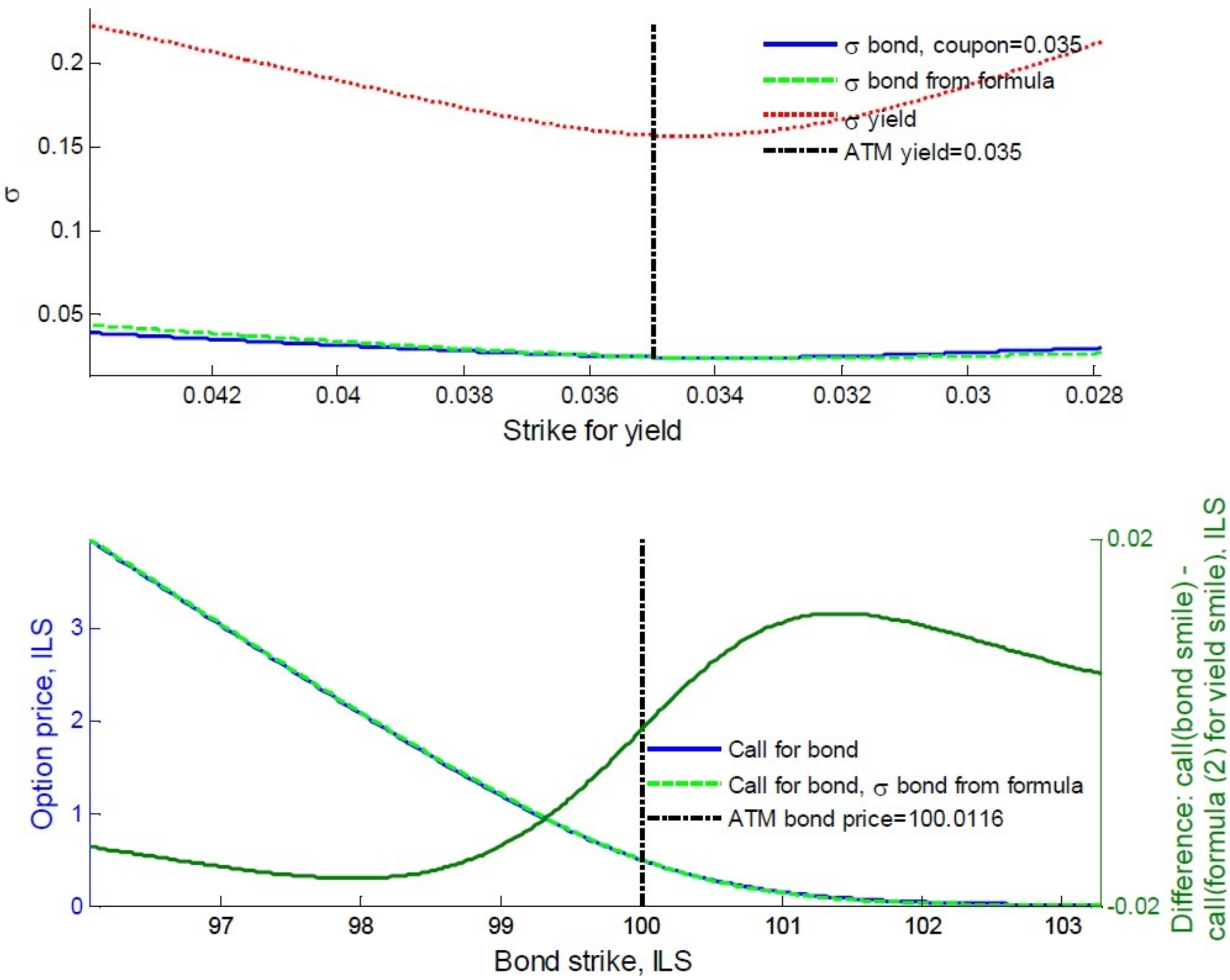

Figure 7: Example of applying the theoretical approach to specific yield volatility smile. In this example ATM yield to maturity is equal to bond's annual coupon being 3.5%, the yield has non-flat volatility profile once observed in the USD swaption market. The options have 3 months to expiry. **Upper part.** Red: implied volatility of bond yield, green: bond volatility obtained by means of formula (1.4), blue: bond implied volatility obtained based on option prices. **Lower part.** Left axis: option prices (based on "green" and "blue" volatilities from the upper panel). Right axis: their difference ("dark green" = "green" - "blue").

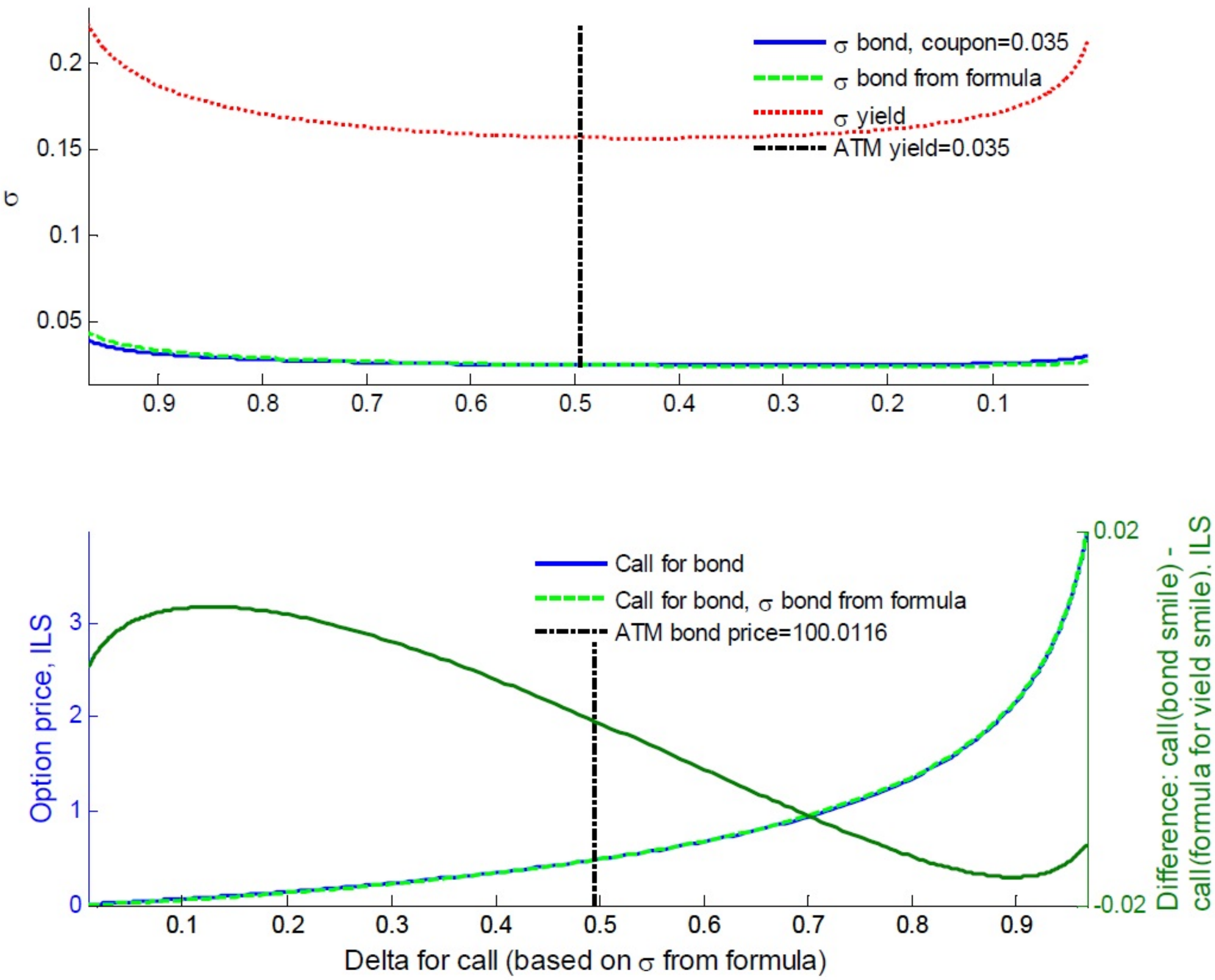

Figure 8: Same example as in Figure 7, but versus delta. In this example ATM yield to maturity is equal to bond's annual coupon being 3.5%, the yield has non-flat volatility profile once observed in the USD swaption market. The options have 3 months to expiry. **Upper part.** Red: implied volatility of bond yield, green: bond volatility obtained by means of formula (1.4), blue: bond implied volatility obtained based on option prices. **Lower part.** Left axis: option prices (based on "green" and "blue" volatilities from the upper panel). Right axis: their difference ("dark green" = "green" - "blue").

Coupon	**Maximal difference between option prices, basis points**	Volatility at the bond price strike corresponding to column 2, based on the "theoretical" approach	Volatility at the bond price strike corresponding to column 2, based on the formula	Bond price (strike) at which value in column 2 is achieved, ILS	ATM bond price, ILS	Yield to maturity correspon ding to ATM bond	ATM yield	Comments
1	2	3	4	5	6	7	8	9
1.000%	**2.110**	1.874%	2.064%	98.949	100.000	0.999%	1.000%	The level for 5Y yield to maturity in the first half of 2012 was near 3.5%; the value decreased since then.
2.000%	**4.060**	3.625%	3.990%	98.000	100.020	1.996%	2.000%	
3.000%	**5.820**	5.265%	5.785%	97.142	100.040	2.992%	3.000%	
4.000%	**7.410**	6.800%	7.461%	96.369	100.060	3.986%	4.000%	
5.000%	**8.840**	8.239%	9.023%	95.671	100.090	4.979%	5.000%	
6.000%	**10.130**	9.587%	10.481%	95.044	100.130	5.970%	6.000%	
7.000%	**11.280**	10.852%	11.841%	94.480	100.160	6.960%	7.000%	
8.000%	**12.310**	12.046%	13.126%	93.929	100.210	7.948%	8.000%	
9.000%	**13.210**	13.161%	14.312%	93.473	100.250	8.936%	9.000%	
10.000%	**14.010**	14.208%	15.420%	93.066	100.300	9.922%	10.000%	

Table 2: Accuracy of approximation with formula (1.4) for the option maturing in 3 months on bond paying fixed annual coupon with 5 years to maturity and constant yield volatility of 0.35. Bonds with coupons varying from 1% to 10% (column 1) are considered assuming that (1) bond's yield to maturity is equal to the coupon, (2) implied volatility smile for bond's yield is constant. Exemplar cases indicate that such differences are larger when the volatility smile for yield is constant versus more "realistic" smile profiles. So by considering cases with constant smile we expect to obtain more extreme differences between option prices computed with two different methods. The estimation of the accuracy is based on comparing "correct" option prices obtained with numerical simulations and option prices based on volatilities from formula (1.4). The simulations were based on the flat yield volatility smile with volatility being 0.15. The maximal difference in basis points (1 basis point = 1/10000 of bond's par value) is shown in column 2. The probability of achieving a close to maximal difference in option prices is not negligible noting that the volatility (columns 3, 4) of the bond price strikes at which the difference is maximal (column 5) is above the relative difference between these strikes and ATM (column 6) bond prices. Yields to maturity corresponding to the ATM bond prices in column 6 are shown in column 7. For comparison, the ATM yields are presented in column 8. One may note that the ATM bond price (column 6), which is the expected future bond price, is not equal to the par value (100) even though ATM yield (column 8) is equal to bond's coupon (column 1). Actually the ATM bond yield is not equal to the yield identified with the future bond price (column 7) due to convexity of the relationship between bond price and bond yield. Exemplar estimates of the approximation accuracy presented in column 2 of the Table are generalized to a wider range of coupons and yields in Figures 9, 10. In particular the figures include cases when the bond coupon and ATM yield to maturity are different. In Figure 9 the flat volatility is equal to 0.15 and in figure 10 the flat volatility is equal to 0.35.

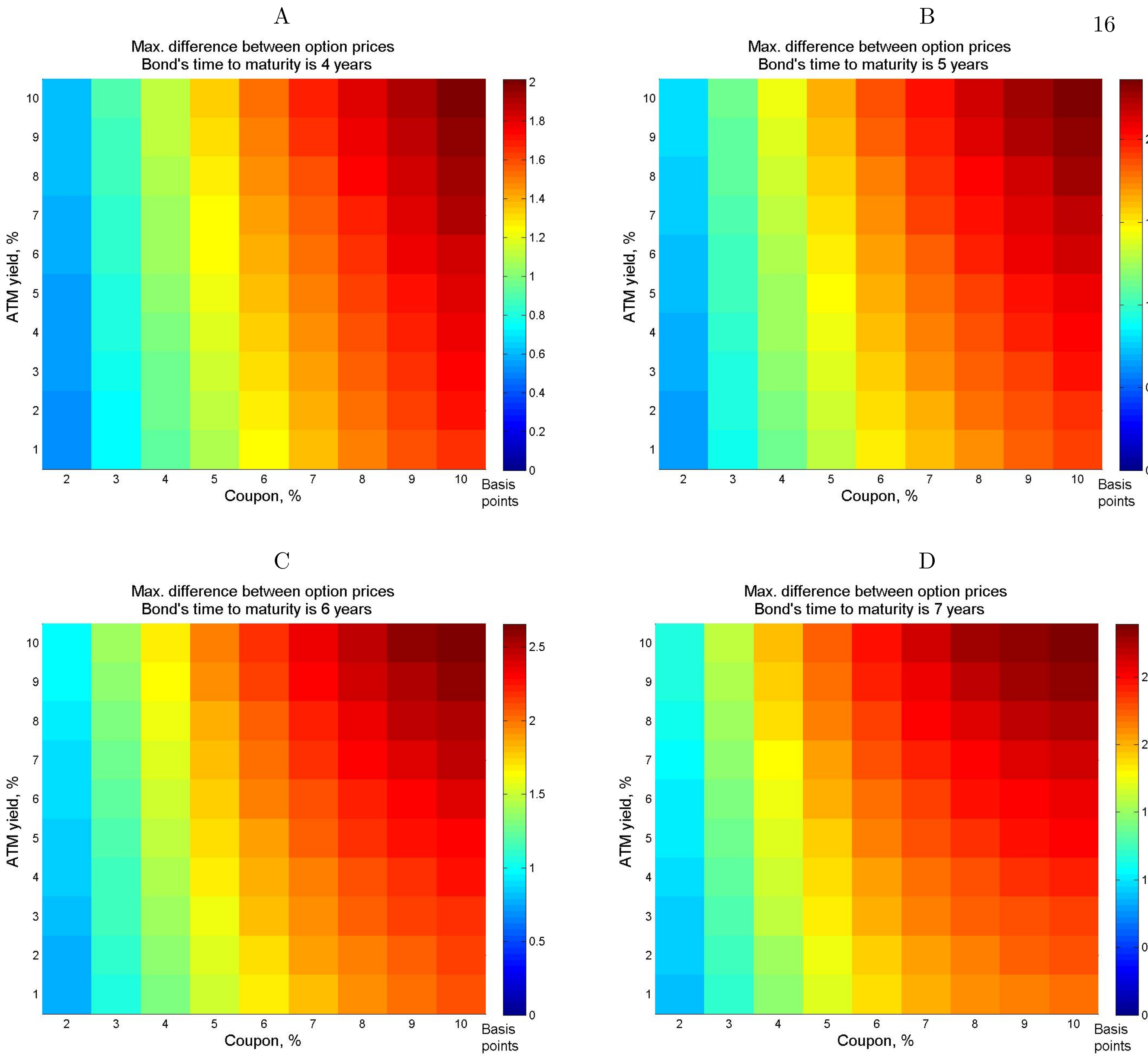

Figure 9: Generalization of the data in Table 1. Maximal difference between option prices computed according to (1) numerical simulation demonstrated in Figure 1 and (2) based on the volatility obtained with the approximation formula (1.4). The differences are provided for the time to expiry of 4-7 years (charts A-D respectively), for at the money rates varying from 1% to 10% (y axis) and bond's annual coupon varying between 2% and 10% (x axis). The difference grows when either coupon or yield to maturity grow. However the growth of the difference is larger for coupon's increase than for yield's increase. The yield to maturity was assumed to have a flat implied volatility equal to 0.15. The differences are smaller than the differences depicted in Figure 10 for the case of yield to maturity being equal to 0.35.

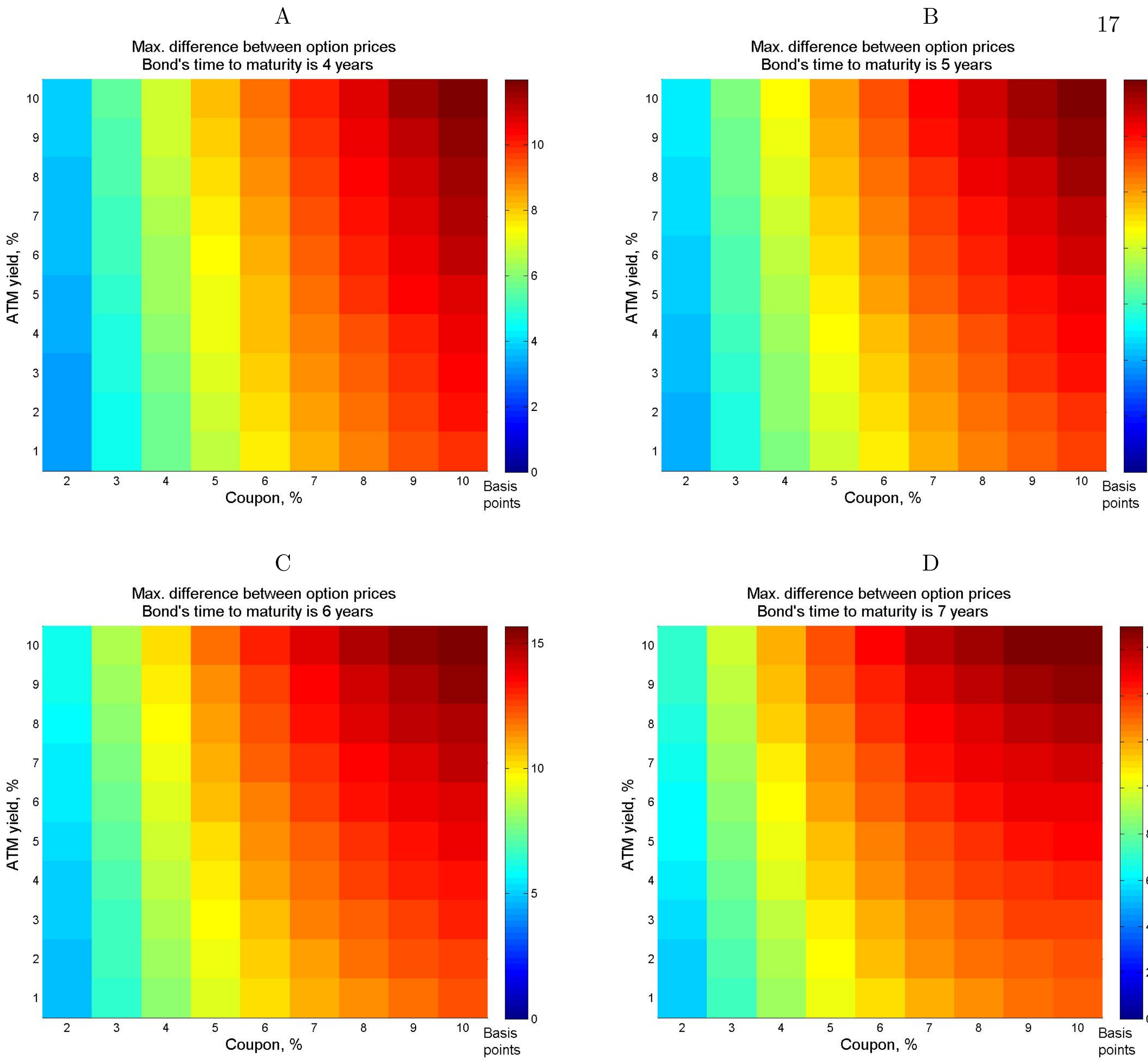

Figure 10: Generalization of the data in Table 1. Maximal difference between option prices computed according to (1) numerical simulation demonstrated in Figure 1 and (2) based on the volatility obtained with the approximation formula (1.4). The differences are provided for the time to expiry of 4-7 years (charts A-D respectively), for at the money rates varying from 1% to 10% (y axis) and bond's annual coupon varying between 2% and 10% (x axis). The difference grows when either coupon or yield to maturity grow. However the growth of the difference is larger for coupon's increase than for yield's increase. The yield to maturity was assumed to have a flat implied volatility equal to 0.35. The differences are larger than the differences depicted in Figure 9 for the case of yield to maturity being equal to 0.15.

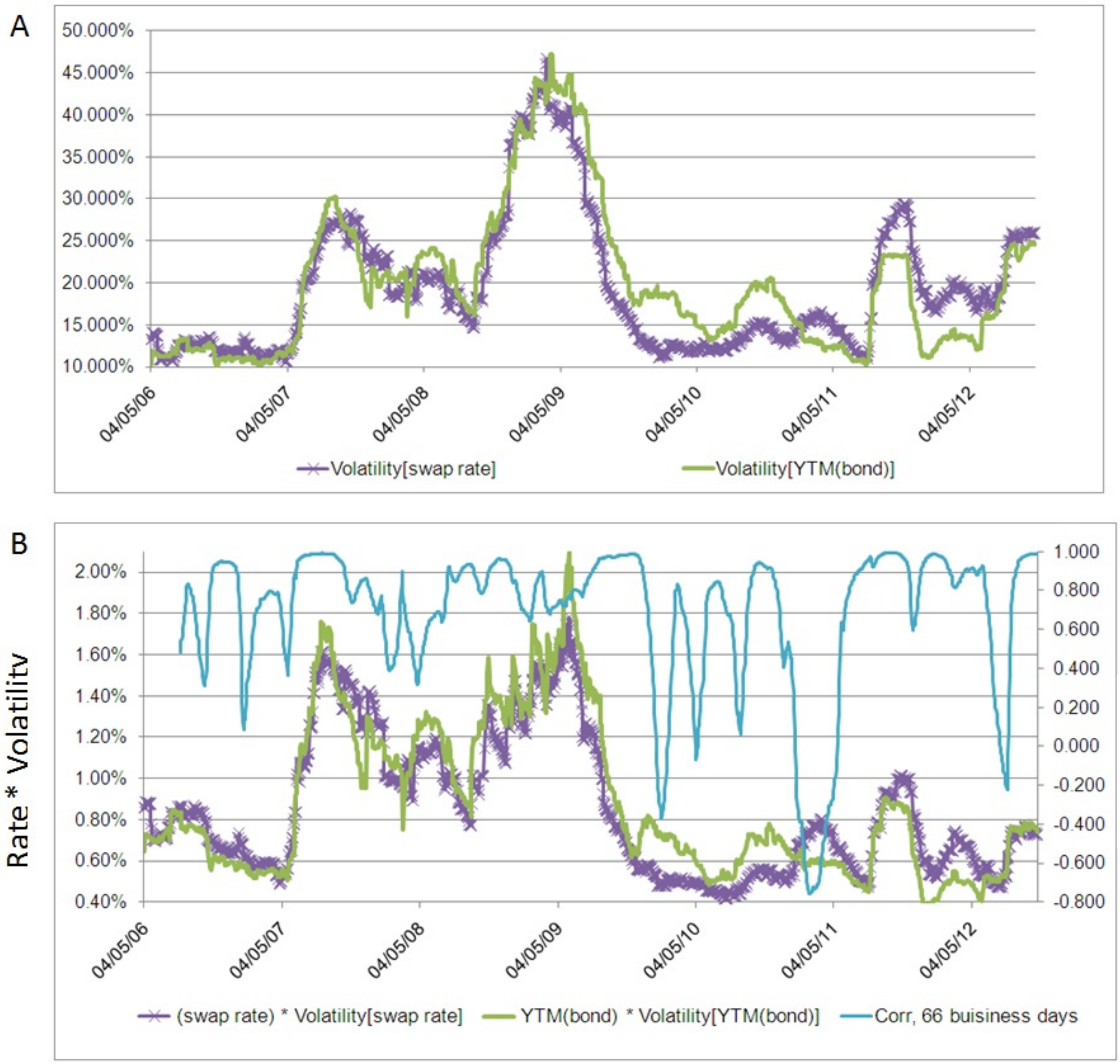

Figure 11: Relationship between historic volatility for 5Y government bond (paying 5% annual coupon) yield to maturity implied from Shahar zero curve and 5Y ILS IRS rate. The volatilities were computed based on 3M measurements. **A.** Historic volatilities. **B.** Same volatilities as in A weighted with the level of the corresponding rates. Blue line depicts the correlation coefficient between the weighted volatilities. Each correlation coefficient was computed over 3 months of volatility measurements. Swap rates data courtesy of Bloomberg LP.

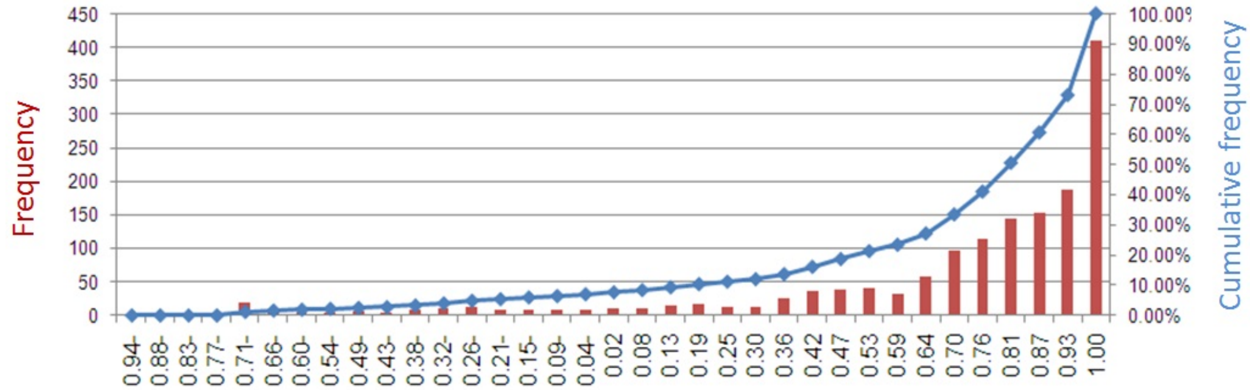

Figure 12: Frequency of correlation values, 3M (red, left axis) and cumulative frequency (blue, right axis).

- Rate volatilities in the swap market weighted with the swap rate level behave similarly to the yields in the bond market weighted with the yield level in the sense that large modifications occur simultaneously in both markets, although rarely during short periods of time swap rate historic volatility may be increasing while bond yield historic volatility is decreasing.

3.2 Swap rates as a proxy to bond yields

The standard approach for pricing European swaptions is based on the Black-Scholes formula with forward swap rate being the forward asset price, e.g. formula (26.5) in [1]. The forward swap rate is equal to the expected forward swap rate in the risk neutral measure used to derive the formula[18]. However the forward bond yield is not equal to the expected future bond yield when the expectation is computed in the world which is risk neutral with respect to bond maturing at time T (Chapter 27.1 of [1]). The discrepancy between the forward yield and expected forward yield is quantified by the notion called convexity adjustment (Chapter 27.1 of [1]). For example, the discrepancies between the ATM bond price and the par value (100) in Figures 5, 7 are caused by the convexity effect. It should be noted here that the situation when the ATM value of the bond option is not equal to bond's par is normal. Discrepancy between the way the swap rate [19] and the bond yield are considered in the pricing formulae was omitted from our current analysis. Whenever the requirements from the inference procedure are more than providing a geometric proxy to the form of the volatility smile, for example for finding arbitrages or for pricing in the market with some non-negligible level of liquidity, the above mentioned discrepancies should be further analyzed.

Is the level of closeness between swap rates and bond yields observed in Figures 3, 4 and 11 sufficient to conclude that they are indeed close and the method we analyze is eligible for producing the proxy to the volatility smile of one of the parameters given volatility smile of the other parameter? We propose that strictly positive non-negligible correlations of daily relative fluctuations (Figure 4) and closeness of the rates themselves[20] (Figure 3) imply that both rates have similar dynamics most of time. Therefore one can apply formula (1.4) using the data from the swap market to produce a proxy to the form of the volatility smile for bond price. Of course the resulting volatility smile should be taken as a proxy only. For example, the level of the "desirable" bond price smile may presumably be corrected as if for the at the money level uniformly along strikes, assuming that the trader has an opinion with respect to the at the money volatility. However when the dynamics observed in the market is not similar (eg. relatively

[18] Annuity is used as a numeraire.

[19] Whose volatility smile is one of the inputs for the inference procedure.

[20] Most of time the level of rates is tens times higher than the magnitude of their difference.

rare cases of low or negative correlation between the rates or between historic volatilities), or the levels of either interest rates or volatilities are high (see inaccuracy estimates in Figures 9, 10) one should keep in mind that the procedure of volatility inference might be less reliable.

The more liquid the market becomes the more critical the inconsistencies between the swap rates and bond yields become for the validity of the inference procedure. Even in case of high level of bid-ask spreads in illiquid markets of options on interest rates we are cautious about the degree to which the implied volatility of swap rates serves as a proxy to bond yields. Our empirical tests can be at best related to the ATM level of volatilities. Extrapolation of the conclusions from the analysis of empirical data to non-ATM strikes along the volatility smile should be considered as an "educated guess". Nevertheless we propose that the geometric form of the volatility smile reconstructed according to the inference procedure should be meaningful for volatility inference between the markets especially when no other source of information is readily available.

A Yield to maturity, coupon and swap rates

Here we analyze mathematically the relationship between the bond's coupon, yield to maturity and the swap rate. *Throughout this appendix we assume that:*

1. The bond and the swap's fixed leg have the same schedule of annual payments.
2. The fixed and the floating legs of the swap have the same payments' schedule. For example, if swap's fixed leg pays an annual coupon then its floating leg provides annual payments as well.
3. Bond and swap are priced based on the same interest rates curve.

Assumptions 1 and 2 ease the formal part of derivations and their relaxing will leave us with the same conclusions or with very close approximations replacing some equalities. However assumption 3 is essential as it justifies the possibility to substitute discount factors from the formula defining the swap rate into the formula defining the bond price. Although the mathematical derivations will not survive relaxing assumption 3, analysis of empirical data shows that the values of the swap rate and yield to maturity are most of time relatively close (Figures 3 and 11). Their difference is always tens times smaller than the level of rate and of course much smaller than the difference between the swap rate and bond's coupon.

A.1 Yield to maturity is equal to annual coupon of the bond when bond's NPV is equal to par value

There is a unique yield to maturity for each bond. Let us show that when yield to maturity is equal to bond's annual coupon, the NPV of such bond is equal to its par value. Let the bond have n years to maturity, its coupon and yield to maturity are equal to c%, and its par value is equal to 1 (=100%). Then we have, noting that the coupon is equal to yield to maturity:

$$NPV = c\sum_{i=1}^{n} \frac{1}{(1+c)^i} + \frac{1}{(1+c)^n}$$

Now use the following propery: $q + q^2 + \ldots + q^n = q\frac{q^n-1}{q-1}$ by setting $q = \frac{1}{1+c}$:

$$\begin{aligned}
NPV &= c \cdot \left[\frac{1}{1+c} \cdot \frac{\frac{1}{(1+c)^n} - 1}{\frac{1}{1+c} - 1}\right] + \frac{1}{(1+c)^n} \\
&= c \cdot \frac{1}{1+c} \cdot \frac{1-(1+c)^n}{1-(1+c)} \cdot \frac{1+c}{(1+c)^n} + \\
&\frac{1+c}{(1+c)^n} = \frac{(1+c)^n - 1}{(1+c)^n} + \frac{1}{(1+c)^n} \\
&= 1 = 100\% \,.
\end{aligned}$$

Therefore NPV is equal to par value in this case. In case NPV is not equal to the par value, the yield to maturity cannot be equal to the bond's coupon as such equality necessarily implies the equality of the NPV to the par value.

A.2 Bond's coupon being equal to the swap rate implies that the bond is at par and the swap rate is equal to bond's yield to maturity

The price of the bond can be directly computed as the net of the future cashflows discounted based on the interest rate curve:

$$B(c) = \text{Par} \cdot \left[c \sum_{i=1}^{N} (T_i - T_{i-1}) e^{-r_i T_i} + e^{-r_N T_N} \right] . \tag{A.1}$$

For the swap rate, and assuming it is equal to the bond coupon, we have:

$$\sum_{i=1}^{N} \left(e^{-(r_{i-1}T_{i-1} - r_i T_i)} - 1 \right) e^{-r_i T_i} = c \sum_{i=1}^{N} \left(T_i - T_{i-1} \right) e^{-r_i T_i} , \tag{A.2}$$

where left and right hand sides of the equality correspond to the present value of the floating and fixed legs of the swap respectively.

So for the bond under consideration, noting that $T_0 = 0$ in our case of computing the present value and substituting the equality (A.2) into (A.1) we obtain:

$$\begin{aligned} B(c) &= \text{Par} \cdot \left[\sum_{i=1}^{N} \left(e^{-(r_{i-1}T_{i-1})} - 1 \right) e^{-r_i T_i} \right. \\ &+ \left. e^{-r_N T_N} \right] = \text{Par} \cdot e^{-r_0 T_0} = \text{Par} . \end{aligned}$$

Noting that the swap rate is equal to bond's coupon and that the bond is at par, we conclude from the section A.1 that the swap rate is equal to the annually compounded yield to maturity of the bond at par.

A.3 Bond's yield to maturity is close to the swap rate derived from the same interest rate curve

Section A.2 shows that the yield to maturity of the bond at par is equal to the swap rate[21]. Traded bonds are rarely at Par. Nevertheless here we show that swap rate is close to bond's yield to maturity although they are not exactly the same as the bond is not necessarily at par.

[21] Again, our mathematical derivations here assume that the swap and the bond are priced based on the same interest rate curve and have the same payments' schedule.

The price of the bond can be directly computed as the net of the future cashflows discounted based on the interest rate curve:

$$B(c) = \text{Par} \cdot \left[(c/k) \sum_{i=1}^{N} e^{-r_i T_i} + e^{-r_N T_N} \right] , \tag{A.3}$$

where c is bond's coupon, k is the number of coupons in a year, r_i is continuously compounded interest rate corresponding the time instant T_i of the i-th cashflow. On the other hand bond's price can be computed based on the yield to maturity discounting:

$$B(y(c), c) = \text{Par} \cdot \left[(c/k) \sum_{i=1}^{N} e^{-yT_i} + \text{Par} \cdot e^{-yT_N} \right] , \tag{A.4}$$

Therefore, after subtracting the left hand side of equality (A.4) from the left hand side of (A.3) and differentiating we have:

$$\begin{aligned} 0 &= \frac{d}{dc} \left[B(c) - B(y(c), c) \right] = \frac{dB(c)}{dc} \\ &- \left(\frac{\partial B(y(c), c)}{\partial c} + \frac{\partial B(y(c), c)}{\partial y} \cdot \frac{dy}{dc} \right) . \end{aligned}$$

So

$$0 = \frac{\text{Par}}{k} \cdot \sum_{i=1}^{N} \left(e^{-r_i T_i} - e^{-yT_i} \right) + DB \frac{dy}{dc} ,$$

where D is bond's duration [1]. Finally,

$$\frac{dy}{dc} = \frac{\text{Par} \ \cdot \sum_{i=1}^{N} \left(e^{-yT_i} - e^{-r_i T_i} \right)}{kDB} . \tag{A.5}$$

The sum in the numerator of the derivative will usually be a small number and almost zero for a shallow interest rate curve. The ratio $\text{Par}/(kBD)$ will usually be close to $1/D$ because the bond with $\text{Par} = 100$ will usually trade above 90. So in most cases the derivative A.5 will be smaller than $1/D$. Considering a bond with 5 year to maturity, to the first order, dy/dC will be at most 0.25 but in common market conditions much-much smaller. Very small absolute value of the derivative in formula (A.5) implies that to the first order bond's yield to maturity will usually be much closer to the swap rate than to bond's coupon[22].

The closeness of bond's yield to maturity to the swap rate rather than to bond's coupon is demonstrated in the example presented in the Table 3. We implemented various simulations with randomly generated zero rate curves and bond coupons and we always got a very close proximity of yield to maturity to the swap rate, always strongly outperforming the proximity to bond's coupon. The results of the simulation are presented in Figure 13.

B Risk neutral probability of the future asset value implied from options' market (following material from [2])

It is known that the price of the call option is computed via the following approach:

$$\begin{aligned} c(t,X,T) =& e^{-r\tau} E\left[\max(S_T - X, 0)\right] \\ =& e^{-r\tau} \int_X^\infty (S_T - X)\pi(S_T) d(S_T) \ . \end{aligned}$$

It can be shown that

$$\begin{aligned} &\frac{\partial c(t,X,T)}{\partial X} = -e^{-r\tau} \int_X^\infty \pi(S_T) d(S_T) = \\ &-e^{-r\tau}\left[1 - \Pi(X)\right] = -e^{-r\tau}\Pi(S_T > X) \ , \end{aligned} \tag{B.1}$$

where Π stands for risk neutral cumulative probability and π for risk neutral probability density.

Whenever bond's yield to maturity grows the bond price decreases and vice versa, that is the relationship between the yield and bond price is monotonous. Therefore whenever we know that the probability of the yield is below some number, say $\Pi(y < a) = A$, we also know that the probability for the bond price to be above $B(a)$ is the same:

$$\Pi(B > B(a) = 100 \cdot e^{-aT}) = \Pi(y < a) = A \ . \tag{B.2}$$

[22] As shown above, equality between bond's coupon and yield to maturity means that they both are equal to the swap rate and that the bond is at par.

In this case, theoretically, whenever we know the option prices on yield (know the volatility smile for yield), we will be able to estimate option prices for the bond price (volatility smile for the bond price). This approach can be used to check the formula for approximating the bond price volatility by the yield price volatility (from [1], p. 643):

$$\sigma_B = D y_0 \sigma_y \ . \tag{B.3}$$

where y_0 is the initial value of the forward yield.

So, given some smile profile for σ_y we can build a smile profile for σ_B by first computing the values of corresponding options on yield and then using approach in formulas (B.1), (B.2) of this Appendix. Further bond option prices are also computed based on (B.3) and the two pricing results can be compared. We have called it the theoretical approach, see Figure 1.

Please note, that using approach in (1.4) in order to build a volatility smile for bond options needs integration implied from the formula (B.1) and therefore a value of the bond option should be known for some strike. We assume that both options (for yield and for a bond price) are near zero for the deepest out of the money strike among the strikes considered in the smile. Formula (B.3) is used to find the value of the volatility and option price (deepest out of the money). Further option prices and volatilities for other strikes are derived based on integration which follows from formulas (B.1), (B.2).

The iterative procedure based on formula (B.1) is as follows:

$$\begin{aligned} c(t, X+\Delta X, T) \approx\ & c(t,X,T) - \Delta X \cdot e^{-r\tau}\left[1 - \Pi(X)\right] = \\ & c(t,X,T) - \Delta X \cdot \left[e^{-r\tau} - e^{-r\tau}\cdot \Pi(X)\right] = \\ & c(t,X,T) - \Delta X \cdot \left[e^{-r\tau} - e^{-r\tau}\cdot \Pi(S_T(X) < X)\right] = \\ & c(t,X,T) - \Delta X \cdot \left[e^{-r\tau} - e^{-r\tau}\cdot \Pi(W(S_T) > W(x)\right] = \\ & c(t,X,T) - \Delta X \cdot \left[e^{-r\tau} + \frac{\partial c(t,W,T)}{\partial W}\right] \ , \end{aligned}$$

where W is the strike for the yield, $W(X) = -\ln(X/100)/T$ for zero coupon bond with continu-

Interest rate curve

Year	Annually compounded zero rates	Discount factors	Swap rate
1	3.00%	0.97087	3.00%
2	4.00%	0.92456	3.98%
3	4.50%	0.87630	4.46%
4	4.80%	0.82900	4.75%
5	5.00%	0.78353	4.94%

Differences

Bond coupon	5Y yield to maturity	(5 Y YTM - 5Y swap rate) / min(5Y YTM, 5Y swap rate)	(5Y YTM - 5Y coupon) / min(5Y YTM, 5Y coupon)
1.00%	4.99%	0.010	3.99
2.00%	4.97%	0.007	1.49
3.00%	4.96%	0.005	0.65
4.00%	4.95%	0.002	0.24
5.00%	4.94%	0.000	-0.01
6.00%	4.93%	-0.002	-0.22
7.00%	4.92%	-0.004	-0.42
8.00%	4.91%	-0.006	-0.63

Table 3: The table contains an exemplar zero rate curve and yields to maturity for 5 year bonds with different coupons. Yield to maturity of those bonds is very close to the 5 year swap rate (4.938%) even though bonds' annual coupons vary from 1% to 8%. This example demonstrates to what degree the swap rate and bond's yield to maturity are close to each other almost irrespective of the value of the coupon, that is the bond may be rather far from being at par. The yield to maturity is much closer to the swap rate than to the coupon as can be seen in the two rightmost columns of the table. The abbreviation YTM in the table stands for yield to maturity.

A

B

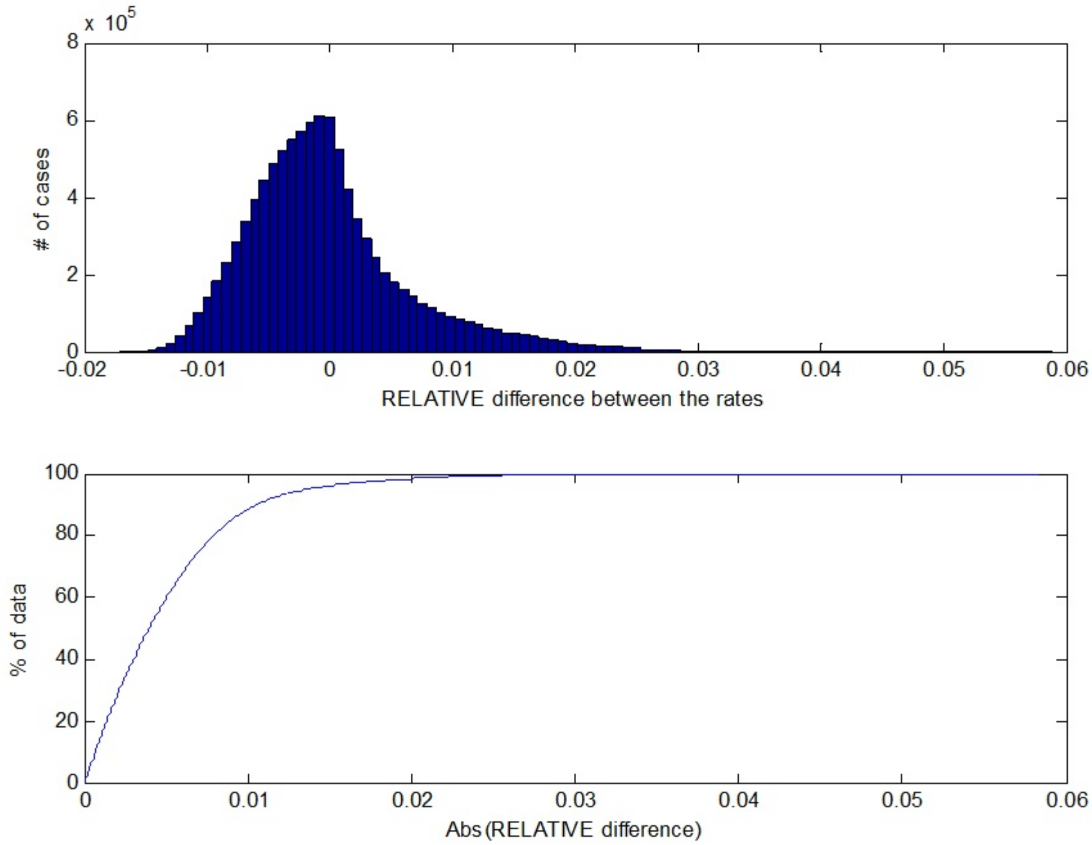

Figure 13: Relative differences between the 5 years swap rate and bond yield to maturity. The compared values of bond yield and the swap rate are implied from the same interest rate curve. Interest rate curves and bond's coupons were randomly generated in 10,000,000 simulations. **A**. Histogram of the relative differences. **B**. Cumulative distribution of the absolute value of the relative difference. For more than half of the data the relative difference was bounded by 0.005. For almost all data the absolute value of the relative difference was bounded by 0.03.

ously compounded yield. Similarly,

$$c(t, X - \Delta X, T) \approx c(t, X, T) + \Delta X \cdot e^{-r\tau} \left[1 - \Pi(X)\right] = \\ c(t, X, T) + \Delta X \cdot \left[e^{-r\tau} + \frac{\partial c(t, W, T)}{\partial W}\right] .$$

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